Sliding Mode Control and Applications

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Welcome back. In previous classes, I was discussing the sliding mode control for linear systems as well as non-linear systems. And in sliding mode control, what is our observation? That we have two phases, particularly if I talk from the second order dynamics onward: one phase is called the reaching phase, and once I reach the manifold, the second phase is called the sliding phase. And what is our general observation? That one can be able to reach the sliding manifold in finite time, and after that, the trajectory moves towards the equilibrium point or towards the objective asymptotically. So, now new idea comes into picture that is called idea of terminal sliding mode control. And what is basic philosophy? Is it also possible to slide along the sliding surface in such a way that after hitting the sliding surface I will land at the equilibrium point in finite time? So, now the trajectory can actually start anywhere inside the state space, and in finite time it can reach the equilibrium point, and somehow this is also a starting point for the higher-order sliding mode control.

So, in this lecture, I am going to cover terminal sliding mode control and, after that, the connection between terminal sliding mode control and higher order sliding mode control. So, the purpose of the discussion is terminal sliding mode control, and obviously, we are going to cover some key questions. What are the key questions? One of the crucial properties that is called the convergence property. So, how does the non-linear sliding surface, it is possible to show that in two-dimensional plane, suppose that this is x_1 and this is x_2 and if we are designing the linear sliding surface particularly or some form of non-linear sliding surface, which has some kind of asymptotic kind of dynamics, it means that once $s = 0$, after that trajectory will converge to the equilibrium point asymptotically.

Now, we are going to look into some kind of surface, some kind of non-linear kind of surface which will pass through the equilibrium point such that I will converge here and in finite time I will converge to the equilibrium point. So, our main goal to improve the convergence property and due to that reason, we are going to look into the question how does the non-linear sliding surface design achieve the finite time convergence compared to the conventional sliding mode control. What is the meaning of conventional sliding mode control? That is nothing but first-order sliding mode control. So, whenever we have higher order system, we are designing the sliding surface and after that our aim to force the trajectory towards the sliding surface and then maintain such that asymptotically I will reach. Now, whenever you are going to design the non-linear sliding surface with some kind of desired convergence property, obviously, some challenges come into the picture that are called singularity challenges.

It means that it is possible to show that somewhere I need infinite gain to converge toward the equilibrium, toward the sliding surface. So, how do we avoid it? How to redesign the sliding surface such that one can resolve this problem? So, that part I am also going to explore and obviously, the main theme is the practical significance because in several practical applications or particularly in high precision applications, we need finite time convergence from any initial condition to the equilibrium point and somehow terminal sliding mode control is going to address that particular problem and for that reason terminal sliding mode control is very, very important. So, now I am going to talk about transition from linear to non-linear sliding manifold. So, what is the basic philosophy of linear sliding mode control? Suppose that I have system like

$$\dot{x} = u + d(t)$$

for simplicity I have taken the linear time invariant uncertain system and this is second order system and after that we are designing many fold like

$$s = x_2 + c_1 x_1$$

or sigma, whatever you want in order to denote sliding variable, we are either using s or σ. So, $s = x_2 + c_1 x_1$ and once $s = 0$ in finite time, then

$$x_2 = -c_1 x_1.$$

So, you can see here that my closed loop dynamics, once we start sliding, so that is governed by this and this is just exponentially stable towards the equilibrium point. In several cases, you are also unable to expect exponential stability. You are only able to expect the asymptotic stability, particularly for the non-linear system. And obviously, the time of convergence is very large because x_1 is tending towards 0 as t is tending towards infinity.

So, what is our basic philosophy now? We are going to introduce some kind of nonlinearity in the switching manifold. It means that now this switching manifold S is not a linear combination of x_1 and x_2; that is some kind of non-linear combination of the x_1 and x_2, and the basic idea is to actually accelerate the convergence when we are nearby the equilibrium point. What happened? You can see this dynamic

$$\dot{x}_1 = cx_1.$$

So, since this is a first-order system, I can easily plot it. So, this is x_1, and if our initial condition is here, so you can see that x_1 is positive, but $\dot{x}_1$ is negative.

So, we are going to move in this direction, but as we move closer to the equilibrium point, the rate of change is going to decrease because the rate of change depends on x_1. So, if you are close to the equilibrium point, at that time the rate of change is very, very small, and due to that reason, we have asymptotic convergence. So, now I have to add some kind of term that will particularly accelerate convergence in this region when we are closer to the equilibrium point, and using terminal sliding mode control, we are able to achieve this objective. It means that in terminal sliding mode control, the equilibrium is a terminal attractor.

What is the meaning of terminal attractor? The term terminal refers to the equilibrium point that can be reached in finite time and is stable. That is the meaning of the terminal, and for that reason, this kind of sliding mode is called terminal sliding mode control. So, let us start with some kind of linear or non-linear system. Here, this dynamics may be linear or nonlinear. Similarly, here I am assuming without loss of generality that this term $b(x_1, x_2)$ is not going to change its sign.

So, that is greater than zero. Otherwise, control design becomes a little more complicated. I have already discussed this. Now, you can see that I have designed a sliding surface like this:

$$s = x_2 + \beta x_1^{\frac{q}{p}}.$$

And here I have a certain kind of guidelines.

So, using Lyapunov theory, I am going to prove why I am assuming that p and q are positive odd integers. That is our assumption, and I am going to justify this assumption with the help of the Lyapunov stability criteria. Another assumption I have is that p should be greater than q. So, it means that this power is going to lie between 0 and 1. Now, again in sliding mode design, I am going to do it like this.

When $s > 0$, then this is u^+; $s < 0$, then this is u^-. I do not care about $s = 0$, because in all practical problems, basically we are in the very, very fine vicinity of the equilibrium point, and due to that reason, this kind of design is practically also giving the same performance even if I define it as equal to 0 between $-u^-$ to u^+. So, from the theoretical point of view, that representation is more elegant, but from a practical point of view, this is okay. So, what kind of modification am I going to do? I am going to make this kind of modification. So, let us try to see what this modification is basically going to introduce so that I will be able to guarantee that once I reach the sliding surface, the trajectory is going to converge to the equilibrium point in finite time.

So, now once $s = 0$, you can see here $s = 0$, then x_2 is nothing but

$$x_2 = -\beta x_1^{\frac{q}{p}}.$$

Here another assumption I have is β, I am assuming that is greater than 0. So, that kind of assumption I am going to maintain. Now, you can see that from this dynamics

$$\dot{x}_1 = x_2,$$

and due to that reason, I will get this kind of dynamics during the sliding. I actually have to solve this equation and try to see the convergence property.

So, since this is a scalar equation, it is very easy to solve. What can you do? You can apply the variable separation method and suppose that when the trajectory has hit the sliding surface, at that time if the initial condition is $x_1(0)$, and now I am assuming that due to this dynamics, this will converge to 0. So, I have to perform integration from $x_1(0)$ to 0, and in this wa So, this term is always finite because x_1 is finite and the denominator is also finite, and due to that reason, in finite time any trajectory is going to converge from any initial condition to 0, but finite-time convergence is depending on the initial condition. Initially, if you hit this surface, which is actually very far from the equilibrium point, then you have to wait for more time; that is the meaning. And obviously, we have parameters like β, so you can be able to accelerate it. So, this idea, this basic philosophy, is actually coined by this guy. This is Professor Xing Yu, who is currently a professor at RMIT, and he is extensively working in this particular area.

I have several collaborations with this particular professor. Now, you can see that I am going to justify why I am assuming that both p and q are odd. So, in order to give justification, what am I going to do? I am going to perform the non-linear analysis, and in particular, I have to show the stability of this non-linear system. Why is this non-linear? Because here $\dot{x}_1$ is equal to x_1 raised to some kind of fractional power I have, because p is greater than q. Due to that reason, I have defined the Lyapunov function as

$$V = \frac{1}{2}x_1^2;$$

I have only one variable during the sliding phase.

So, $\dot{V}$ is $x_1\dot{x}_1$, and if you substitute the dynamics, then you will get this. It is possible to show that this dynamics is finite-time stable. Before that, I have to prove that $\dot{V}$ is negative definite. So, this is negative definite only if $\frac{p}{q}$ is even and after that p is greater than q. Due to this condition, I have actually kept the same condition when I defined the sliding manifold, and you can see that this is the generalization of the linear manifold.

Why? Because if you set $p = q$, I will again end up on the linear manifold. So, what is the basic idea? This term $x_1^{\frac{q}{p}}$ is responsible for improving the convergence towards the equilibrium point. That is the idea. So, you can also able to justify this particular observation with help of the Jacobian calculation around the equilibrium point. So, now I have dynamics like this:

$$\dot{x}_1 = \beta x_1^{\frac{q}{p}}.$$

You can calculate the Jacobian about $x_1 = 0$, and you can see here what the Jacobian is suggesting. Since here $\frac{p-q}{p}$, this becomes positive. So, as x_1 tending towards 0, then this Jacobian—Jacobian is somehow related to the eigenvalue. So, this is somehow the idea of a non-linear eigenvalue. So, as x_1 tends towards 0, λ tends towards minus infinity.

So, with infinite speed, I am going to hit towards the equilibrium point, and due to that reason, in finite time I can able to land up; that is the basic idea. So, same kind of things I have actually written which indicate that at the equilibrium point the eigenvalue tending towards the negative infinity, and of course, the system trajectory which will have an infinite negative eigenvalue will converge to the equilibrium point with infinitely large speed. Now, what am I going to do? One more important point before actually generalizing this: in several previous classes, I have already established that if $\dot{V}$ is given by this kind of differential inequality, some kind of fractional power, where α lies between 0 and 1, then this dynamics is finite-time stable. Due to that reason, I have not included that part here. Easily you can able to show this that this is finite-time stable.

Now, what am I going to do? I am going to generalize this for any higher-order single-input single-output system. So, the output variable here I am assuming is x_1, and the control I have is just control u that belongs to $\mathbb{R}$. So, whenever I have a linear or non-linear system, we have already seen that if I have a single-input single-output non-linear system, I can apply input-output linearization, and if the relative degree of the system is exactly equal to the order of the system, then the system can be fully represented in chain-of-integrators form. A similar kind of methodology is equally applicable for the linear system. So, I am assuming that our system is either linear or non-linear and that it is fully input-output linearizable.

It means that by using the output, I can express the whole system, or you can apply some kind of feedback linearization technique so that your system will be converted into a chain-of-integrators form. So, from Khalil book you can able to see how to convert a system into feedback linearizable form, where the system can be represented in the form of a chain of integrators. What now I am going to tell you is how to design terminal sliding mode control for that higher-order single-input single-output system. So, now, here I am assuming that output x_0 is x_1. Now, I am going to define

$$x_1 = \dot{x}_0 = x_2 + \beta_1 x_0^{\frac{q_1}{p_1}}.$$

So, this is exactly the same as the second-order system. Now, from here onward, I have to do cascading. So, now, I am going to define another variable x_2, and that is the differentiation of this variable, and this term again comes into the picture. Similarly, I can do this cascading up to $n-1$, and every time I am going to preserve the same kind of structure, all gain β_1 up to β_{n-1} that should be greater than 0, p_i should be greater than q_i.

So, that kind of condition I am going to maintain, and everything I am going to maintain as an odd integer. So, what is observation now? Again, observation is like this. So, if the initial condition of the system is far from the equilibrium point, so suppose that you can take any higher-dimensional space and suppose my equilibrium point is very, very far, and if you are going to calculate this term as $\frac{q}{p}$.

So, since q/p, p is greater than q. So, this somehow gives us the fraction that is going to lie between 0 and 1. So, the magnitude of this term is very, very small, and due to that reason, the concept of fast terminal sliding mode control comes into the picture. So, what is the basic idea? Now, you can see that this is somehow a fusion of the linear as well as the terminal sliding mode control. So, in order to actually increase the speed of convergence towards the equilibrium point or towards the origin, what is the basic idea? You can also add this linear part.

In after several years, new idea of the fixed-time stability concept comes into picture, and idea is actually generalization of this. So, in place of x_1, now you can put some kind of power that is greater than 1, and in that particular case, you can be able to get some more convergence. So, you can see that using terminal sliding mode control, several new ideas actually pop up in the literature on sliding mode control or nonlinear control theory. So, if you see recent development where people are assuming this as a some they are defining manifold like $\dot{x}_1 + \alpha x_1$, and here they are defining some kind of term like γ_1, and here they are taking β and $x_1^{\gamma_2}$. So, now, what people are doing in order to remove the restriction that q and p are actually the odd integers is that they are now redefining this as $x_1^{\alpha}\,\mathrm{sign}(x_1)$.

So, now there is no restriction on α, α should lie just between 0 to 1, and then we can able to proceed. So, in this way, people are now defining the sliding surface, where γ_1 is greater than 1 and γ_2 lies less than 1. Obviously, that is lower bounded by 0, and in this way, people are designing the fixed-time sliding mode control. What is the basic property? That now you can start anywhere in the state space, in fixed time you can able to reach here; that is independent of the initial condition. Now, once you define the first terminal sliding mode control, then during the sliding, the trajectory is going to be governed by this particular dynamics.

So, if you are far from the equilibrium point, your dynamics are actually approximated by this, and once you are very close to the equilibrium point, your dynamics are approximated by this. And in this way, equilibrium becomes the terminal attractor; that is the basic idea. And obviously, in this differential equation as well, you can easily calculate the time of convergence. This time of convergence calculation I have also did when I was discussing about the unit vector control.

So, please try to do it by yourself. And since every term is finite, so we can able to show that time is also finite. Now, what I am going to do, I am going to improve the cascading because I have to design first-order terminal sliding mode control. So, what am I going to do? Now, I am going to cascade the structure. So, initially in higher sliding mode control, you can see that this term is not present.

Now, I am going to add this term everywhere. Again, what am I going to do? In order to design the control action, I am going to take the last dynamics. I will calculate the derivative of this, and I am going to apply the control because whenever you are taking x to the power of $n-1$, obviously $\dot{x}_0$, $\dot{x}_2$, and $\dot{x}_1$ actually contain the information of $\dot{x}_2$. Similarly, in the last dynamics where the control explicitly appears, it will appear. Now, you can design an on-off control; just K sign(s) you can take, and K you can keep very, very large. Then, you will also be able to show that it is possible to maintain $s_{n-1} = 0$, and once $s_{n-1} = 0$, then $s_{n-2} = 0$.

So, the next dynamics is going to collapse, and in this way I will be able to progress, and finally I will reach here. So, cascading kinds of things are going to happen in this particular methodology and time of convergence. So, first you have to calculate the time of convergence when control will explicitly appear. So, some finite time is required to collapse s_{n-1}. Similarly, from s_{n-1} to s_{n-2}, some finite time is required.

But every time, the form of time calculation is exactly the same. The total time of the convergence is given by the summation of all times of convergence. And in this way, I can be able to get this kind of time of convergence. So, please check by doing the calculation.

Calculation is not that difficult. Each time you have to keep $s_{n-1} = 0$, then you can calculate time. After that $s_{n-2} = 0$, you can calculate the time and simply you can do the summation, because now how basically we are converging? First, this dynamics will collapse. Then, actually, the next s_{n-2} dynamics will collapse. Similarly, I am going to progress. And for that reason, the total time of convergence is just the linear summation of the T_i.

Now, this philosophy looks very good. You can design the non-linear sliding surface, and the philosophy is exactly the same. What can you do after designing the sliding surface $x_2 + \beta x_1^{\frac{p}{q}}$, and what is our aim? So, let us just check for a moment. I am defining q in terms of p. So, due to that, I have to define q in terms of p here. And what I am assuming is that p should be greater than q, and both p and q are odd.

How do you calculate control? In order to calculate control, I have to take the derivative of this. So, I have to calculate $\dot{x}_2$, and after that, you can see that once you calculate the derivative of this, then this kind of term is going to appear, and obviously, $\dot{x}_1$ also comes into the picture. So, what is the difficulty with respect to this term? If you see carefully, then since denominator p is greater than q. So, this term power of this is negative.

So, this term will appear negatively in the denominator. So, as x_1 tending towards 0 and $x_2 = 0$, at that time this term is going to create some kind of singularity problem. And due to that reason, in literature, you can see that several criticisms come into the picture for the terminal sliding mode control. During practical implementation, it is possible to show that this kind of situation will not occur several times. Why? Because once you are in sliding phase, at that time automatically $x_2 = \beta x_1^{\frac{q}{p}}$. And if you substitute here, you can see that this kind of singularity, singularity kind of things is automatically compensated.

It means that now the singular term is actually not present, and due to that reason, what happens is that if you are able to hit the sliding surface, then your job is over. And due to that result in several practical applications, you cannot see the singularity. So, singularity is just some kind of mathematical problem, but still, mathematically we have to resolve this. And due to that reason, Professor Jing Yu came up with another modification of the terminal sliding mode control, which is called non-singular terminal sliding mode control. And here, the idea is now in place of power on x_1; you can keep the power on the x_2.

And here initially we are putting power on x_1, which is $\frac{q}{p}$; now on x_2 we have to put power, which is the reverse of that, which is $\frac{p}{q}$. So, you can see here that it is $\frac{p}{q}$. Again, once you are in the sliding phase, you can see that I am able to write $\beta x_1 = x_2^{\frac{p}{q}}$ and p and q both are odd. So, easily I can able to write

$$x_2 = -\beta^{\frac{q}{p}} x_1^{\frac{q}{p}},$$

and obviously some power will appear here and x_1 and this whole to the power $\frac{q}{p}$. So, now again x_1 has some kind of power that is fractional power and after that I will replace x_2 by $\dot{x}_1$.

So, the same kind of characteristic as singular terminal sliding mode control comes into the picture, and with this modification, it is possible to show that whatever problems are encountered during the first singular terminal sliding mode control can be removed.

And here is the same kind of calculation we have done. What have I done? I have taken the second-order system. And I have calculated $\dot{x}$, and then you can see that now control is fully non-singular, and the property of control is exactly the same.

This will go through the equilibrium point. So, this is x_1 and x_2. So, in finite time you can able to converge here and after that in finite time you can land up to the equilibrium point. So, all trajectory will land up. We have to show whenever you will do modification, you have to show that in finite time, trajectory will reach to the equilibrium point and due to that reason, so for that you have to first show the reachability condition. And what is the reachability condition? $\dot{s}$ should be less than 0, but we have to show something more. If I want to show the finite time, then $\dot{s}$ should be $-|s|$, and some constant should be here.

So, those are the kinds of things we have to show. Here, whenever we are calculating $\dot{s}$, you can see that this term also comes into the picture. So, $|s|$ is fine, the constant is fine, but x_2 also comes into the picture. So, now we have to actually check that when $x_1 \neq 0$, $x_2 = 0$. So, each trajectory is going to be stuck somewhere in this line or node, because once $x_2 \neq 0$. Then everything is fine; in finite time, I can converge; this will act like some kind of gain part.

So, I will get some kind of η reachability condition, this is also called η reachability condition, people are putting η term here. So, in finite time I can able to reach, by Lyapunov function you can able to show, but here during modification we get difficulty here when $x_1 = 0$, and it is possible to show that even if $x_2 = 0$ and $x_1 \neq 0$, it means that on this line trajectory never going to stay on this line. Why? You can see the dynamics, and in dynamics $x_2 = 0$; if you set this equal to 0, then our dynamics is now

$$\dot{x}_2 = k\,\sigma.$$

So, I cannot stay here. Again I have to leave, and once I hit here, our job will be over. And due to that reason, it is now possible to show that when $s > 0$, $\dot{x}_2 = -k$. If $\dot{x}_2 = -k$, then after infinite time, you can see that $x_2 = 0$. And similar kinds of things mean that in finite time I can be able to hit this particular surface. And in this way, Professor Xinghuo Yu has established it. Now, a new era comes into the picture; that is the era of higher-order sliding mode control.

I will formally define what the meaning of higher order sliding mode control is in the next class. So, now what Professor Levant has done is tell us about the singularity problem that occurs during the control calculation; basically, we are assuming that I have a second-order system. So, this system can be represented by

$$\dot{\sigma}_1 = \sigma_2, \quad \dot{\sigma}_2 = u + d(t),$$

something like this. Now, here σ_1 is nothing but σ, and σ_2 is $\dot{\sigma}$.

So, you can design this kind of sliding surface. Now, during the control calculation, what are we basically doing? We are calculating $\dot{s}$ that is nothing but $\ddot{\sigma}$ and after that we are taking the derivative of this $\frac{1}{2}\sigma^{-\frac{1}{2}}$ and after that this is $\dot{\sigma}$ that comes into pictures, $\dot{\sigma}$ means σ_2. Now, here we are replacing u and, after that, we are designing u such that we have to compensate for this term. So, he was saying that there is no need to compensate for this term. Why? If your dynamics look like this and if you design $-k\,\mathrm{sign}(s)$, then this term will just give you the sign. So, this is either positive or negative and whenever you are defining this kind of manifold, then whole plane is separated into either $s > 0$, then this side become $s < 0$.

What is the meaning of s here? That is the non-linear combination of $\dot{\sigma}$ and $\sigma^{\frac{1}{2}}$. So, your differential equation is converted like this:

$$\ddot{\sigma} = -k + d.$$

So, now if you start anywhere, in finite time you will be able to reach it because now the dynamics look like this:

$$\dot{\sigma}_1 = \sigma_2, \quad \sigma_2 = u + d,$$

where σ_1 is σ and σ_2 is $\dot{\sigma}$. So, always if $u = \pm k$ it is possible to show that you can able to converge here in the finite time and due to that reason he has simply removed the singularity term. And in this way, he showed that both σ and $\dot{\sigma}$ are equal to 0 in finite time, and finally, they have also designed the cascaded structure for third order, fourth order, or any arbitrary order, and this kind of thing will satisfy some kind of special property that is called the homogeneity property because if you scale σ by λ^2 and $\dot{\sigma}$ by λ, you will get exactly the same kind of sliding dynamics.

So, this property is called the homogeneity property. So, based on this property, he has actually defined the higher order sliding mode control. So, I have not actually talked about the meaning of higher order sliding mode control; I am just telling you that you can start anywhere, and now our aim is to converge here in finite time. And obviously, a more formal definition I will discuss in the next part of this course. So, terminal sliding mode control actually opens several avenues.

Avenue of fixed-time stability, avenue of the higher sliding mode control. Also, they are talking about how to control the time of convergence and due to that reason, terminal sliding mode has a special respect in the area of the sliding mode control. So finite time convergence is granted, singularity free you can able to do and also you can able to prove the high precision application. So, obviously after that, higher sliding mode control became very, very popular; several new things came into the picture. So, terminal sliding mode control delivers fast, robust performance and overcoming any traditional restriction.

So, with this remark, I will end this lecture. So thank you very much.