# Sliding Mode Control and Applications

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**Week-03**

**Lecture-12**

So, welcome back. In the previous class, I talked about the theory of input-output linearization. So, we have established the definition of relative degree, and with the help of the definition of relative degree, we have actually applied the theory of relative degree for the linear system, and we have converted the linear system into input-output representation form. But what is the main goal of this particular course? We have to design sliding mode control for the non-linear system, and for that reason, I am now going to take the system in a specific form that is the control affine form. Obviously, the system is non-linear, and I will then show you how to represent the control affine form exactly in the form of the input-output linearization form, similar to the linear system. So, that is the goal of this lecture, and this is the continuation of the previous lecture.

So, the purpose of the discussion is exactly the same as in the previous class. I am going to start with controlling a fine non-linear system and then I am going to convert it into a linear-like system. Several times, you are not able to convert into a linear system, but you can convert a system that is very close to the linear system, and after that, we are trying to solve the tracking problem again. Obviously, you can also solve a stabilization problem.

With the help of this particular way of input-output linearization, but since we are very much concerned with the output, and due to that reason, whenever we are doing input-output linearization, we are trying to solve some kind of tracking problem, and most practical problems actually lie in this category of the tracking problem. So, we have system sometime we have to maintain some kind of voltage, some plant we have to maintain some specific temperature. So, objective that whatever output that is coming from the system, we have to maintain either that as a constant or time varying. So, that falls in the class of tracking problems. We have already understood this. And now, what is our main objective? The main objective is to give a non-linear system; first, we have to understand how to transform that into input-output linearization form.

So, this lecture is a continuation of the previous one, but here our system is more general than the linear system. And due to that reason, I am going to start again with the linear system. In previous class, what we have observed that if I have some LTI system that is represented by the transfer function $h(x)$ and that is nothing but the ratio of the transfer function of numerator and denominator, then we can able to represent this transfer function in some kind of feedback form. And once we represent that, then by the realization of a chain of integrators, $p$ number of chains of integrators, as well as the minimal realization, I will get this particular form. So, here relative degree of this system is actually $\rho$.

What does it mean? That if I take or if I start with the output and if I will take $\rho$-th derivative, then control will explicitly appear. So, if you have attended the previous lecture, then you might have seen that for nonlinear system, this is the input output representation. But now, in this lecture, I have system that is in control affine form. And what is our objective? Now, our objective is to represent this control affine system in input-output linearization form. So, here I am going to define our output $y = h(x)$.

Again, I am assuming that the relative degree of this control affine system with respect to this output, which belongs to the one dimension that is $\rho$, is what it means. That $h(x)$ and their $\rho - 1$ Lie derivative, if I calculate, will be present. So, at the $\rho - 1$ derivative of $h(x)$, the place control is explicitly going to appear. So, this $\psi$ is basically coordinate system of $\psi$ is nothing but output and their derivative up to $\rho - 1$-th derivative. So, that is nothing but this $\psi(x)$ function.

And what is $\phi(x)$ ? $\phi(x)$ since the order of the system is $n$. So, now I need some kind of $n - \rho$ coordinate transformation such that I will get some kind of dynamics like this. So, this dynamics $\dot{\xi}$ is coming due to the relative degree and this extra dynamics that is coming due to some kind of coordinate transformation. And I have to define this coordinate transformation such that using this transformation $z = T(x)$, this control affine system can be converted same like some kind of linear like system.

Obviously, a structure is not exactly the same as a linear system, but it looks like the linear system. So, now our main aim to understand how to select $\phi$ because selection of this part $\psi(x)$ that is very easy, why I will take the output, I will calculate their derivative and in this way I can able to fix $\psi(x)$ and that I will define as a $\xi$. Okay so I have $\eta$ and $\xi$ coordinate. And now, whenever we are doing this, we have already understood during the regular form transformation, since I am transforming the system from one domain to another domain, so that transformation should satisfy the property of diffeomorphism such that I can apply analysis here and after that, I am able to come back from here to here and again from here to here, and due to that reason, whatever transformation we are going to consider should be one-to-one, onto, differentiable, and invertible.

So, now since we know that in this particular form up to $n - \rho$, dynamics control does not explicitly appear. So, how to restrict control such that that will not appear up to $n - \rho$. So, for that I have to generate some kind of condition.

And we know that $\eta = \phi(x)$. So, if I calculate $\dot{\eta}$, then I have nothing but

$$\dot{\eta} = \frac{\partial \phi}{\partial x} \dot{x}$$

and after that $\dot{x}$ comes into the picture. So, how to make sure that control will not explicitly appear? For that, I have to maintain

$$\frac{\partial\phi}{\partial x}g(x) = 0$$

and due to that reason, you can see here that this kind of condition comes into the picture. So, I hope that you are able to understand why $\frac{\partial\phi}{\partial x}g(x) = 0$.

At that time, you have to maintain this kind of equality. So, you have to solve partial differential equation and you have to make sure that this equal to 0 for all $x$ where our system is going to lie. The second part is very easy. I will take the output and its higher order derivative. So now, I am going to state a theorem without proof.

The idea is easy to understand. Suppose, now if I have non-linear system and we are lucky enough that relative degree of this system is exactly same like the relative degree of the state. What does it mean? That if I calculate the $n$-th time derivative of the output $y = h(x)$, then control will explicitly appear. So, now I have system $n$-dimensional system you can assume and I have another coordinate frame in terms of output where $n$ number of variable it means that easily I can able to control the system. Relative degree of system equal to $n$, then system is called fully input output linearizable.

It means that from controlling a frame, I can generate a linear system in terms of the output. You can also interpret it this way; due to the bad coordinate frame, the system is nonlinear. Now, if I just select the output and its higher order derivative, we will somehow get a very good coordinate frame where everything looks linear. But if the relative degree is not equal to $n$, then what happens? I cannot fully linearize the system. So, in that particular case, I need some kind of extra function that will compensate for the remaining dynamics.

So, up to $\rho$, I can compensate using a linear system, but for the remaining dynamics, I have to fix it so that the dynamics is free from control, and we have already understood how to do that. So, for that, I have to make sure that whatever $\phi_1$ up to $\phi_{n-\rho}$ I am going to select, that should be 0 . So, now if I have some kind of control affine system $f + gu$ and $y = h$, you can see that up to $\rho$ that is the relative degree of the system and here I can able to define $\xi$, and $\xi$ is nothing but $h(x)$ and their higher order derivative, least derivative you can tell that I am going to keep here, and in this way I can able to get some kind of exactly the same kind of linear system. Now, here I have to define these two terms, that is, $\gamma(x)$ and $\alpha(x)$. So, how do I get this term? So, we already know that whenever I am starting with $h(x)$, every time I am going to calculate the Lie derivative.

So, once control will appear, at that time form of Lie derivative is like this. Here $b_c$ is nothing but $[0,0,0, \ldots ,1]$. So, in this way, I will get $\gamma(x)$. For $\alpha(x)$, what do we have to do? We have to calculate the ratio of the Lie derivative of this to this. You can also easily understand this if you calculate it; if you start with $y = h(x)$ and once control appears at the $\rho$-th derivative, $\dot{y} = \rho$, at that time I basically have two terms that are going to be involved.

The first term is this, and this term is going to be associated with control. What is our objective? Our objective is to linearize this system, and due to that reason, I have to remove this part. So, how do you remove this part? You can see here $L_g L_f^{\rho-1} h(x)$, which is exactly equal to this. So, this will cancel out and after that, whatever dynamics here that is not associated with control, that is exactly like this. So, that is going to compensate.

So, this part is actually going to compensate this particular part. So, by calculation, you can easily check. Now, I know that for the remaining dynamics I have to calculate the partial derivative because the coordinate frame I have selected is nothing but the $\eta$, and $\eta = \frac{\partial \phi}{\partial x}\dot{x}$, and that should be free from the control. So, this condition is required for that. So, the remaining term, once you substitute this particular condition, you will be able to see that the remaining term is like this, and I have actually applied the transformation here. $Z$ is nothing but what it is going to contain. Again, you can check, $Z$ is $\eta$ and $\zeta$. So, and that is equal to $T(x)$, but I have to do all substitution in terms of $x$, because our original coordinate frame is $x$, and due to that reason, I have written

$$x = T^{-1}(z)$$

So, in this way, you can be able to get this dynamics.
In the case of a linear system, you can see the dynamics are like this. Here, I am also going to comment one more important thing that you can be able to design $u$ such that

$$\zeta = 0$$

and once $\zeta = 0$.
So if $A_0$ and $A_0$ is nothing but the eigenvalue of the numerator part, and due to that reason, the pole of the system, in the case of a linear system, should lie in the left halfplane. Obviously, 0 should lie for stability, but in order to maintain the extra dynamics that are tending towards 0 . So, whenever you have a stabilization problem, $\zeta$ is already collapsed, and due to the system dynamics, you can notably design because $A_0$ is coming from the system.

If that is, their behavior is sufficiently good, it means that if the numerator pole or numerator zero is going to lie on the left half-plane, then you can easily be able to stabilize the $\eta$ also, and $\eta$ and $\zeta, \xi$ both tending towards 0 as $t \to \infty$.

What does it mean? That if the system is in minimum form, by input-output linearization, you can be able to control that system.

For tracking, actually, here $\zeta^\star$ will come into the picture, and at that time, it is possible to show that you have to make sure that this dynamics is bounded, as $\zeta^\star$ is bounded, because our prime objective is to control $\xi \to \xi_r$.

We have already seen in the previous class that $\xi_r = r$. So, in this case, I am not concentrating on this particular dynamics, but I have to make sure that this dynamics will not blow up.

And due to that reason, what do we have to do? We have to ensure input-output stability. What is the input here? $\xi^\star$ is the input. So, we have to show that this dynamics actually remains bounded. And if $A_0$ is negative, then it is possible to show that any bounded input will give us a bounded output. It means that if $\zeta^\star$ is bounded, then $\eta$ is also bounded.

So, in this way, you can solve the stabilization problem and the tracking problem in the case of a linear system. A similar kind of philosophy can also be applied to the nonlinear system. So, suppose that if you have to just solve the stabilization problem, now you can design $u$ such that this $\xi = 0$, and once $\xi = 0$, then you have dynamics that are

$$\dot{\xi} = f_0(\xi, 0).$$

So, if this is dynamics, you can also interpret this dynamics as an internal dynamics because there is no control here. So, if this is stable, then the overall system is stable.

So, now what am I going to do? I am going to look into the equilibrium point of the converted coordinate frame. So, everything I have expressed in terms of the output or some kind of transformation $\phi(x)$ is the definition of an equilibrium point. We are assuming that all higher derivatives of output are equal to 0 . And in this way, I can match

$$\eta = 0 \text{ and } \xi = 0 \text{ with } x = 0.$$

So, whenever we do the input-output linearization, we have two different dynamics. One is an external part that we can control using the input. You can see here that control is explicitly appearing. And the dimension of $u$ we have to match with the dimension of $\zeta$. So, if that does not match, then the theory is like this: suppose that in this particular example I have just one control input. So, using one control, I have to control the growth dynamics, and that is possible using pole placement or any other method you can use to do it.

After that, I have internal dynamics. I have already told you that dynamics is unobservable by the same control input and due to that reason, our internal dynamics is good enough such that once $\xi = 0$, then that will tend towards 0 or at least that remains bounded if $\xi \rightarrow 0$ for the practical purpose. So, now I am going to define what the meaning of zero dynamics is.

So, once the output and their higher-order derivatives, if you are going to substitute them equal to 0 , what does it mean? It means that whenever I am talking about output and their higher-order derivative equal to 0 , it means that I am talking about

$$h(x), \dot{h}(x), \ldots, L_f^{\rho-1} h(x) = 0.$$

So, once $\zeta = 0$, whatever dynamics come into the picture is called the zero dynamics. Now, for a nonlinear system, the definition is exactly the same as that of a linear system. Now, we are just going to look into the stability of this zero dynamics. If zero dynamics is stable or asymptotically stable, then the system is minimum phase. If that is not the case, then the system is in non-minimum phase. So, in this way, you can extend the definition of minimum phase from linear to nonlinear systems.

You can see here that whenever we are talking about zero dynamics, at that time I am forcing control equal to zero. So, at that time $\zeta = 0$, and because $\xi$ contains $y(t)$ and their higher derivatives, so $\xi = 0$ and

$$u(t) = \alpha(x(t))$$

I have to maintain. Then only I can be able to match with the equilibrium point. So, how to do that? For that, this condition is required.

Now, I have already told you that if we are lucky and the order of the system and the relative degree of the system are both the same, then our system is perfectly input-output linearizable.

It means that I can express the system in terms of the output and their higher-order derivatives - so up to ( $n-1$ ) derivatives - and there is no zero dynamics. And this is a minimum phase system by default because there is no zero dynamics. Now, you can easily design control as

$$u = \alpha(x) + v$$

where $v$ is a new control input that you can select to achieve your desired closed-loop behaviour.

After that, try to compensate for this, and in this way, you can easily design the control action. Obviously, you have to make some more adjustments here whenever you are going to design the control.

Now, what I am going to do is to try to understand minimum phase using some examples. I am going to start with exactly the same example as the previous class, and I have selected the output as $y$ and let $y = x_2$. If you calculate the relative degree, then the relative degree is

$$\rho = 1.$$

So, now, if the relative degree is 1 , then by designing control, I can stabilize $x_2$. Once $x_2 = 0$, you can see that this is our dynamics, and we already know that $x_1$ is not tending to 0 . Due to that, this system is not a minimum phase system. However, $x_1$ will not blow up, so we can guarantee boundedness from here. Now, I am going to take another system, which is a little more complicated.

After that, I am going to calculate the relative degree. As we have already discussed, you start with the output and take successive derivatives. The first derivative is

$$\dot{y} = x_3,$$

where control does not explicitly appear. The second derivative is $\ddot{y} = \cdots$ (control appears here), and hence, for this third-order system, the relative degree is $\rho = 2$. I have already told you how to calculate gamma and $\alpha$. So, $\gamma$ and $\alpha$ can be calculated using the standard formulas. Please verify that $\gamma = 1$ for this system, and $\alpha$ is given as above. This is assigned as homework. Now, if we explore the zero dynamics:

Here, we have a second-order system. We can force

$$x_2 = 0, x_3 = 0$$

by designing a suitable control. For example, you might use sliding mode control to ensure $x_2 = x_3 = 0$. Once $x_2 = x_3 = 0$, the remaining dynamics reduce to

$$\dot{x}_1 = f(x_1),$$

which can be checked for stability using Newton's law. Since $x_1$ is one-dimensional, you can plot the phase line and verify that

$$x_1 > 0, \dot{x}_1 < 0,$$

so the system will converge to the equilibrium point from both sides. Hence, this system is a minimum phase system. Next, we consider the coordinate transformation. For a third-order system, we have

$$x_2 \text{ as output, } \quad \text{and } \dot{x}_2 = x_3$$

appearing in the derivative. Thus, ( $x_2, x_3$ ) forms a coordinate frame because

$$y = x_2, \dot{y} = x_3.$$

So, ( $x_2, x_3$ ) is the coordinate frame because $y = x_2$ and lead derivative of the $\dot{y} = x_3$. So, that is the coordinate frame. So, I just need one extra coordinate frame, and I have to maintain this kind of equality. Why is $\phi$ equal to 0 ? We only need one additional coordinate frame, $\phi(x)$, which must satisfy

$$\phi(x) = 0$$

so that $x = 0$ is the equilibrium point. We also need

$$\frac{\partial \phi}{\partial x} g(x) = 0,$$

where $g(x)$ is associated with the dynamics and the control input. By solving this partial differential equation (PDE), we can determine $\phi(x)$. There are various software tools available to solve such PDEs efficiently.

So, in this way I can able to generate $T(x)$ and in that particular coordinate frame I can able to express the system. So, now this system is in input output linearizable form. Now, you can design control $u$, and you can collapse $\xi_1$ and $\xi_2$ to equal 0 , and once $\xi_1$ and $\xi_2$ equal 0 , then you are able to see the 0 dynamics of this system. If you have to solve the tracking problem, then specify $\xi_1$ tending towards $r$, and if $r$ is constant, then $\xi_2$ becomes 0 . It might be possible that $\xi_1$ and $\xi_2$ will tend towards some other $r_1$ and $r_2$; that kind of problem you can also solve using this structure.

Now I have taken another example. So, this is an example of field control DC motor and here, again, same as in the previous class, I am assuming $x_3$ is not equal to 0 . Why am I making this assumption? Because I have to control the speed of the DC motor, I will take the first derivative. So, you can see here that in first derivative, control will not appear. So, I will take the second derivative. And once you take the second derivative, the $\dot{x}_1$ term comes into the picture, and then control is going to come into the picture here.

So, at that time, this term $x_3$ is also associated, and due to that reason, I have to make sure that $x_3$ is not equal to 0 . Again, you can apply this formula. So, here I have written otherwise we have to go to previous slide and then you can able to calculate the $\gamma$ and $\alpha$. Zero dynamics easily you can able to calculate once you understand the coordinate system output and their higher derivative you can keep equal to 0 and whatever remaining dynamics that is represent the 0 dynamics. And you have to make sure that 0 dynamics should be stable, asymptotically stable that will converge.

So, here our objective is to maintain the value of $x_2$ at some specific voltage; we have to see the physical interpretation, and I want to maintain $\frac{k}{b}$. So, if that is maintained, then I can... Therefore, there is no need to force $x_2$ to equal 0 because our objective is to maintain some constant kind of current or voltage.

And due to that reason, this system, so by seeing physical interpretation, we have to interpret system as a minimum or non-minimum. If our objective will meet, then that is minimum phase system. Now, what do we have to do? So, I am able to generate two coordinate systems. So, out of the third order, I actually have to search for just one coordinate system now. How do I search? Whatever control component associated with that, that I have to force equal to 0 .

So, only one control component and due to that reason that you have to force that equal to 0 . So, by solving this easily, you can get the value of $\phi$. So, here is the $\phi$I have generated like this. So, in this way, even if we have a practical system, I can be able to convert it into input-output linearization form.

So, it is time to conclude this part of the lecture. So, what we have seen is that either we have a linear system. So, linear case we have actually looked into previous class and in today's lecture, we have taken the control affine system. Using the notion of the lead derivative, we have seen how to represent the system in some kind of input-output representation form. And once the system is in input-output representation form, then it is always easier to solve the tracking problem. And due to that reason, whenever you have to solve a tracking problem, most of the time this kind of development you have to do in advance, and after that, you can be able to design the sliding mode control.

We have also defined what the meaning of minimum phase is. So, now in the next class, with the help of this theory, I am going to talk about the tracking control using the sliding mode, and I am going to take examples of the non-linear system because the linear system is very easy to develop theory for, but it is a little bit difficult to develop theory for the non-linear system, and due to that reason, I am going to consider the non-linear system, while the linear system is a special case of the non-linear system. It means that if you understood the development for the non-linear system easily, you are able to extend that to the linear system. So, with this remark I am going to end the today's lecture or this lecture. Thank you very much.