Linear Systems Theory
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Module - 06
Lecture - 04
Supplementary Lecture: Comparison Lemma and Lyapunov Stability

Hello everybody, so this is a little Supplementary Material to lectures on week 6 where we essentially defined for ourselves the concept of Lyapunov Stability and conditions to verify stability in terms of eigenvalues. And some other LMI like conditioned like the A^TP + P A = -Q and so on. So, in the proof of one of those we have to use something called the comparison lemma. So, I will do a small proof of that and also in addition I will give you a little more physical interpretability of the solutions. Especially of the condition 5 that we had for stability of which was like A^TP + P A < 0.

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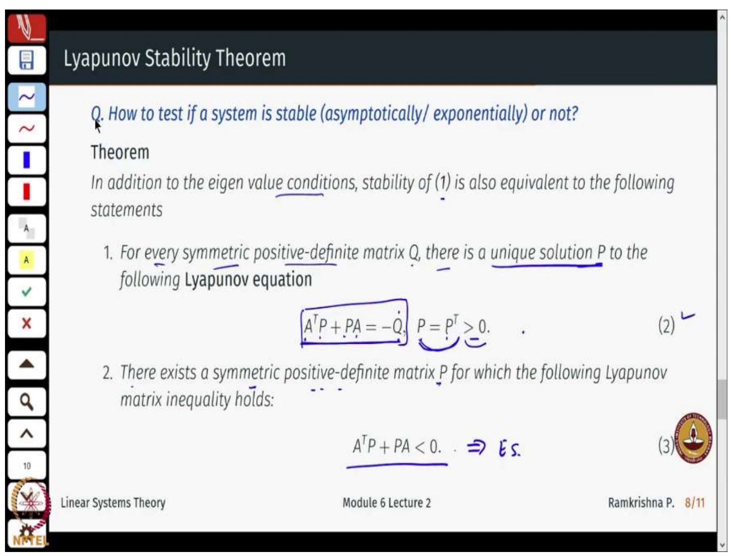

So, essentially a conditions which look like this, so we will try to see if there is any physical interpretation to this condition number 3 ok.

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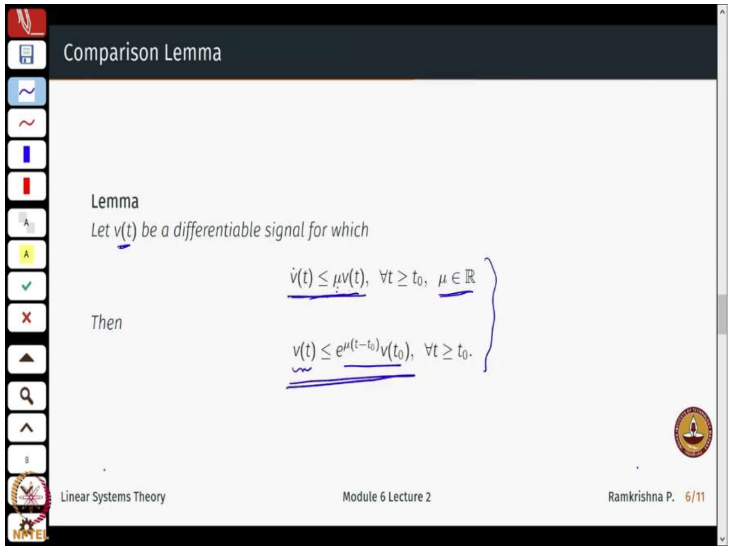

So, we will start with proving the comparison lemma, so the statement I will read out again. So, given v(t) which is a differentiable signal which satisfies a condition like this at $V(\dot{t}) \leq \mu V(t)$, then essentially v(t) solution of it looks something like this ok. So, I will just do small proof of that and it is sometimes useful to know how this proof techniques were derived.

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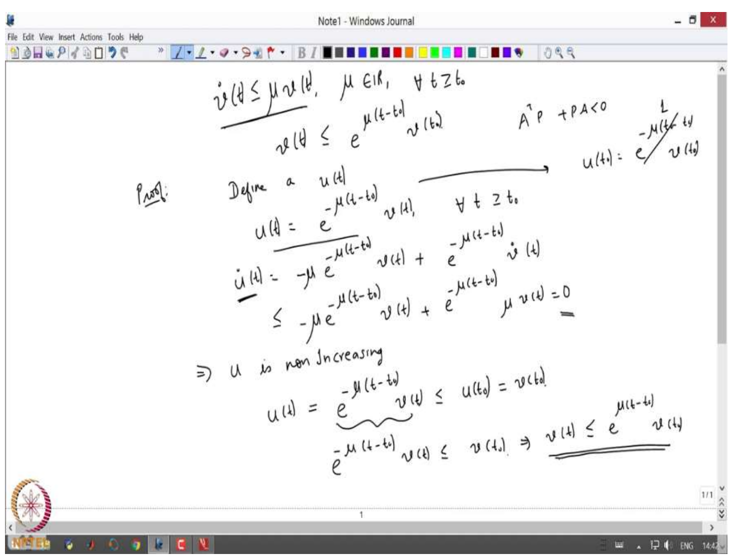

So, what are we given is v is a differentiable signal in such a way that $V(\dot{t}) \leq$ μV(t). This mean μ could be anything in R and for some or should all times t greater than some defined initial time at t_0 .

So, if this holds then v(t) $\leq e^{\mu(t-t_0)}V(t_0)$ ok. This inequality was essentially a useful in proving the condition that the 5th condition on the Lyapunov stability namely A^TP + PA < 0 ok, so we will do a little proof of this ok. So, what does the proof say? Let us define a new signal u(t) which looks like this, u(t) = $e^{\mu(t-t_0)}V(t)$; again for all times t ≥ t_0 ok.

So, I will just take the derivatives on both sides, so $u(\dot{t})$ = -μ$e^{\mu(t-t_0)}V(t)$ + $e^{\mu(t-t_0)}V(\dot{t})$ $V(\dot{t})$ ok. What do I knew of $V(\dot{t})$? So, $V(\dot{t})$ it is satisfies this is actually given to me.

So, this will be less than or equal to -μ$e^{\mu(t-t_0)}V(t)$ + $e^{\mu(t-t_0)}$ μV(t). What is $V(\dot{t})$? $V(\dot{t})$ is μV(t), so, this is equal to 0 ok. So, what do we know that, so $u(\dot{t})$=0; this means that u is non-increasing, right. And therefore, u(t) is which is this is how we defined it right $e^{\mu(t-t_0)}V(t)$ starting from here ok.

So, this is less than or equal to the value of u(t_0) ok. Now what is the value of u at t_0? Just substitute for t_0 here u(t_0) = $e^{\mu(t_0-t_0)}$V(t_0). So, u at t naught, so this is this goes to 1, now this is V(t_0) ok. Now, take a look at this, so what do I have $e^{-\mu(t-t_0)}$v(t) ≤ V(t_0) ok. This is essentially what I wanted to prove right, so I just can just rearrange and write this as a following right.

And this is what we wanted to proof, and this is this concludes a proof ok. So, is a very nice intuitive proof here and of course, we saw how this was instrumental for us in deriving stability conditions for linear systems ok.

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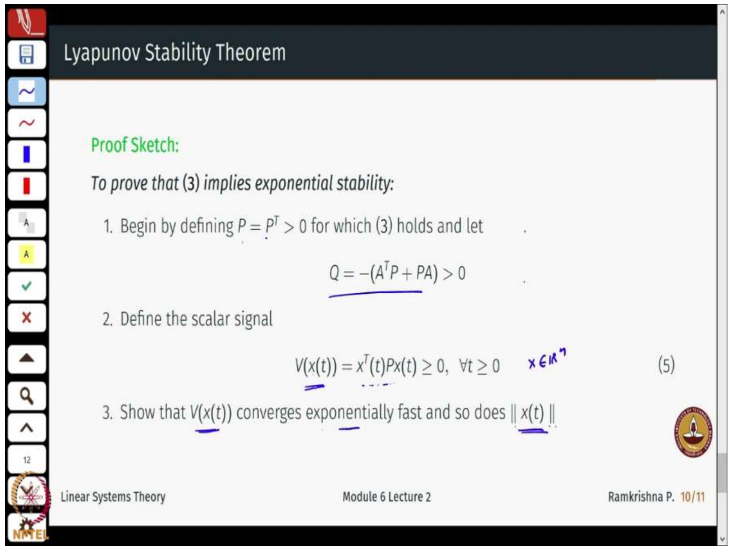

Now , so if I can go back to this thing here the equation number 3 this essentially. So, how did the proof go? The proof required defining a scalar signal v(x(t)) as $x^T(t)$P x(t) it is a quadratic function it is always greater than or equal to 0.

And we showed that the time derivative of $\dot{v}$ or the time derivative $\dot{v} \leq 0$ or which also had some bounds. And therefore, since $\dot{v} < 0$; v was converging exponentially fast and so does the solution ok. So, what does this mean, in terms of does it is have any physical interpretation right?

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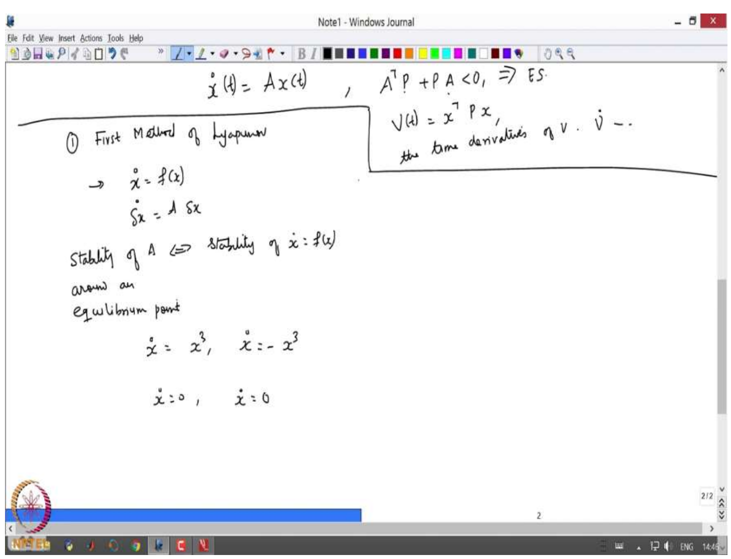

So, again we start with the autonomous LTI system $\dot{x}$ = A x(t) ok. And then we were we had this condition A^TP + P A < 0, not only that we defined a function v dot sorry v as sorry this I (Refer Time: 07:48) V(t) = x^TP x. And then we were interested in the time derivatives of e of V.

So, we computed $\dot{v}$ and then derived a bunch of properties based on the system dynamics based on this condition. And what we should essentially that this condition satisfaction of this condition was actually equal to or actually implied exponential stability ok. So, if we ok we are trying to submit and then we look at some literature from how do they prove stability in the non linear case.

The first one is what they call as the first method of Lyapunov and this is essentially involved linearization of the non linear system of what we solved right. So, stability so you start with the non linear system $\dot{x}$ = f(x) we have some linearization δ$\dot{x}$ = Aδx. And stability of A had some direct implications on the stability of $\dot{x}$ = f(x); again this is all around an equilibrium point.

Of course we here rule out the condition when A has a 0 eigenvalue, I mean a counter example to that was $\dot{x} = x^3$. And $\dot{x}$ = -x^3 which both had the same linearization namely $\dot{x}$ = 0 and x dot is 0 and we could not say anything about the stability of the non linear system ok.

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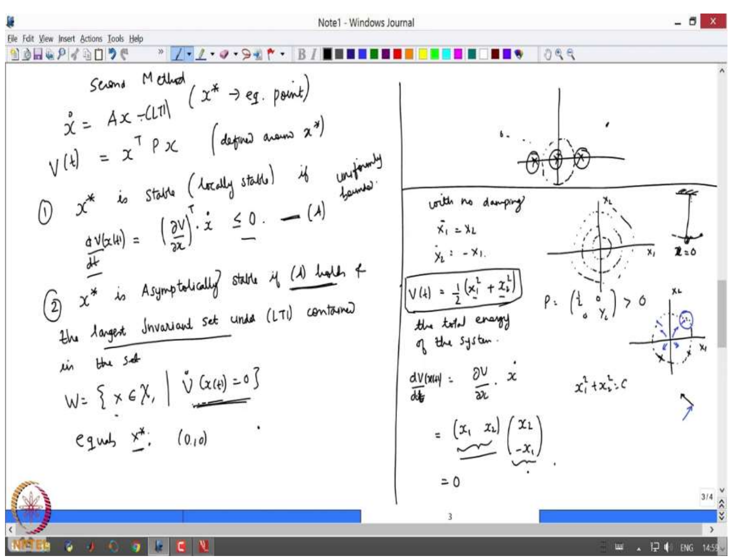

So, that is that we dealt with quite extensively this is also something called the second method of Lyapunov. We will not analyze this thoroughly in the context of non linear systems, but we will try to interpret that in our in within the context of a linearized system ok.

So, I have I start with this dynamics $\dot{x}$= A x right, so this could be linearization of a non linear system around an equilibrium point and so on. So, let V(t) be, so this is a positive definite function x^TP x, so this is usually in the non linear setting we will say we will define it.

So, this function is defined in a small neighborhood around the equilibrium point, but in the linear setting well we can assume it to be if it is the system is linear by nature we can assume it to be valid for all x ok. So, if x^* is an equilibrium say x^* is an equilibrium point.

So, the theorem says that x^* is stable or if I am looking of a non linear system I am just looking at locally stable. Locally stable because non linear systems could have say isolated equilibrium points where for example, this could be stable this equilibrium could be stable this could be stable and so on.

So, I am just looking at it is behavior around an equilibrium point within a small neighborhood and therefore, I call it locally stable this is not globally stable. Because, all trajectories if the trajectory start here or here they may not necessarily converge to the origin or even at this point at this equilibrium if I start and so on ok.

So, this system is stable if I am looking at this quantity $\frac{d(V(x(t)))}{dt}$ which is $\frac{d(V)}{dx}\dot{x}$ ok. Just put a little transpose here if I just say this I am just taking the gradient of this vector, if this quantity is less than or equal to 0 then the system is locally stable.

Second, so x^* is asymptotically stable if I will call this a condition A, if A holds and the largest invariant set under well this linear systems I will go let me call this LTI. Under this dynamics of LTI systems contained in the set, so let me define the set, W as all x belonging to some to the state space such that $\dot{V}$(x(t)) = 0 equals x^*x star right.

Which means that the only solution that starts only solution that starts in this set remains in W for all sets right and that actually coincides with the equilibrium point ok. So, we slowly see what this means in the context of some physical systems true ok. So, let us take the case of a simple pendulum all right.

So, and say just look at the linearize version of this, so it is dynamics around this equilibrium point x = 0 with no damping would look something like this right $\dot{x_1}$ =x_2 and in normalize all parameters to be 1, $\dot{x_2}$ =-x_1 ok. Now, how does how does the phase space of this look like the phase space of this looks like this, so we just concentric circles around the origin.

So, here the region is the equilibrium point or here or here or here ok, additionally if I now define this function right. So, V(t), so let me call this $\frac{1}{2}({x_1}^2+{x_2}^2)$, so this is exactly this function with P being the $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. And this you can see is symmetric and greater than 0 right.

So, this if you has also the interpretation of the total energy of the system the kinetic energy plus the potential energy like $\frac{1}{2}$mv^2 + $\frac{1}{2}$kx^2 kind of stuff ok. So, this is the total energy of the system, now when I say $\dot{V}$ ie $\frac{dV(x(t))}{dt}$. So, I am computing this quantity right $\frac{dV}{dx}\dot{x}$, so what did this mean.

So, what is $\frac{dV}{dx}$ if I just take the gradient of this, so I will get $[x_1\ x_2]$ and here I have x_2 and I have -x_1 this essentially is 0 ok. So, what does this mean if I look at just this quantity which is the definition of energy that the rate of change of energy is constant right that the energy does not change with time? That is also from the physical interpretation right.

If I just have a have a lossless pendulum and I subjected to some initial condition it will just keep on oscillating infinitely ok. Now, I can also look at it from the phase space right just take any point in the phase space. So, just take one initial condition say over here and this is our trajectory will go right in a in the in the circle [vocalized noise].

So, at any point here if I compute the energy here it will be the same as a computing the energy at this point or this point or this point why because even if I look at the solutions the solutions will be of the form ${x_1}^2$+${x_2}^2$ is constant ok. So, that also is consistent with the

the phase space right, so this will give me tell me that this function ${x_1}^2$+${x_2}^2$ is constant along the system trajectories, but this is x_1 this is x_2 right and therefore, the $\dot{V}$ is 0.

Now, a third interpretation right of, so this essentially can be looked upon as dot product of 2 vectors. So, how does a gradient vector look like if I put this to plot this is gradient vector will simply look like this right something like this something like this something like this. So, essentially if I look at the so if I just zoom in over here.

So, the gradient vector will be something like this and the original phase space of the system will be just perpendicular to it, in such a where the angle between these 2 is 90 degrees right. And then well they are like perpendicular to each other and therefore, the dot product will be a 0 right. So, so that is that is another interpretation of it is the it is a gradient of the energy function times the phase plane or the vector field of the system how it looks like. Now, the second thing here is to do with asymptotic stability right.

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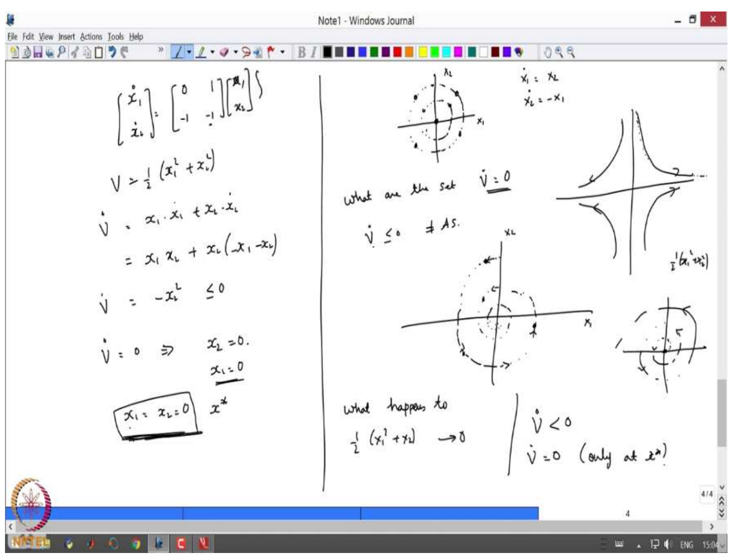

So, what does what does the theorem say that additionally, so let us say I take a system which is $\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix}$. I just add some friction term or some damping term $\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ok.

Now, if I again look at V = $\frac{1}{2}({x_1}^2$+${x_2}^2)$, I compute $\dot{V}$ that = $x_1\dot{x_1}$ + $x_2\dot{x_2}$. So, this will be, so x_1x_2 + $x_2(-x_1-x_2)$ So, this is a -${x_2}^2$ and this is less than or equal to 0 ok. So, the statement says that well the system is stable as long as $\frac{dV}{dt}$ is either less than or equal to 0, so in this case it was just equal to 0. So,, so this also means that, so this statement a here the condition a means that your solutions are uniformly bounded well that is the first condition we check for stability.

Second condition we check for stability is that first that the system should be stable and it should also be in such a way that as t goes to infinity the solutions go to go to the equilibrium point which is the origin and in this case ok. Now, why is this step here important ok, so ok, so what am I looking at I am looking at the largest invariant set such that $\dot{V}$ = 0 ok. What does it mean by $\dot{V}$ being equal to 0, $\dot{V}$ being equal to 0 implies I am looking at x_2 being equal to 0 ok. And in turn also if a if x_2 is 0 also have a x_1 is also equal to 0 ok.

So, what is and why am I obsessed with the largest invariant set right, so here the statement says that the largest invariant set such that $\dot{V}$ such that $\dot{V}$ = 0 should only be the origin which is (0, 0) in this case. So, in this case the only possibility for $\dot{V}$ being equal to 0 this x_1 is x_2 = 0 ok.

So, why is this important to check this additional condition why and why am I actually calling it the largest invariant subset. Now, if I go to the previous example where I had no damping $\dot{x_1}$ = x_2 and $\dot{x_2}$ = -x_1 what do I have is well my phase plane is something like this ok.

Now, what are the points, or what are the set or what is the set such that $\dot{V} = 0$. Now, I see that at this point $\dot{V}$ is 0 at this point $\dot{V}$ is 0 at this point $\dot{V}$ is 0 this point also $\dot{V}$ is 0 this point also $\dot{V}$ is 0. If I start with this initial condition, so at each point in this curve the $\dot{v}$ = 0 in the x_1 x_2 plane.

And therefore, now this system even though $\dot{v} \leq 0$ is not asymptotically stable. Because additionally I need to verify a condition like this right that the largest invariant set such that $\dot{v}$ = 0 is only the origin which is actually happening in the second case right. Though when $\dot{v}$= 0; that means, $x_2 = 0$ 0 if I plug it in here I get x_1 = 0. And the only possibility here is x_1 = x_2= 0 which is exactly my equilibrium point ok.

Now, in the that was about why we are obsessed with the largest invariant set. Now, if I look at the phase space in the second setting where I have some damping in the system the phase space is usually are the ones which go to the origin in the $x_{1,}x_2$ right this. So, possibly in this in this direction, so they will spiral to the origin again based on what their eigenvalues are if it is a stable focus and so on ok.

So, if I know what happens to this function $\frac{1}{2}(x_1^2 + x_2^2)$ which essentially is the total energy of the system? At this point it has a certain value, and at this point the value diminishes it diminishes further here further diminishes here until it actually reaches to 0.

So, from calculated this is also means that $\dot{V} < 0$ and $\dot{V} = 0$ only at x^* ok. Now, the third condition right of an unstable system ok; you if you if you are looking at a saddle point than this it is how my phase space will look like. Or if I have a say in some cases if the it might just spiral away from the origin right with this direction of the arrows ok.

Now, what happens to the energy well at this if I start slightly next to the origin? Then you see that the energy the value of $\frac{1}{2}(x_1^2 + x_2^2)$ it increases in a and in becomes unbounded right here also here right. So,, so [vocalized noise], so stability can also in a in a way be interpreted as a certain energy function decreasing along the trajectories of the system and eventually going to 0.

So, when my pendulum is swinging, it switches between it is kinetic and potential energies and at some point of time if there is dissipation the energy will all dissipate and just go to 0. You can also talk off it in terms of an electrical circuit of an RLC circuit the energy will oscillate between the electric and magnetic part and from the R will keep on dissipating the energy until it reaches the state of 0 energy.

Instability will amount to the energy growing and bounded at least mathematically, physically well you have some physical limitations that the energy cannot grow to infinity ok. So, that is the a little interpretation of the Lyapunov stability in terms of the physics of the system.

And that is what the motive of this entire supplementary material was to actually give a energy give a little relation between stability and the loss of physics which essentially comes from the laws of conservation of energy. And I hope this explanation was useful to and it will be more useful when you even do the non linear setting of Lyapunov functions.

Thanks.