

Sliding Mode Control and Applications

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Tracking Problem based on Sliding Mode Control for Nonlinear Systems

Welcome back. In the previous class, I was talking about input-output linearization of linear systems as well as nonlinear systems. And in this class, I am going to extend that result to design sliding mode control for the nonlinear system, and particularly I am going to focus on the tracking problem. So, let us start today's discussion. So, purpose of the discussion to explore the application of sliding mode control to solving tracking problem. So, for several times, I have actually discussed the stabilization problem for the sliding mode control prospect and what our main intention is: if the system actually contains some kind of uncertainties and disturbances, then we have to solve the stabilization problem.

In this class, I am going to tell you how to extend sliding mode control to solve the tracking problem despite the uncertainties and perturbations, and the expected outcome is a clear understanding of the sliding mode control methodology for the tracking problem and robust control that ensures asymptotic tracking of the desired reference trajectory. So, let us start today's discussion. In the previous class, I was talking about the input-output linearization of the control affine system. Here, you can see that our system is a little bit different because I have two additional terms.

This is Δ_1 , and here, this is Δ . So, this uncertainty Δ is entering through the control channel, and this is not going to enter through the control channel. So, you can treat this system as a control-affine uncertain system, and exactly like the same class, I am

assuming that I have a single-input single-output system. So, output $y = h(x) \in \mathbb{R}$. But x can contain n number of states.

So, for that reason, I have written $x \in \mathbb{R}^n$. And I need some kind of assumption. Why are assumptions required? I have already told you that not every differential equation will represent the physical situation. For that, I need some kind of extra condition such that the differential equation will represent the physical system. So, for that, we have to talk about existence and uniqueness of the solution.

So, mathematically, if you give this kind of guarantee that f , g , h , and Δ_1 are sufficiently smooth in their domain, it means that they are differentiable and piecewise continuous in t , this term is obviously sufficiently smooth in x and u , then you can talk about existence and uniqueness. Let us discuss about this particular uncertain system. Here, I am assuming that f and h are defined. So, h is our output and f represents natural dynamics. What is the meaning of natural dynamics? You can keep all uncertainty equal to 0 and control equal to 0.

So, natural dynamics and h are known, but all other quantity g , g is the matrix through which basically control is going to enter that is uncertain, Δ is uncertain and Δ_1 are unknown. Let us review the results of the previous class. So, in the previous class, I started with the control affine system, and I defined the relative degree and how to define the relative degree. I will start with the output and I am going to take the successive derivatives of the output. And if the relative degree of the system is ρ , then control is explicitly going to appear on that particular derivative, which means the ρ -th derivative control is going to appear.

At that time, we have to make sure that this term should not equal to 0. In this particular development, I am going to assume that this is greater than 0, but equally, you can also assume this is negative, but the sign of this term should be certain because this is associated with the control input. Now, let us talk about the objective. So, our control objective is to design sliding mode control because our system is uncertain. We have already seen that the original system contains three different classes of uncertainty.

One uncertainty is entered through the control channel. The control matrix through which control is basically going to enter the system is also uncertain, and one uncertainty may be an external uncertainty that we have represented using Δ_1 . So, what I am assuming is that I have a reference signal $r(t)$, and this signal is ρ -time differentiable; ρ is nothing but the relative degree of the system, and that is also bounded. We again need this kind of assumption because what is our plan? Whenever I have some kind of nonlinear uncertain system, using input-output linearization, I am going to transform that system into a linear system as well as some kind of internal dynamics. So, in order to give guarantee that linear system has solution, this assumption is required.

And further, I am assuming that all these different signals and their derivatives up

to ρ are available because I am going to use their information to design the sliding mode control. So, in the next part of this course, you are going to learn how to calculate successive derivatives if you know some signals, even if that signal is uncertain. So, for that, we are going to explore the robust exact differentiator given by Professor Levant, and you will be able to utilize it here; if you know $R(t)$, you will be able to calculate the successive derivative.

In the previous class, we have already seen that if I have a control affine system and if the relative degree with respect to the output $y = h(x)$ is equal to ρ , then I can select a coordinate system like this: $h(x)$ is the output and they are successive derivatives. And remaining dynamics, I can, I am going to maintain this condition and using that condition, what is meaning of this condition? That η dynamics here, that is basically independent from the control.

So, only control is going to appear across this ξ dynamics. And we already know that I have just a single control because I am talking about a single-input single-output system. So, on last derivative, control is going to explicitly appear. Now, let us come to our original system. So, when I have started this class, I have taken this system and I have already assumed f and h are known.

So, this is just the repetition. And after that, what do I have to do? Now, our aim is to restrict Δ and Δ_1 such that I can again able to achieve the input-output form or some kind of normal structure such that I can able to design the tracking control. So that is our main agenda. Now, in the presence of the perturbation, if we are going to convert or if we are guaranteeing that there exists some kind of normal form or input-output linearization, then some extra assumptions are required, and several practical systems are going to satisfy these assumptions.

So, what is our assumption? You can see here that if I have some variable that is ζ_1 , then ξ basically contains $\zeta_1, \zeta_2, \dots, \zeta_n$.

Several times this is actually ξ and this is ζ . So, you have to differentiate that. Sometimes I am actually confused between these two. So please be careful. So, now whenever I am going to calculate $\dot{\zeta}_1$ at that time you can see that in order to express as a linear system, what is our assumption? That Lie derivative of h along g should be 0.

So, this term is zero. So, this disturbance is not going to create any matter during the normal form transformation or input-output linearization, but this disturbance is still present. So, we made an extra assumption. So, what kind of extra assumptions are required? I have to assume that

$$\Delta_1 \in \text{null space} \left(\frac{\partial h}{\partial x} \right),$$

due to this assumption,

$$\frac{\partial h}{\partial x} \Delta_1 = 0.$$

So, in this way, $\dot{\zeta}_1 = \frac{\partial h}{\partial x} f(x)$. So, this assumption has to be explored every time you are dealing with some kind of practical system, and based on my expertise, I have seen that several systems are basically satisfying this assumption.

Now, I am going to define

$$\dot{\zeta}_1 = \zeta_2$$

and then again I will take the successive derivative same like whenever I am going to define the relative degree. So, I have to assume that again a higher-order derivative and this Δx always satisfy this kind of condition; then only I can express this system in the input-output linearization form, which is also called the normal form.

Now, in the previous class, we have already seen that whenever we have some kind of control-affine system, we have internal dynamics, and after that, we have a structure like a chain of integrators. So, this is a linear-like system, not exactly a linear system, but by designing proper control, you can see that this system is completely converted into a linear system. Here, there is basically a disturbance, and for that reason, this is a linear uncertain system. And once ζ is either 0 or bounded, we need some kind of extra assumption to guarantee that this dynamics is not going to create any problems; that is, internal dynamics or zero dynamics is not going to create any kind of problem.

What is the meaning of zero dynamics? By putting $\zeta = 0$, whatever dynamics comes into the picture is defined as zero dynamics. So, I have to make some further assumptions on this dynamics. So, we will state that assumption in the next part of this lecture.

So, first I am going to convert this problem into the form of a tracking problem. So, what is our objective? I have to track

$$\zeta_1 \rightarrow r, \quad \zeta_2 \rightarrow \dot{r}, \dots$$

and in a similar way, I have to proceed. And due to that reason, what have I done? I have defined $\zeta_1, \zeta_2, \dots, \zeta_{\rho-1}$. In terms of error, you can see that this internal dynamic or zero dynamic remains the same. Here,

$$E_1 = \zeta_1 - r, \quad E'_1 = E_2$$

and similarly, if I proceed, then I will get this kind of structure.

What is our goal? I have to do asymptotic tracking. So, in order to perform asymptotic tracking, if $E \rightarrow 0$ asymptotically, then I can able to give guarantee that $\zeta_1 \rightarrow r$, $\zeta_2 \rightarrow \dot{r}$ and similarly, $\zeta_{\rho-1} \rightarrow r^{(\rho-1)}$.

Now, you can see the second thing that we have already assumed: that the derivative

of the reference signal is bounded. So, if I assume that ζ is bounded, then its difference is also bounded, which means the error is bounded.

Why this assumption is crucial? Because once this dynamic is represented in terms of the error, if you see the zero-order dynamics, that is represented by somehow this is if ε is tending towards zero, before that I have this kind of structure. So, and both systems are running parallelly. So, in the presence of this bounded quantity, I have to show that this remains bounded throughout the time interval.

So, how do you give a guarantee? So, for that, we need some extra assumptions, that is, the assumption of input-to-state stability. So, in this particular course, I am not going to give much time to input-to-state stability. But I will just give you an overview. So, overview is suppose I have some kind of linear system and in absence of control, this linear system is stable. Now, if you apply some kind of bounded input to this linear system, then you will always get bounded output.

So, this property is called the bounded-input, bounded-output property. Obviously, this property is not satisfied by all non-linear systems, but specifically some classes of non-linear systems have exactly the same behavior as linear systems, and that class of systems is called input-to-state stable linear systems. So, by solving this, you can also establish that this system, if our system in absence of control, if that is stable, then what class of control if I will apply for non-linear system, then that remains bounded. So, those are the kinds of things you are trying to learn from the Khalil book. I have also recorded a video lecture in the course of non-linear control, which is available on YouTube.

If you have any confusion, please go through it. Now, since I have already assumed that this part remains bounded, so I have to just now worry about this linear system. Obviously, this is not linear provided if somehow we are going to cancel this term, this term as well as this term. And we know that by designing feedback, it is possible to compensate for this non-linear term. So, suppose I have some kind of methodology such that I can compensate for this, then I can assume that this is exactly some kind of linear system.

Now, I have to design a sliding mode control. And what is the philosophy of sliding mode control? Since in this chain of integrator, this is also called chain-of-integrator dynamics and any linear controllable system can be basically converted into this form. Where control, suppose that if I have just single input, then control is explicitly going to appear on the last dynamics. Now here, since I have just one control and due to that reason, whenever I am going to apply sliding mode control, then whatever sliding dynamics, σ , I am going to assume that should belong to one dimension because I have only one control. So, what am I going to do? I am going to transform the system into two different parts.

One part is called the reduced-order or somehow the sliding phase. And lastly, I am

going to compensate based on the design of this sliding mode control and after that, take the derivative. Since I have to control, I have to make sure that $\sigma = 0$ for all finite time $t \geq t_0$. So, how do I do that? I will design the sliding surface, take its derivative, and after that I will design the control: one control which compensates all other terms, and the second part of control is based on switching. The switching part, we are going to design; the gain of the switching part is designed based on the bound of the uncertainty.

That is our philosophy, and due to that reason, you can see that once the system is in the sliding phase, obviously this drawing is not appropriate because I am just considering I have two errors E_1 and E_2 , but basically here you can see that this is $\rho - 1$ dimensional system. Now, what is our objective? I have to force this system to remain on this particular sliding surface in higher-dimensional space. And we already know that if I have a linear system, then I can design a state feedback either based on pole placement or any kind of control technique. So, this is just a linear problem. So, what can you do? How do you select the gains $k_1, k_2, \dots, k_{\rho-1}$? You can just make sure that this polynomial is Hurwitz.

Then, it is possible to show that if you have this kind of system and assume this as ρ as some kind of virtual control, so if you apply here, then e_1 to $e_{\rho-1}$ are 0 asymptotically. And since e_1 to $e_{\rho-1}$ equal 0 asymptotically and e_ρ is the linear combination of that where k_1 and $k_{\rho-1}$ are finite, so I have only one choice: $e_\rho \rightarrow 0$.

In this way, I can stabilize the whole system. Now, I have only one part. So, I have already discussed how to maintain a stable sliding mode. It means that if I am on the sliding manifold, at that time I have the reduced-order dynamics and know how to design them such that I will not go in the wrong direction. I will converge towards the equilibrium point:

$$E_1, E_2, \dots, E_{\rho-1}, E_\rho = 0.$$

Now, I have to design some kind of switching feedback such that I can converge to 0, converge to $\sigma = 0$ in finite time. So, for that, I have calculated the rate of change and then I am going to design the control. As I mentioned, the control contains two parts. So, you can substitute some known parts.

There are several known parts. Why? I am going to substitute the known parts; if you do not do that, then unnecessarily I have to increase the gain of the switching function. Due to that reason, if there is a known non-linear part, I am going to compensate for that first. I am also assuming that g is not completely known, and due to that reason, here I am using g and \hat{g} ; \hat{g} I am going to use here. Now, you can see that if you apply this kind of control, you can decide the control just by seeing the dynamics.

I have to compensate for this term, this term, as well as this term. For that, I have taken the inverse, and now you can easily understand why I am assuming this is greater than 0: because if this is greater than 0, then it will not change the sign. I can also

calculate the inverse since it will give a scalar number. Once you substitute this control in the dynamics, you get a first-order dynamics.

So, one control means that you have to just design a first-order sliding mode control. And when I was talking about the sliding mode control of a non-linear system, I had mentioned that if this dynamics occurs, then \hat{g} comes into the picture, and this is uncertain. Even if this is g , that does not matter because I am assuming that in the first-order sliding mode control, this may be uncertain. So, here \hat{g} is not required; g is completely fine.

Now, I have to bound this term. This structure is exactly like

$$\dot{\sigma} = h(x) + g(x)u.$$

Both terms are uncertain, so I have to bound them. Here, I have an extra term, and due to that reason, I have to use it.

We have already discussed when talking about regular form stabilization. Practically, I am solving just a one-dimensional problem. The remaining problem is solved by linear control theory, pole placement, or any other control technique. Now, I am going to apply this strategy for the underwater vehicle. From the first module onward, I was talking about this particular example.

Our objective here: second-order dynamics. In module 1, I am expressing the system in terms of the velocity coordinate frame. Here, I am going to control the heading angle. So, in terms of heading angle, this is second-order dynamics.

τ is the normalized torque input, parameter A . I am assuming it is not completely known, but its nominal value is 1. Suppose this is the bound of the parameter or some kind of perturbation; you can increase or decrease this bound based on your convenience or physical situation. Our objective is to track this trajectory. This trajectory is bounded because it is sinusoidal, which takes values between -1 and 1 . Its derivative is also bounded because \cos and $2 \cos$ appear, which are automatically bounded.

Now, I am converting it into a state-space representation:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = f(x) + u,$$

where u is the torque. Take output as σ and ψ . With respect to ψ , the relative degree of this underwater vehicle is 2, and the system is already in input-output linearization form. No need for lead derivative.

Define errors between x_1 and ψ_r , and x_2 and $\dot{\psi}_r$, and express the system in terms of error dynamics. If A is known, control can cancel it, and we get a linear system:

$$\dot{E}_1 = E_2, \quad \dot{E}_2 = U.$$

Any linear control will work, but practically, this term is unknown. So we substitute the estimate, choose k_1, k_2 to make the system Hurwitz.

Feedforward term uses the known derivative and second derivative of the reference. For simplicity, assume gain unity. Bound the parameter:

$$x_2^2 \leq \dots$$

Since $x_2^2 = E_2^2 + \dot{\xi}_r^2$, and $\dot{\xi}_r = 2 \cos(2t)$, we can calculate the bound.

Linear state feedback cannot reject all disturbances, but under assumptions, error remains bounded. Assume error evolves within a certain range; this value can be chosen based on practical specifications. Compute bounds using Lyapunov function:

$$V = E^T P E, \quad P > 0.$$

Derivative:

$$\dot{V} = \dot{E}^T P E + E^T P \dot{E}.$$

If $\|E\| \geq \mu$, \dot{V} is bounded. Calculate ultimate bound from the Lyapunov function. Choose $k_2 = 4$, or any practical value.

Now, stabilize at equilibrium using sliding mode control. Choose sliding surface:

$$S = E_1 + E_2$$

Take derivative, substitute dynamics, design control: known terms substituted, and switching function V . Design V to maintain $S = 0$ for all time. Some actuators cannot handle high switching frequency, so use a saturation function instead of sign function, achieving convergence within a band.

Then reduced-order dynamics:

$$\dot{E}_1 = E_2, \quad S = E_1 + E_2$$

System is uniformly ultimately bounded. Precision increases by decreasing ε . Using sign function, $S = 0$ exactly, and $E_1 = -E_2$.

Finally, you can compare three control methods: linear feedback, saturation-based, and sliding mode control.

Homework: design and compare these methods.

Conclusion: we discussed sliding mode control for tracking, demonstrated robustness, applied input-output linearization, expressed system in normal form, and designed sliding mode control.

Thank you very much.