Sliding Mode Control and Applications

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Week-03

Lecture-15

So, welcome back. So far, I have discussed the sliding mode control for linear systems as well as non-linear systems. And what is our core assumption that I assume all states in formations are available? So, practically, it is very difficult to measure all the states. And due to that reason, people are exploring the static output feedback-based design. What is the meaning of static output feedback-based design? If I only know the output, is it possible to achieve our objective or not? That kind of problem we are going to look in this particular lecture that is it possible design sliding mode control based on the static output odd mode and what kind of challenges, what kind of restriction comes into picture that I am going to address in this particular lecture. So, in this lecture, we will explore hyperplane design since we know that in sliding mode control, we have two phases.

Suppose that if I have a second-order system, then this phase is called the reaching phase and this phase is called the sliding phase, and the design of both phases is equally important. So, basically, the hyperplane design, also called the sliding phase design, is equally important. Why? Once we reach the sliding surface, the overall dynamics are just governed by the design of the hyperplane. And what is our assumption in this particular lecture? I am assuming that I only have information about the output.

Using that, I am trying to design the sliding manifold or sliding hyperplane. So, the challenge is that since I do not have information on all state variables, because I am assuming that I do not have enough sensors to measure the entire state vector, it is possible to show that we have an alternative design methodology, where if you have just information about the output, by designing a dynamical system called an observer, you can estimate those kinds of states that are not actually measurable. So, that kind of idea I am not going to actually discuss in this particular lecture; I am just going to assume that if I have only information about the output, it is possible to design a stable or suitable sliding surface or not. Once this part is over, we are also going to look at the design of the reaching phase. And obviously, I am also going to maintain robustness as well as stability.

So, just like in the previous class, I am going to take the linear uncertain system. I have also told you that even if you have no linear system, you should just take some nominal part from that system and push all other nonlinearity inside this term. Obviously, this is not always possible, but particularly for some classes of systems, this philosophy is justifiable; you can justify it. And I am assuming that I have output that is represented by

$$y(t) = Cx(t),$$

that is, the output is not directly connected by the input. I am assuming that I have n number of states and I am assuming that all states are not directly measured.

Now I have measurements; so p number of measurements I have and m number of controls I have. Okay, so since I am talking about the static output feedback design, i.e., control design based on the output information only, and due to that reason, I need some kind of assumption. Okay, I am going to justify that assumption in the next subsequent slide. So, I am assuming that here p is the measured output, which is actually the number of measured outputs, and that p is greater than the number of controls, but less than the number of states. So, under that assumption and under the matching condition, since I know that sliding mode is insensitive with respect to the matched uncertainty, I am assuming that this particular uncertainty can be expressed in terms of the matching condition. Due to that reason, B and ζ I can express in this form, where ζ is bounded by the control. This is the norm k_1, which again lies between 0 and 1, such that the control is not amplified, and

$$\alpha(t, y)$$

is some kind of non-positive function. So, that's the kind of assumption I have. And since a state vector $x(t)$ is unavailable, only y is available here, and for that reason, this problem is a little bit challenging, so I have to address it carefully.

Same as in state feedback control, we are transforming the system into regular form. Here, since I have to do design in the form of the output, i.e., control design with the help of the output, I am now going to impose more restrictions. I am going to convert my system into a controllable as well as observable form. So, that form is called the canonical form. Suppose that I have a linear system (A, B, C); you can just forget about this term because sliding mode control is going to take care of this.

During sliding, only the A, B, and C matrices play a role, and for that reason, I am now going to work with A, B, and C. What am I going to do? I am going to represent the system in some convenient form such that I can design the sliding phase. So, I have exactly completed the regular form transformation. Initially, our coordinate frame is x. Now, I am going to convert it into some kind of z coordinate frame.

Here, I am assuming that T exists. So, if you see the dimension of C, x is an $n \times 1$ matrix and y is a $p \times 1$ matrix. So, C should be a $p \times n$ matrix. Out of $p \times n$ and $p \times p$ matrices, I am assuming that this particular matrix T is orthogonal. And this part B_2, I am going to assume as non-singular because during control design, I have to apply B_2^{-1}.

Now, we are also assuming that there exists an orthogonal matrix T that can expand the output space. During transformation, once I apply the transformation, I am still assuming that by using some other coordinate frame, I will be able to map the output space. So, how do you map? It means that all vectors of T should be linearly independent. So, I have p number of outputs, and you can see here that I still have p number of column vectors. A $p \times p$ matrix means I have p number of column vectors.

So, using these column vectors, I can again assume that this is the first column vector, the second, and up to the p-th number of column vectors. So, using p number of column vectors, I am again assuming that I can expand the whole space. So, now I need some kind of condition. You can see here that I am assuming that A_{11} and C_1 are some kind of fictitious matrices; I am stating that the $p-m$ entry is an entity and the remaining entries are 0. So, I am assuming that this is detectable. You can see here in this structure, A_{11}, A_{21}, and A_{22} are directly controlled by this guy B_2. If you apply control in the transformed coordinate frame, then you can control A_{21}, A_{22}, but you cannot control this. Because in the transformed coordinate frame, this is

$$\dot{Z}_1 = A_{11}Z_1 + A_{12}Z_2.$$

Here Z_2 acts like the control, and this control again I have to design based on the output feedback, such that I can force Z_1 towards 0. If I cannot do this, then our whole system somehow gives poor performance.

Sometimes we will actually diverge towards infinity, and due to that reason, I need another assumption, which is the assumption of detectability. What detectability basically ensures is that if the output is equal to 0, then

$$x(t) \to 0 \quad \text{as} \quad t \to \infty.$$

So, even if I do not know $x(t)$, but somehow if we are able to establish that the system is detectable, then this property will be satisfied. But during the sliding phase design in transformed canonical form, you can see here that basically this matrix A_{11} is important. I have to design Z_2 based on the output.

So, some kind of information, some kind of gain, and things are coming from this matrix C. So, that matrix I am assuming is C_1. These two pairs should be detectable. Then only I can give guarantee that whatever dynamics are not directly controlled from the output, that actually helps us to maintain stability, and due to that reason, the detectability property is very, very important. So, A_{11} represents the dynamics of the state whose variables are not directly controlled by the input; I have already discussed this, and C_1 observes only part of the state vector, which means that since I have fewer numbers of states to control A_{11}.

Because this is output feedback based control, and due to that reason, the detectability condition is very important. Detectability guarantees that any eigenvalue of A_{11} corresponding to an unobservable mode, whether it is in the left half-plane or at the origin, means marginally stable. And I have already talked about this: if you do not put this kind of assumption, then overall system dynamics may be unstable even if the output behaves well. Now, under this assumption, i.e., the detectability assumption, I am going to design the sliding manifold. So, I have already assumed that I only have information about the output. So, now I am going to scale that output by some kind of matrix $m \times p$, and you can easily justify why this is $m \times n$ matrix. Now, control: I have m number of controls and p number of outputs, and Cx represents the output, and due to that reason, the dimension of F should be $m \times p$. Now I have to prove this is stable. One more important observation you have to understand here is that once we are actually in the sliding phase, at that time, we know that obviously the sliding variable

$$\sigma = 0,$$

but the average value of

$$\dot{\sigma}$$

should also be 0, because every plant will act like some kind of low-pass filter, and this is also called Utkin's equivalent method. So, according to the observation, whatever control is actually maintaining the sliding manifold is calculated by keeping $\dot{\sigma} = 0$. So, now I have to actually prove that this guy, F_{CB}, is not equal to 0. It means that it has full rank.

Then only I can maintain the sliding phase, because in the sliding phase, whatever control is directly applicable to the system to maintain sliding is given by the equivalent control—some kind of average control you can assume. And since

$$\text{rank}(CB) = m,$$

and due to that, the full-rank assumption and obviously, we have also assumed that we have the canonical form, and due to that reason, this assumption is already satisfied. It means that I am able to design a sliding surface like this. Now, the hyperplane matrix, I am going to show you that hyperplane.

Once we are in sliding, even if the state is unobservable, it is also stable. So, for that, what I am going to do: F matrix is such that if I apply F and T, T is $p \times p$ orthogonal matrix, we have already seen this. F_2 I am assuming is non-singular; I am going to justify why F_2 is non-singular. Now, what I am going to do: you can see that if I am able to express like T^{-1}, I will multiply both sides; then

$$TT^{-1} = I.$$

And since T is an orthogonal matrix, due to that reason,

$$T^{-1} = T^{\top},$$

and due to that reason, I have substituted here $T^{\top}$ and taken F_2 out. So, F_2 is just a scaling factor, but here k, gain k, is very, very important, and you can see that in order to take F_2 out, I have to make sure that F_2^{-1} will also exist.

So, k is somehow the design parameter. Now, if I calculate F_C, because you can see that primarily during the sliding, some control is going to appear. So, for that reason, I am going to calculate F_C. Equivalent control is basically going to apply during sliding. And C_1, which I have already defined for the sake of clarity, I have repeated again.

And now, you can see that

$$F_{CB} = F_2 B^2.$$

I have already assumed that

$$\det(F_2) \neq 0 \quad \text{and} \quad \det(B_2) \neq 0.$$

So, that kind of thing I have actually assumed, and due to that reason,

$$\det(F_C) \neq 0.$$

During the sliding phase, I have this kind of condition.

Now, you can substitute F_C. And after that, T-dynamics is now converted into Z-dynamics. If we are going to talk about the canonical structure, due to that reason, you can see now that I have null-space dynamics:

$$\dot{Z}_1 = A_{11} Z_1 + A_{12} F_2^{-1} F C_1 Z_1.$$

Now, this particular term $F_2^{-1} F_1$ is nothing but the gain of the output feedback. Now, I have to show that this gain always exists. So, for the existence of this gain, I have to express this system in the form of a triplet.

So, A_{11} is here, A_{12} is here, and C_1 is the output. So, you can also think like this: I have some system where the output is nothing but C_1Z_1, and the system is given like

$$\dot{Z}_1 = A_{11}Z_1 + A_{12}u,$$

and u you are going to design by C_1Z_1 multiplied by k. So, k you have to design based on the idea of output feedback. And obviously, pole placement, this is also called pole placement problem because I have to keep all poles in the left half-plane for this particular matrix

$$A_{11}, A_{12}, kC_1$$

such that Z_1 tends towards 0 as $t \to \infty$. That kind of condition is required.

And this is not always possible. If you have information about all states, then things are very easy. You are able to do pole placement in any desired way, but here you only have the information of the output, and for that reason, I need some kind of extra condition. So, this is the Kimura deviation condition. And what is this condition suggesting? They are saying that although this problem is very restrictive, if the number of controls and the number of outputs is greater than or equal to $n + 1$, where n is the dimension of the system, then your system has enough flexibility such that you are able to place the pole based on the output. Observability of A_{11} and C_1 is also very important because somehow if the system is not observable, we cannot actually get other states just by looking at C_1Z_1.

So, that kind of thing is also required. And what is the meaning of observability? By the Kalman test, we know that in this way we are actually proving observability. So, you can see that whenever we are designing the sliding surface based on the output, many restrictions come into the picture. And several times in practical systems, checking so many restrictions will create a lot of difficulty. Several times you are not able to achieve it either. So, if that is not observable, you cannot do the pole placement.

So, somehow the Kimura division criteria is giving us the flexibility or some kind of condition that you can check without solving whether the output feedback best pole placement is possible or not. And why is pole placement important? Because by using pole placement, you can control stability, speed, and transients. And state feedback always has maximum flexibility because you know all the states. So, you can manipulate the poles. But whenever you just have output feedback, you have limited freedom.

But the main theme of this lecture: just try to understand why output feedback is so restrictive and what alternative solution we have to find; that is the main motivation of this lecture. And this condition: what is it somehow, why this condition, what is the physical interpretation of that condition? You can see here n is the number of states. If you know the state, then you can influence the dynamics, but what is m? m is the number of inputs, meaning the number of actuators I have such that I can manipulate the state. If m is too small, I cannot manipulate all states.

Similarly, p is the number of outputs. It means the number of sensors I have attached to the whole system such that I can get information of the state. If p is too small, then I also cannot get the desired behavior, the desired feedback behavior. And due to that reason, this condition comes into the picture. So, I am not going to give you the formal proof, but I am just going to give you the idea. Basically, if this criterion is satisfied, the main theme of this whole lecture finally lands up at the pole placement, output pole placement.

And due to that reason, this criterion is very, very important. If you get this kind of criterion, then simply there is no need to go for the output feedback. You have to design an observer and then try to estimate all other states and then you can design the control. Somehow, this criterion is very, very important because it saves you a lot of time. So, you can see that suppose I have a mass-spring-damper system, and now I am exploring whether it is possible to go for output feedback control. The answer is simply no, because you have $n = 3$ states, you have to control, you have one actuator, and suppose you have just one sensor. So, using one sensor, you are not able to basically design the output feedback. A similar kind of thing you can conclude: you have to control the aircraft's attitude. So, $n = 6$ here, and this is the physical interpretation: $n = 6, m = 3$.

So, again, you can see here that the condition is not satisfied. So, it is not good to actually go through the output feedback. Now, up to here, we understood that in sliding mode control how to design a sliding surface based on the output feedback. Now our job is not over: what I have to do is I have to force the trajectory towards the sliding surface. For that, again I have to design the control law based on the output feedback. If you see this structure in previous classes, you become familiar with this structure: this is nothing but unit vector control. So, this guy is going to give you the direction in which you are going to move. So, this is

$$\sigma/|\sigma|.$$

Here again I am assuming that when $\sigma = 0$, then this term will not be present; only this linear part will be present.

This will represent the nonlinear part. Now, here R matrix I am going to select exactly same like previous class, some kind of positive symmetric definite matrix. And since the system is uncertain, I have to select this term properly. And that is actually greater than k_1 times $g(y)$, and I will justify why I am assuming that ρ also depends on this linear part. The main intention of this linear part is to suppose that if your state is far away from the equilibrium point, you can design output feedback such that you are able to enter some kind of sliding patch, and after that, by applying this kind of control, you are able to maintain it. And obviously, I have designed this stable manifold such that I am able to maintain it.

So, the sliding patch is somehow decided by this ϵ. So, ϵ should be sufficiently large. Now, here it means that a static output feedback gain is also equally important because using that gain, I can create some kind of sliding patch. Now, in order to discuss the actually a static output feedback, I have to perform some kind of transformation that is switching function dependent coordinate transformation, and what is our switching function $f(c)$ and $x(t)$, this is the switching function.

So, I have to perform transformation based on that. So, again you can see that what I have done is convert the system exactly into regular form, but here I am going to guarantee that the regular form is observable as well as controllable. So, this is the canonical form. Where A_{11} matrix, A_{11}, A_{12}, and KC_1 I am going to assume, and by designing K, I am able to check the stability because I know that A_{11} and C_1 are actually detectable or observable as well. And T_{cap}, I am going to define it like this. So, whenever you are going for a static output, a lot of transformation you have to do.

A transformation matrix is defined; one can define it like this. And after that, now since I have to converge to the manifold, the necessary condition is that

$$\dot{\sigma}^{\top}\sigma < 0.$$

So, that condition I now have to satisfy, and most of the time it is possible to show that the analytical design of G is very tough whenever you are dealing with a practical system, and for that reason, people are using some kind of well-known numerical method. A numerical method is based on linear algebra, and obviously, you have to apply the transformation, and after that, you have to proceed. And it is also possible to show that if you are not willing to do a lot of calculations, then simply what you can do is to let this ϵ decide the sliding patch, and if you know how to design f from the sliding manifold design and hyperplane design.

Now you can put G that is sufficiently large, and then you can easily prove that this condition will be satisfied. So in finite time, you can be able to reach here, and after that, you can be able to maintain. So now this is just a step to show that in finite time you can able to reach in the sliding patch. So, G matrix, since I am going to apply this kind of transformation, the control is also transformed like this, and after that, now P matrix, because if you see the control term, our control term contains P_2^{-1}.

So, I have to design the P matrix and the P_2 matrix. So, you can see here that the P_2 matrix I will select is positive symmetric definite, where this is a block diagonal matrix, and after that, I have to ensure this. If we are able to ensure this, then I can able to give guarantee that our overall system is stable. So, that is the philosophy, and in this way, I can maintain the sliding patch. You can see that if ϵ is very, very close to 0, then this patch is very, very small. But if I take ϵ very, very large, it means that I am able to increase the sliding patch.

So, this is our design requirement. And obviously, this design requirement, if you take the help of the diagonal matrix and the canonical form, will lead you to this structure. And finally, you have to show that this combination is necessary because the system should be controllable as well as observable. So, this should satisfy the condition of strictly positive reals. So, now what we have observed that a static output feedback based design is very very restrictive. A lot of transformation you have to apply before designing the sliding mode philosophy based on static output feedback.

But obviously, design is possible for some classes of systems. And if design is possible, then this somehow decreases the burden of the number of actuators. Due to that reason, whenever you have some kind of problem where design is possible based on static output feedback, please go for it. But whenever this design is not possible, what do we have to do? We have to design an observer, and that kind of thing we are going to look into in the subsequent part of this course: how to design an observer or differentiator such that, using the output information, I can obtain all other information that is not available due to one or only a few sensors. So, those kinds of things I am going to look into.

So, now it is time to conclude this lecture. So, in this lecture, we have basically explored the static output feedback based design for the linear linear time invariant uncertain system and we have also addressed the several design challenge. Thank you very much.