Sliding Mode Control and Applications

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Week-03

Lecture-14

Welcome back. In previous classes, I was talking about sliding mode control design for linear system as well as non-linear system using the regular form approach and input output based linearization. In this class, I am going to discuss a very efficient approach that is called state feedback unit vector control. So, now let us come to the purpose of the discussion. So, what is our main aim? Again, our main aim is to talk about state feedback, and at this time, I am going to talk about unit vector control. So, we are trying to understand what the meaning of unit vector control is in the context of state feedback.

And again, I am going to talk about sliding mode control. And for that reason, philosophy is exactly the same. If I have some kind of sliding surface. So, the sliding surface can be linear or non-linear.

I have to reach this sliding surface, and after that, I have to maintain along this sliding surface, and finally, if I am solving a stabilization problem, then I have to converge to the equilibrium point. If we are solving the tracking problem, we have to make sure that all errors will converge asymptotically or in finite time to the equilibrium point. And obviously, why are we applying sliding mode control? Because the system is actually subjected to uncertainty or disturbances, that is our main goal. And since sliding mode control is inherently a non-linear control and due to that reason you might have seen that everywhere we are using this tool, tool of Lyapunov function. So, what is the key focus area of this lecture? We are going to talk about unit vector control for the uncertain system, and this philosophy is actually applicable to various practical problems.

So, in this lecture, I am going to talk about the attitude control of a satellite using unit vector control. So, in this lecture, I am going to start with you can interpret this system as a linear time invariant system with some kind of state dependent or time varying uncertainty. You can also be able to do another interpretation. Suppose I have some kind of non-linear system like

$$\dot{x} = g(t, x, u).$$

So, what am I going to do? I am just going to take the linear part out of this, and whatever uncertain part in terms of control, in terms of state, and in terms of time, I am going to actually keep here.

And then I am going to design sliding mode control here such that I can achieve the exact same objective. So, somehow a non-linear problem is reduced to the linear problem. Obviously, this is not always possible, but there is several practical evidence where we can do this. Here, I am assuming that I have n number of state and m number of control input. So, basically, this is a multi-input, multi-state stabilization problem.

So, you can interpret it like this: all output is going to reflect on the state, and all state is going to reflect on the output. So, I have to stabilize all state. So, in that particular terminology, this is fall in the class of multi input, multi output stabilization problem. I have a certain assumption for this particular uncertain part. So, in the first part of this lecture, I am assuming that this is going to satisfy the so-called matching condition.

What is the meaning of matching condition? Mathematically, we are going to interpret this as follows: physically, we are interpreting that whatever uncertainty is going to enter is only through the control channel, and for that reason, I am going to express this uncertain part, and B is actually the matrix through which control is going to enter. So, now, if you substitute $B\phi$ here, then you can see that all uncertainty is only going to enter through the control channel, and we all understand from the previous discussion that sliding mode control is insensitive with respect to matched perturbations. Now, since here this ϕ is containing term in terms of time, x and u and due to that reason, further I am assuming that this kind of bound will exist. Here k is the scaling factor. You can see that I am assuming that k should lie between 0 to 1,

$$0 \leq k \leq 1.$$

Why? Because if that is greater than 1, then further they are going to amplify the control. And this α, obviously this is positive, always positive. So, that is going to contain the information of the all time dependent or a state dependent uncertainty and we are assuming that this is known function. So, this particular uncertain part is actually bounded by a control-dependent term as well as a state- and time-dependent term. Now, let us come to the control structure.

The control structure actually consists of two parts. So, this is state feedback control, and we are going to design state feedback control based on the Lyapunov strategy, and after that, this is the non-linear part. So, this is switching wise, exactly same like the previous lectures, but here structure is little bit different. Linear part, I have already told you that this is state feedback. So, you can see that up to here, I have some kind of matrix where the entry is constant.

So, S is a constant matrix, A is the system matrix, θ I have to design, and I am assuming that θ is some kind of stable matrix, or I am going to consider θ as a Hurwitz matrix. Using S, I am going to design the sliding surface. So, in this lecture, I am going to design the sliding surface like $Sx(t)$. So, this S is going to belong to $\mathbb{R}^{m\times n}$, and since x is $n \times 1$, due to that reason, σ will belong to $\mathbb{R}^{m\times 1}$. So, I have an m-dimensional control and an m-dimensional sliding surface.

I am going to define the multiplication of S and B as λ, i.e.,

$$\lambda = SB.$$

Now, you should try to see the non-linear component of control. Since, in the previous slide, I am assuming that uncertainty depends on the state and what our goal is? You can start anywhere in the state space and finally, I have to converge to the sliding surface and then we have to converge to the equilibrium point and due to that reason, whatever gain I am going to consider that should be depending on the state, that is the philosophy. And λ^{-1} is coming here; I am giving that multiplication of S and B. I will select S in such a way that λ^{-1} exists.

And this P_2 is a positive symmetric definite matrix. In the next slide, I will tell you how P_2 and this θ are connected. So, this P_2 and θ are actually solutions of the Lyapunov equation. Now, from here you can easily understand why I am telling this as unitarity feedback, because you can see that $P_2\sigma$, and after that this vector is divided by $P_2\sigma^T$. So, finally, just unity comes into the picture.

So, this will give you the direction in which we have to basically move. Here, I am assuming that P_2 is a positive semidefinite matrix and $\sigma \neq 0$. So, this is only defined when $\sigma \neq 0$. So, I am not worried because $\sigma = 0$ is our equilibrium point. Now you can see that this is nothing but the Lyapunov equation,

$$\theta^T P_2 + P_2\theta = -Q,$$

where Q is some positive definite matrix.

P_2 and θ, since θ is a stable matrix, I will select, and this is negative definite, and due to that reason, by the condition of the Lyapunov equation, it is possible to show that P_2 is always unique. So, in this way, I am able to design this part, the non-linear part of the control. Further, I am assuming that this scalar function is the gain of the non-linear part that will satisfy this kind of inequality,

$$\psi(t, x) \geq \|SAx\| + \alpha(t, x) + \varepsilon,$$

where $\psi(t, x)$ is the gain.

So, this particular inequality is required because I have to prove that if I start anywhere in the state space, then I will converge towards the sliding surface and then I will move towards the equilibrium point. So, since the non-linear part is only responsible to force the trajectory towards the sliding surface and due to that reason, we have to select gain, a state-dependent gain, in such a way that it is able to give a guarantee that if I start from anywhere, I will converge here and after that I will maintain.

If you see carefully, then this $\psi(t, x)$ is containing the upper bound of the linear part, after that $\alpha(t, x)$, $\alpha(t, x)$ we have already defined, that is the upper bound of the uncertainty, and some ε we are keeping. Why? Even if linear control and α both equal to 0, then still I can able to maintain the sliding mode control. So, this part always should be finite. Due to that reason, I have intentionally added ε, and this part we already know that $\sigma^T x$ is always positive.

So, I have to maintain that this should be less than. So, this should lie to be 0 to 1. This is always positive; k_1 is positive. So, I will maintain $k_1 < 0$; then automatically this is greater than or equal to 1. Now, what am I going to do? I will take this particular scalar bound, and after that, I am going to rearrange.

I am going to multiply and after rearranging and you can see here that if you take this guy as a common and after that $\psi(t, x)$ will come here, I have $\alpha(t, x)$. So, finally, by rearranging, I can able to express like this. So, this is homework for you. What can you do? You know the bound of uncertainty, you know the bound of control, and then please verify whether this inequality will be satisfied or not. I have already checked; this is satisfied.

Now, what am I going to do? Our main goal is to force the trajectory to the equilibrium point; that is our ultimate goal. So, for that I have defined the sliding surface and now philosophy is exactly same like the previous classes, I am going to take the rate of change. So, S is a constant matrix, and due to that, $\dot{x}$ comes into the picture, and I am going to substitute $\dot{x}$. Here, since I know $\lambda = SB$, and for that reason, instead of SB, λ comes into the picture. A similar kind of thing, since this will satisfy the matching uncertainty condition.

So, here again SB comes into the picture. After that, now I am going to substitute overall control. So, this contains a linear part as well as a non-linear part. So, if you substitute this control, let us just look at a time regarding the control.

So, this is our control. So, this is the linear part, and this is the non-linear part. So, I have to put the combination of the addition of these two. So, if you substitute here and again, and if you put $\sigma = Sx$, then finally you will get this expression. Now, what is our objective? Our objective is to show that this particular dynamical system.

Again, this is a dynamical system. Why am I telling? Because σ and I am expressing everything in rate of change of σ. So, I have to show that this is finite-time stable. So, how do we show that? So, for that, I am going to take a Lyapunov function, and this is a classical Lyapunov function that is quite useful in the case of linear systems. I am taking

$$V(\sigma) = \sigma^T P_2 \sigma,$$

where P_2 is a positive symmetric definite matrix, and I have already discussed how θ and P_2 are actually related by the Lyapunov equation.

Now, if you take $\dot{V}$, then

$$\dot{V} = \dot{\sigma}^T P_2 \sigma + \sigma^T P_2 \dot{\sigma}$$

comes into the picture, and now this calculation again you have to do, okay. So whenever some calculation is appeared, it is better that you can do by pen and paper, then easily you can able to understand each and every term. Since, now here you can see that $P_2\theta + \theta^T P_2$ that is unity and due to that reason $\sigma^T\sigma$ comes into the picture, here I am going to use the norm inequality and after that I am rearranging the term and then finally, you can see from this bound I can able to get this expression. So, now here one can able to show that this σ, because our ultimate goal is to stabilize that σ that is quadratically stable. What is the meaning of quadratic stability? You can take some kind of quadratic Lyapunov and then you can show that $\dot{V} \leq 0$ for all classes of perturbations.

So, whatever classes of perturbation I have taken, so for that this inequality is satisfied and due to that reason we are telling that σ is quadratic stable, but that is not enough. I have to show that σ is finite-time stable, and for that reason, I now have to do some further calculations. This inequality is very famous and is called the Rayleigh inequality. So, what is the Rayleigh inequality telling? Suppose that if you define the Lyapunov function in terms of

$$V(\sigma) = \sigma^T P_2 \sigma,$$

then this $V(\sigma)$ is upper bounded and lower bounded by the maximum and minimum eigenvalues of the matrix P_2 multiplied by $\|\sigma\|^2$. In other words,

$$\lambda_{\min}(P_2)\|\sigma\|^2 \leq V(\sigma) \leq \lambda_{\max}(P_2)\|\sigma\|^2.$$

And this inequality is very useful; several times you might have encountered this inequality.

Now, what I am going to do, since $P_2\sigma$ is here, so σ I will calculate, P_2 I will multiply and then I will get this inequality. And after that, I am going to substitute this inequality at this particular position, and then you can see that again this term is coming from the Rayleigh inequality, and then I am going to substitute it inside $\dot{V}$. Now, $\dot{V}$ contains the expression V; this is constant, and this is also constant. So, the square root of V, and it is possible to show by solving it that the settling time is always finite.

Why? c_1, c_2, everything is finite. In this way, you will be able to prove that in finite time, I will be able to reach the equilibrium point. Now, since in finite time I can able to reach to the equilibrium point, so $\sigma = 0$ and how basically I have defined the sliding surface, S — basically I am defining $\sigma = S\,x(t)$. Since $\sigma = 0$ and S matrix I have designed in such a way that

$$Sx = 0 \implies x(t) = 0.$$

So, all states are tending towards 0 as $\sigma = 0$. In this way, basically we will give the guarantee that our system is stable in spite of the disturbance.

Now, it means that our key result is that one can ensure global stability in the presence of mass trans-certainty, those kinds of trans-certainty which will actually enter through the control channel. Control contains two parts: the nominal part. So, the nominal part is responsible for stabilizing the nominal system, and the nonlinear part is actually able to compensate for the uncertainty; that is the philosophy. And after that, we have applied the Lyapunov way to prove that we can able to reach to the sliding surface in finite time, and once $\sigma = 0$, so S I have designed in such a way such that $Sx = 0$ also implies $x(t) = 0$. Now, whatever theory we have developed, I am going to apply this kind of theory to one application, and this application is quite useful, because India is now at a progress in the sector of space with very efficient speed and due to that reason our young generation should be aware about the control problem which is associated with the space and due to that reason I have taken the attitude control problem.

And what is the meaning of that? So, for attitude control problem distance does not matter, I have to align the satellite angle such that I can able to get some kind of proper application, because every satellite is actually sent to the earth orbit or some other orbit to do some kind of certain kind of job. And due to that reason, we have to ensure that the satellite maintains the correct orientation. This will also allow for precise alignment of the antenna with a ground station. And this alignment is crucial for achieving optical signal direction and maintaining reliable communication. And due to that reason, the control problem is very, very important.

So, I have just actually drawn a block diagram to show you what the meaning of the attitude control problem is. So, here this box represents the satellite, and you can see here that we have three axes: the axes of roll, pitch, and yaw, and this yaw axis is along the direction of the Earth. This is the direction of the Earth. Similarly, the pitch axis P is normal to the orbital plane, and after that, roll is actually in this direction. So, due to several uncertainties or depending on the application, several times we have to tune our satellite.

We are in space. So, there are several different kinds of uncertainties or perturbations. And, if you see the dynamics of the attitude of the satellite, that is a highly nonlinear equation. So, due to that reason, we have to design the sliding mode control. So, first try to see how to express this problem mathematically. So, in order to express this problem mathematically, I am going to use the conservation of angular momentum as well as Newton's laws.

To develop the mathematical model of the attitude control problem, we begin with basic physics. A rigid satellite can be represented by three principal axes corresponding to roll, pitch, and yaw. Let the inertia tensor about these axes be denoted by

$$I = \mathrm{diag}(I_1, I_2, I_3),$$

where I_1, I_2, and I_3 are the moments of inertia along the principal axes.

Using the conservation of angular momentum, the angular momentum vector is given by

$$\mathbf{H} = I\,\boldsymbol{\omega},$$

where $\boldsymbol{\omega} = [\omega_1,\ \omega_2,\ \omega_3]^\top$ is the angular velocity vector.

According to Newton's second law for rotational dynamics, the time rate of change of angular momentum equals the applied torque:

$$\dot{\mathbf{H}} = \boldsymbol{\tau}.$$

Substituting $\mathbf{H} = I\boldsymbol{\omega}$ and noting that I is constant for a rigid body, we obtain

$$I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}) = \boldsymbol{\tau}.$$

This nonlinear vector equation represents the attitude dynamics of the satellite. Because the dynamics are highly nonlinear and subject to uncertainties and external perturbations (e.g., gravity-gradient torques, solar radiation pressure, magnetic disturbances), sliding mode control is often adopted to ensure robustness and maintain the desired orientation of the satellite.

In the next slide, we explain the motivation behind the previously derived expression. Since the rotational dynamics are governed by

$$\frac{d\mathbf{H}}{dt} = \boldsymbol{\tau},$$

the time derivative of the angular momentum must be generated by an external control torque. For a rigid satellite with three rotational degrees of freedom—roll, pitch, and yaw—we require one control input along each principal axis.

Thus, the control torque vector is expressed as

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_r \\ \tau_p \\ \tau_y \end{bmatrix},$$

where τ_r is the control torque about the roll axis, τ_p is the control torque about the pitch axis, and τ_y is the control torque about the yaw axis.

These three components correspond directly to the three-axis actuation required for full attitude control. This formulation is essential because, in space, various uncertainties and perturbations (such as environmental torques) disturb the satellite, and each axis must be independently controlled to maintain the desired orientation.

Now, let us derive the expression

$$\frac{d\mathbf{H}}{dt} = I\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}).$$

The question is: why does the time derivative of the angular momentum contain both $I\dot{\boldsymbol{\omega}}$ and the cross product term $\boldsymbol{\omega} \times (I\boldsymbol{\omega})$?

This arises because the satellite frame is rotating. If the reference frame were inertial (i.e., stationary), the relation

$$\frac{d\mathbf{H}}{dt} = I\,\dot{\boldsymbol{\omega}}$$

would be sufficient. However, in a rotating frame, the rotational effect of the frame itself must also be accounted for.

For any vector $\mathbf{v}$ expressed in a rotating frame with angular velocity $\boldsymbol{\omega}$, the following identity holds:

$$\left(\frac{d\mathbf{v}}{dt}\right)_{\text{inertial}} = \left(\frac{d\mathbf{v}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{v}.$$

This relation captures the contribution of the frame rotation to the rate of change of the vector.

Now, if we substitute $\mathbf{v} = \mathbf{H}$, where

$$\mathbf{H} = I\boldsymbol{\omega}$$

is the angular momentum, we obtain

$$\frac{d\mathbf{H}}{dt} = I\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}),$$

which is the well-known rotational dynamics equation for a rigid body.

To express the system in state–space form, we now define the state variables as

$$x_1 = \omega_1, \qquad x_2 = \omega_2, \qquad x_3 = \omega_3.$$

And what is our ultimate objective? I have to maintain ω_1, ω_2, and ω_3 equal to 0 in finite time. So, somehow angles θ_1, θ_2, and θ_3 become constant. So, that is our objective. So, I have expressed like this $f(x)$, all kind of non-linear term I have keep inside the $f(x)$ and then I have designed the control based on the unit factor. And obviously, our objective is to reject all uncertainty, so this will act like uncertainty maintains sliding mode in finite time, and after that, using Lyapunov function, we have to show that we will get this objective $\sigma = 0$ in finite time.

So, I have designed the control like this. So, this part is a linear part and this is a non-linear part:

$$u = u_{\text{linear}} + u_{\text{nonlinear}}.$$

So, now that you can check the theory, you can easily understand each and every term. Now, I will take the Lyapunov function exactly as in the previous discussion:

$$V = \sigma^T P_2 \sigma,$$

and after that, I calculated the rate of change of V:

$$\dot{V} = \sigma^T P_2 \dot{\sigma} + \dot{\sigma}^T P_2 \sigma.$$

So, I skipped some steps. Why? Because you will do homework, and after that, fill the space, and then you will get this. And in this way, I can show that

$$V = 0$$

in finite time. So, if $V = 0$ in finite time, what does it mean? σ contains x_1, x_2, and x_3. So, all equal to 0 in finite time.

So, in finite time I can be able to do the attitude control. So, in order to show the efficacy of the results, what have I done? I have taken some parameter $\varepsilon = 0.5$, $\psi(x) = 3$ and these are the initial conditions, and after that you can see that in MATLAB I have simulated. So, in finite time, you can see that I have achieved

$$\omega_1 = 0, \quad \omega_2 = 0, \quad \omega_3 = 0.$$

This is the control, and this is the sliding surface.

And after that, this is the MATLAB code. So, I am going to share this slide with you. So, you can just copy and paste in MATLAB software. You can easily retrieve the result.

And this is not difficult. To actually write the MATLAB program. You can write the initial condition, you can give the span, and after that you can call `ode45` and then plot it, and this is the function. Function means dynamics. Now, so far I have just talked about if I have matched uncertainty; then how to tackle matched uncertainty using the unit feedback control. But several systems are affected by unmatched uncertainty. And it is very easy to show that the same result can also be extended for some classes of unmatched uncertainty, not for all classes.

So, this is a regular form transformation. And if I have unmatched uncertainty, you can see that some terms will also appear. There is no typo here. Now, matched uncertainty can be exactly canceled by the sliding mode control. We have already seen in the previous discussion that for unmatched uncertainty, a special treatment is required. And what are we assuming? We are assuming that

$$\sigma = z_2 + mz_1.$$

So, once $\sigma = 0$, at that time

$$z_2 = -mz_1.$$

So, that kind of thing I am going to keep in this equation and try to calculate the condition under which the sliding dynamics, i.e., the sliding dynamics, is asymptotically stable. So, it is possible to show that for all classes of uncertainty, I am not able to do or solve this problem. So, only vanishing uncertainty. What is the meaning of vanishing uncertainty? In the first module, I have already discussed that class of uncertainty which is 0 at $z = 0$, that is 0, and is called vanishing uncertainty. So, I am assuming that whatever unmatched uncertainty is vanishing in nature.

So, I am going to use this bound. After that, since I just have to stabilize the reduced-order dynamics, and reduced-order dynamics is just expressed like this, and due to that reason, I have defined the Lyapunov function like this:

$$V = z^T Pz,$$

where again, P is a positive symmetric definite matrix, and this is again the Lyapunov equation. And here, you can see that M, I have to design it such that I can reject the unmatched uncertainty. And m is our design parameter.

You can also see that the sliding surface I have to design is like this:

$$\sigma = Cx,$$

here I have to write

$$\sigma = Sz.$$

So, what is S? S is the identity matrix I_m, and this is $[z_1, z_2]^T$. In this way, I am able to design S. Now, since I am assuming that during sliding, this kind of dynamics comes into the picture, because sliding mode control is completely able to reject the matched uncertainty.

And due to that, I can achieve design

$$\sigma = 0$$

in finite time. After that, I have calculated the 2-norm and then I have substituted it inside the Lyapunov equation. So, now, you can see that this expression comes into the picture. So, this expression suggests to us what kind of m; m is nothing but, as I have already told you, that is nothing but some kind of $z_2 + mz_1$.

So, I have to design. So, this is a parameter of the sliding variable. So, I have to select it based on this particular inequality, and if I substitute that, then I will achieve this. It means that I have already shown that if σ is selected like this, then our reduced-order dynamics are also asymptotically stable in the presence of unmatched uncertainty. Okay, so you can see here that I have started somehow with this system

$$\dot{x} = Ax + Bu + f(t, x_1, x_2),$$

and after that, I have done a regular form transformation. What is the meaning of z_1 and z_2? By regular transformation, you can know, you are already knowing that I will define some kind of transformation such that

$$z = Tx.$$

So, in this way z and x are related. Now, just our problem; our problem is exactly like the matched uncertainty problem because I know that if I select m based on this particular inequality, then even in the reduced-order dynamics, I can achieve asymptotic stability, and due to that reason, now again I have expressed the problem exactly in the same way as the matched uncertainty case. I will design the control; this control contains a linear part and a non-linear part, this kind of inequality should be satisfied, and again I can achieve

$$\sigma = 0$$

in finite time. So, once $\sigma = 0$ in finite time, then asymptotically I can stabilize z_1 and z_2. So, in this way, I can also reject the unmatched uncertainty.

So, it is time to conclude this lecture. So, what have we observed in this lecture? Using unit feedback control, I can manage both classes of uncertainty. We have done Lyapunov analysis, and finally, we have also taken one practical example that is very, very relevant, particularly for the satellite problem, on how to stabilize the satellite dynamics in finite time, i.e., the attitude of the satellite in finite time. So, with this remark, I am going to end this lecture. Thank you very much.