

Sliding Mode Control and Applications

Dr. Shyam Kamal

Department of Electrical Engineering

IIT(BHU) Varanasi

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Welcome back. In previous classes, I talked about the sliding mode control for non-linear systems as well as linear systems. I have also extended the concept of terminal sliding mode control to achieve finite-time convergence from any initial condition. Now, during this development, what we have observed is that system dynamics is insensitive with respect to any kind of meshed uncertainty. meaning of matched uncertainty we have already discussed that is some class of uncertainty which is going to enter through the control channel. So, our system becomes insensitive with respect to matched uncertainty whenever we are in the sliding phase, but in the reaching phase, I cannot give that kind of guarantee, and due to that reason, a new development comes into the picture that is known as integral sliding mode control.

So, for the purpose of integral sliding mode control, I will justify the name. So, the name we are mentioning here is integral and sliding mode, which you are already well aware of. So, why term integral comes into picture, you can able to understand in the subsequent slide. Let us first explore the limitations of conventional sliding mode control, which means whatever we have learned, either sliding mode control achieved by on-off control with a very high switching frequency or unit vector control, as well as terminal sliding mode control.

In all previous development, robustness comes only after the reaching phase. Actually, whenever we are talking about robustness, in sliding mode control, we are talking about the insensitivity with respect to matched perturbation. So, that will only achieve during the reaching phase. What does it mean? That suppose I have a second error system $x_1 = x_2$, and I have designed this as our sliding surface, which is S . Obviously, if I start

from here, so our dynamics is actually perturbed by the perturbation, but once we are on this particular surface, then now you can see that our system is free from the matched uncertainty, and due to that reason, the robustness or higher level of robustness that is called insensitivity is only during this phase.

Due to that reason, what kind of key questions come into the picture? Can we eliminate the reaching phase? So, if I am somehow able to remove this phase and we know that we have insensitivity with respect to matched perturbation during the sliding phase, then I can achieve a higher level of robustness that is called insensitivity throughout the system evolution. And obviously, second point I have already discussed that no disturbance rejection during the initial transient, that is during the reaching phase. Now, we have also seen the order of reduction. So, now here one challenge comes into the picture that whenever we are designing some kind of sliding mode control, you can just take the example of the second-order system

$$\dot{x}_2 = u + d(t).$$

So, $d(t)$ is nothing but a kind of matched uncertainty.

So, you can see here that during the sliding, what happens is that x_2 is replaced by some kind of control such that I am able to achieve some kind of objective with respect to x_1 . It means that I have to design x_2 and some kind of function $\phi(x_1)$, and this term should be differentiable and also should be bounded. If that is not differentiable and bounded, then some kind of unnecessary singular term appears inside the control. We have already seen the terminal sliding mode control. So, now this particular phenomena appear due to the somehow due to the order reduction, because we are designing some kind of control for the reduced order, and then we are extending the philosophy of sliding mode control to maintain $s = 0$.

But if we actually remove the reaching phase, it is possible to show that one can remove the singularity. And due to that reason, in several cases, order reduction is not going to give us the benefit. And obviously, our key question is, can we achieve robustness starting from $t = 0$? The answer is yes, because whatever control I am going to implement, which is high switching frequency-based control, means that we are going to switch at very high speed. I have just done a comparison between integral sliding mode control and conventional sliding mode control. So, you can see that reaching phase is required during the conventional or classical sliding mode control, but integral sliding mode control one can able to remove the reaching phase and that is the basic requirement or basic philosophy of the integral sliding mode control.

Robustness only after; robustness means we are talking about insensitivity with respect to the matched uncertainty, and this is going to start from $t = 0$. So, what we are going to learn in this particular lecture is that order reduction comes into the picture.

Here, there is no order reduction, and I have already told you that “sometime” means reduced order, not “become” beneficial. Obviously, most of the time this is beneficial, but for some particular applications, it is not always beneficial. And disturbance rejection is partial, because we are only talking about the insensitivity whenever we are in the reaching phase and due to that reason disturbance rejection is partial.

Here, actually sliding starts from $t = 0$, which means when I started the system, and due to that reason, throughout the time interval, we have insensitivity with respect to the matched uncertainty. In the next lecture, I will also talk about when unmatched uncertainty comes into the picture, what kind of phenomena actually appear during the integral sliding mode control. Obviously, this is basically a requirement of all applications. Now a day we are in the era of AI robotics and there we are trying to develop several critical application for medical purpose, for defense purpose and for that it is possible to show that integral sliding mode control is very helpful and precision is also guaranteed because we are assuming uncertainty rejection from the initial point onward. So, let us try to observe the development.

I am going to consider the very generalized system, and during the definition phase, we have already seen that this class of system is called the non-autonomous system. Why is that called a non-autonomous system? Because you can see that this dynamics f that is explicitly depending on t and here I am assuming that control is going to appear like bu , control can be time varying, time dependent or state dependent and this ϕ is the perturbation. Here, in this lecture, in particular, I am assuming that the perturbation is bounded and that this perturbation is also matched. What is the meaning of matched perturbation? We have already discussed this several times; I have actually mentioned that if you are able to express this perturbation as some kind of $\xi(t, x)$, and where ξ is bounded, then this perturbation is called the matched perturbation. So, for all t and x , we are assuming that this perturbation is bounded and rank of b , we are assuming m .

In this particular development, I am assuming that I have n number of state and m number of control. Integral sliding mode control contains two parts. So, one part is called the nominal part, and u_1 , which is called the switching part, is responsible for disturbance rejection, and the philosophy is very, very simple. You can just return back to the sliding mode control for the first order system, at that time what I have discussed that suppose $x \in \mathbb{R}$ and here I have disturbance that is also belongs to \mathbb{R} . So, if I design $k \geq d(t)$, then I can able to give guarantee that $x(t)$ tending towards 0 as t tending towards infinity, that can the t to some finite value T .

So, this kind of guarantee I can able to give. Now, as t tends towards T , since $x(t)$ is 0. It means that in a one-dimensional plane, I cannot move from here. This is $x = 0$. What does it mean? Instantaneous velocity is non-zero, but the average velocity is 0.

So, if you see the equivalent value of this $k \text{sign}(x)$, it is exactly equal to $d(t)$. Due

to that reason, sliding mode control acts like a disturbance observer. And the same philosophy I am going to utilize here. So, nominal control you can design from any standard method, you can apply PID controller that is widely used in industry, pole placement method. So, in this lecture, I will give you some introduction to pole placement and the linear quadratic regulator, which is based on the optimal control principle and model predictive control.

So, these kind of control is widely applicable in the industry. What I am telling you is that you can add one more component, which is from the sliding mode control. Now, if you are going to add this component, then you will get extra benefits. So, your system is free from the matched uncertainty from the initial point onward. So, that kind of beauty you will get, and why that kind of beauty I will get due to this, you will be able to extend this kind of result for the multi-input, multi-output case by defining the unit vector-based control.

So, same philosophy I have utilized in this particular lecture, because I have already assumed, you can see here that I have m number of input. And due to that reason, I have to generalize case $\text{sign}(x)$ for multi-input, multi-output case. So, using the unit vector philosophy, I am going to generalize it. Now, here you can understand the significance of the name. I have already told you that control contains two parts.

One part you can design based on any standard method. Nowadays, AI is very popular for data-driven control. So, you can design a control action by removing uncertainty. You can remove matched uncertainty, and then what can you do? You can assume that your system

$$f(t, x) + bu(t, x)$$

now this is free from the uncertainty, and now what is our goal? Suppose that I have to solve a stabilization problem. So, either you can be able to solve a stabilization problem in finite time or infinite time asymptotically; you can design control u_0 , and this is called nominal control.

Now, if you are going to solve it, then you can see what the solution is. The solution to this is obviously that you have to integrate both sides, and due to that reason, here $x(t) - x(0)$ comes into the picture. And after that right-hand side, this whole dynamic from here to here comes into the picture. Now, you can see that as disturbance is going to enter. So, if there is no disturbance, then solution $x(t) = x_0(t)$ and after that plus the integral of this whole term that is stated here, that is exactly match, but what happens? Since, $x(t)$ is generated from the system and due to that reason, now if I have disturbance, then there is mismatch between $x(t)$ and the right hand side of this solution and that error is nothing but the sliding variable. Now, here you can see that I have just m number of control, but if you see the dimension of this, so $x(t) \in \mathbb{R}^n$ and due to that reason I have to map to \mathbb{R}^n .

So, how to map? So, you have to actually create some kind of projection matrix that will actually transfer n dimensional space to m dimensional space and due to that reason, I am going to design $m \times n$. In several application, it is possible to show that simply if you will take $g = b^\top$ that will work. But in some cases, that will not work; then you have to actually design, but you have to make sure that g should be invertible. I will tell you why this is the restriction that comes from the sliding mode control. So, now you can see that this becomes our sliding variable, which is the error between the actual trajectory and this trajectory, generated if there is no disturbance.

Now, here, integral is involved during the design of sliding mode control, and due to that reason, it is so-called integral sliding mode control. That is only the motivation to give a new name like integral sliding mode control. So, the same kind of interpretation I have written here is that this particular term is by keeping the nominal control. Nominal control and I have told you that you can now design, free to design. So, several classical control methodologies that are applicable for linear or non-linear systems that are not performing well in the case of perturbations, but now sliding mode is going to provide support.

If in the absence of disturbance, if you are going to design, then I will guarantee that it will also work in the presence of the disturbance, and that kind of guarantee is offered by the sliding mode. So, the same kind of things I have actually discussed here. So, $x(t)$ is nothing but some kind of penalizing factor or error you can tell that is the difference between actual trajectory and nominal trajectory that is actually generated by the nominal control and g is nothing but the projection matrix. Now, you can see here that during definition, if you will substitute $t = t_0$ and at that time initial condition is $x(t_0)$. So, what kind of result do I have? Everywhere in place of t , t_0 comes into the picture.

So, this $x(t_0)$ and $x(t_0)$ cancel out here, integral from t_0 to t . So, from the initial point onward, $s(t) = 0$. So, now let us interpret this. Suppose that you are starting from the equilibrium point, and if you apply on-off control with very high speed, it means that the disturbance observation property comes into the picture from the starting point onwards.

That is the philosophy. So, it means that you can maintain $x = 0$ from the initial point; then this guy $k \text{sign}(x(t))$ actually takes care of any matched perturbation. So, based on this philosophy, we have essentially designed the integral sliding mode control, and since sliding will start from the initial point onward, because we are already at $s = 0$, in higher-dimensional space you can think of it this way: suppose that a trajectory is generated by this, so you are always on the trajectory, and you are not going to leave this trajectory. Why? In the vicinity of this dynamics, one high switching frequency is always acting. So, for that reason, you are always going to slide along the nominal trajectory that is the philosophy. And now we have to make sure how to achieve that.

So, for that, I obviously have to maintain the η reachability condition. In another way, you can take Lyapunov function and try to prove that

$$\dot{V} \leq -V^\alpha, \quad 0 < \alpha < 1.$$

So, this will give the guarantee about the finite time stability. You are already familiar with this kind of development whenever I was talking about a state feedback based unit vector control. So, this is nothing but the unit vector control, and this guy $\rho(t, x)$, which is a scalar function, is responsible for taking care of all kinds of uncertainty and perturbation, particularly matched uncertainty.

So, since uncertainty is matched, I am able to represent it like this. Now, what I am going to do, I am going to follow the exactly same principle, since I have to maintain $s = 0$ and due to that reason I have to push \dot{s} with some kind of control such that average value of \dot{s} equal to 0, that kind of things I have to put. So, now if you take \dot{s} and this is our s , you can see that $\dot{s}g$ is a constant matrix. So this is \dot{x} , this is the initial condition that equals 0. Now here, t you have to substitute, and dt/dt that equals 1, okay? Due to that reason, you can see that $\dot{x}(t)$ comes into the picture and \dot{x} .

Is nothing but the dynamics of the system. So, I have substituted dynamics, and this part is due to the nominal control. Now, you can see that this term and this term is going to cancel each other, and after that here u , u contain two part. I have already told you that u contains u_0 and plus u_1 , and u_1 is nothing but some kind of switching part. So, u_0 part is going to cancel out and just I have $gbu_1(t, x)$ and $gb\phi(t, x)$. Now, what I am going to do? I am going to substitute this control.

Now, if you substitute control, then this kind of dynamics comes into picture. Now, what do I have to prove? I have to prove that the η reachability condition will be satisfied. So, either you can proceed by taking the Lyapunov function

$$V = \frac{1}{2}s^\top s,$$

or you can directly do this kind of manipulation; both are fine. So, $s^\top \dot{s}$, and then it means that \dot{V} is nothing but $s^\top \dot{s}$. Now, if you substitute, then you can see here that I have gb into $x(t)$ and here gb .

So, a square comes into the picture, and after that, I have to divide by norm. I will end up here. Again, I will apply the norm inequality, and after that, what will I actually do? I will separate this term and I am assuming that $\phi_{\max} > \rho(t, x)$ and in this way, I can able to ensure the reachability. It means that if I am at some kind of nominal trajectory, I cannot going to leave and nominal trajectory that is generated without the uncertainty that is now our sliding manifold in this particular case. So, I am going to apply this kind of methodology for the LTI system.

You are able to apply this kind of methodology for any class of system because you can see that during development I have considered the non-linear time-varying system and the forced non-linear time-varying system. So, all other systems are subclasses of this system, and for that reason, I have taken this LTI system. Again, this assumption means I have massive uncertainty, and I am assuming that it is upper bounded. So, and after that philosophy is exactly same, I will design two part u_0 is nominal control, u_1 is that is going to take care about the matched uncertainty. So, in one of the philosophy courses during your modern control theory or linear control theory course, you have already learned about pole placement or the linear quadratic Gaussian method.

So, in this lecture, I am going to talk about pole placement. So, for pole placement, this formula is very useful. If you have an LTI system, it is called Ackermann's formula. And what this formula is telling us is that if you know the complete state, it means that I am assuming that all states information are available to us. And then, what I am going to do, if I have a system like

$$\dot{x} = Ax + Bu_0,$$

is design

$$u_0 = Kx,$$

and obviously K is some kind of matrix, because the dimension of B is now m , and the dimension of x is nothing but n .

So, again, some projection matrix is required here as well. So, either you can write K^\top or K ; that does not matter. So, this is basically the state feedback. So, what Ackermann formula is suggesting that you can able to design the gain just based on the notion of controllability.

So, let us talk sometime about controllability. So, what is the meaning of controllability? Controllability give guarantee that if system is controllable that you can able to check using this matrix B and A and applying the Kalman test. If that is controllable, then you can always be able to design some kind of control such that if you start anywhere in the state space, you can converge to the equilibrium point. So, the Ackermann formula suggests that you can design K like this: P^{-1} is nothing but the controllability matrix, which I will define in the next slide, and $\phi(A)$ is the characteristic equation evaluated at A . So, what am I going to do? I am going to take the example of the inverted pendulum on a cart, and this arrangement is quite useful whenever you are doing a space mission; intentionally, due to that reason, I have selected this. Here, due to the simplicity of presentation, I have assumed several things.

I am assuming that whatever friction force is here is negligible. I am assuming that the friction force at this particular joint is also negligible. I am assuming that for the time being, there is no external disturbance. Just in this development, I am going to assume some kind of matched disturbance that will enter through the control channel.

But even if unmatched uncertainty comes into picture, you can able to show that our state, I will define state for this system that will not deviate much.

So, in this lecture, I am going to show you the insensitivity with respect to matched perturbations. So, during mathematical modeling, we have to make several assumptions, and this pendulum is moving in a 2D plane. Now, these are the modeling assumptions. You are already well aware that no mathematical model is actually universal. You always have to make several assumptions in order to apply Newton's law or its equivalent, and for that reason, I am assuming that the mass of the pendulum rod is 0, with no friction.

No air resistance, and I am assuming that the pendulum bob has a point mass. And what is our objective? Now, our objective is to maintain this pendulum in a vertical position, and I have just one force. I can only move the cart. I have that kind of restriction.

So, I have to force the reference to x . In this particular difference, in this particular development, I am assuming that $x_{\text{reference}} = 0$, but you can assume anything. Now, I have to do a force analysis. So, you can see here that in the horizontal direction, what kind of force is acting. So, one force is acting like whatever you have applied, and due to that reason, I have now written it like this.

So, I have applied Newton's law, and now I have a cart. So, you are able to see that the horizontal force I have to calculate is a projection. So, the projection along this is $l \sin \theta$, and for that reason, here $x + l \sin \theta$ comes into the picture; you have to take the derivative. Now, the vertical direction projection is $l \cos(\theta)$. So, you have to calculate $m \frac{d^2 x}{dt^2}$. So, from the origin, I have a distance of $x + l \sin \theta$, and due to that reason, these are the horizontal forces, okay.

Now, I am going to simplify it, okay? Please do these kinds of calculations by yourself, and this way I can be able to model the cart-pendulum coupling, okay? Now, in the vertical direction, I am also going to solve it, and after that, we will get the force along the vertical direction, but try to see what our objective is here. Our objective is to maintain $\theta = 0$; it means that θ is very, very small, and for that reason, I can also do some kind of simplification. So, this is the torque balance because now the pendulum is actually dancing on the cart. So, we have to control that, and these are the couple dynamics.

Now, these are the state and u is the control. Now, what do I have to do? I am going to assume that θ is actually very, very small. So, $\sin \theta$ is approximated by θ , $\cos \theta = 1$, and the higher order term is 0. So, somehow I am linearizing about $\theta = 0$. So, the vertical force is balanced by the mg , and in this way, you can see that this is the simplified model. Obviously, you are able to design sliding mode control for a non-linear model, but that is complicated and useless for this application because our θ is not changing much.

After that pendulum motion, you can see that it is further simplified like this, and

this is the linear coupled dynamics. Now, we are able to apply any kind of linear control theory, and if you are a student of control systems, then in every lab in India, we have a pendulum system. So, you can apply the inverted pendulum system. So, you will be able to apply the control that is actually developed in this particular lecture. And here, x is the position, \dot{x} is the cart velocity, θ is the pendulum angle, and angular velocity is $\dot{\theta}$; this is the linearized model.

So, I have actually expressed this in terms of the first-order dynamics, and then I will get this kind of model. And, if you see the open-loop pole, what is the meaning of open-loop pole? Just you can set $u = 0$. Then, this pole is unstable. And, due to that reason, now I have to design control. Obviously, I have just one control, and I have to control two variables, x and θ .

And, you can see that this system is fully controllable. How do you check controllability? $[B \ AB \ A^2B \ A^3B]$ that should be full rank. And how is this system actually equivalent to a space? You can see that cart is equivalent to launch vehicle, pendulum is nothing but attitude deviation and control force is thrust. So, whatever control you have designed for cart inverted pendulum system, you can equally able to apply that for a space vehicle system, because a space vehicle system I cannot able to build inside the lab. So, you will be able to test it, and after that, you can scale it out and apply it to the space vehicle.

I have taken this kind of mass, and then I will get this matrix. So, please check the calculation. And after that, I have to calculate the controllability matrix again because, in order to design where I will place the pole, gains are required. So, the simple gains I have selected are $-1, -2, -3, -4$, and this is the characteristic polynomial, and this is the characteristic polynomial evolution that is given like this, and finally, I will come up with this gain. This is the gain calculation for a fourth-order system.

I assume there is no disturbance. Then, if you apply this kind of control multiplied by Kx , it is possible to show that all states are tending towards 0 as t tends towards infinity. Now, I am assuming that I have disturbances as well, and obviously, because whenever you are going to launch the process, air is present along with several other kinds of uncertainty. So, you have to actually apply, you have to not rely on the state feedback control, state feedback control or pole placement based control or linear control. You always need some kind of nonlinear control. So, how can we extend this theory using integral sliding mode control? Again, I have this integral sliding manifold.

So, I have already discussed a lot about this. Nominal control I designed with an Ackermann formula, and after that, I took the \dot{s} , and you can see that I have a nominal term and a switching term. And, after that, based on the unit vector control, I have designed control; the ρ I have selected is greater than D plus the maximum value of the disturbance and η , which is responsible for maintaining the sliding mode control. So, this

is the complete control. Now, what task do we have to do? You know what S is.

You know, you have to design G . So, I have already told you that you can select $G = B^\top$. That is the obvious choice. And then, u_0 , I have already calculated the gain. So, I have not done the simulation here because that is homework for you.

You can apply that to your practical system and then see the result. So it is time to conclude. This is the very, very innovative kind of the sliding mode control that is called integral sliding mode control and why this sliding mode control is very elegant because it is also cooperate with all other classes of controller. So, if you have any classes of controller model predictive control or any control that is extensively used in the industry, you can now add this control philosophy to sliding mode control to achieve instant robustness, which means you can eliminate the uncertainty from the initial point onward. Obviously, with full system dynamics, we have universal protection from zero because nowadays our system is on the network.

So, always we have chance of the disruption. So, using this philosophy, you can able to mitigate the uncertainty or also you are able to understand what kind of uncertainty is hitting to the system. And due to this reason, in all safety-critical applications, whenever people are designing sliding mode control, in practical applications you are able to see that it is integral sliding mode control. The design philosophy is also simple because it is very cooperative in nature, and obviously, any kind of extreme uncertainty will be addressed in the next class. I will also tell you that it is comfortable with some classes of unmatched uncertainty. So, with this remark, I will end this lecture. Thank you very much.