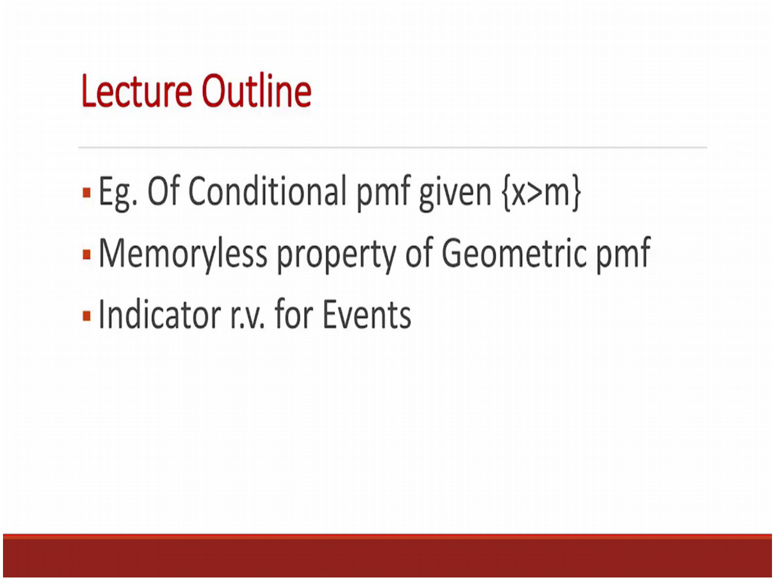
Probability Foundations for Electrical Engineers
Prof. Aravind R
Department of Electrical Engineering
Indian Institute of Technology, Madras

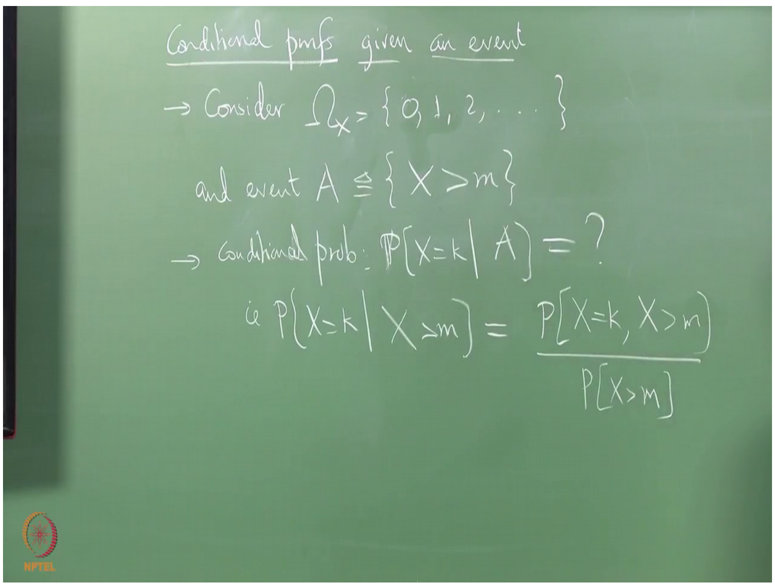
Lecture - 11
Part 1

(Refer Slide Time: 00:14)



So, today we are going to do the, look at this conditional pmfs, given some event.

(Refer Slide Time: 00:19)



So, I am going to restrict this to the case that omega X to start off with this a bunch of is a set of non negative integers and A is an event of the type X is greater than m. This could be 0 also, does not matter. Let me write it like this, does not matter. Whatever we are saying here can easily be extended to arbitrary events of the type als and xls and b and so on. I am just doing it for X greater than m because I need to take some concrete example and I do not want to just do it in very general terms.

So, I have this event A, X greater than m and I am interested in we are interested in finding the probability how the probabilities of X vary, if given the additional information that X is greater than m, that is, this event A has in fact, occurred. So, when this event with the information that A has occurred and how does the pmf of X change. So, we have the conditional pmf, let me write the conditional probability P of let us say X equal to k given, this is what we want. So, if I take any A it is clear this is not you cannot go beyond this, but for this particular A and for example, the geometrical pmf we can derive a very interesting result. So, it is what I am going to look at now.

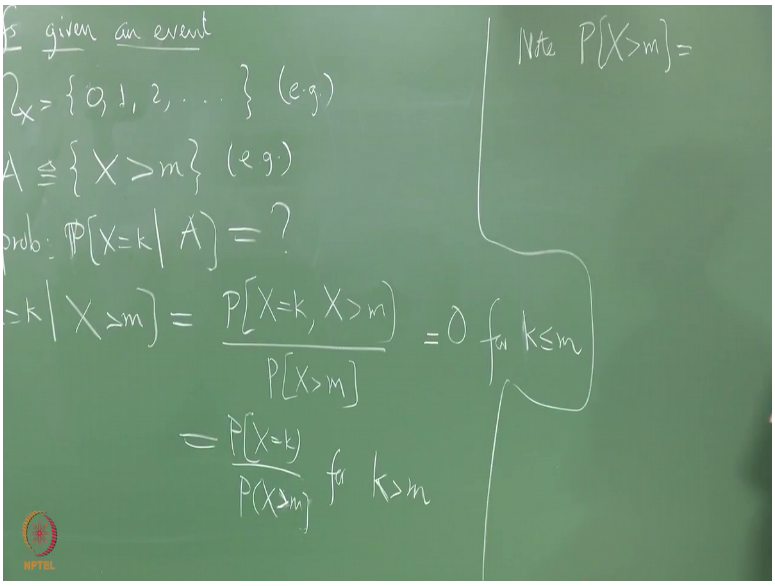
So, basically, we have i e. this conditional probability by the rules by the definition of conditional probability is what? So, is clearly, exactly, the same as what we had earlier. The numerator what can you say about the numerator? What will be the value of the numerator if that number k is smaller than or even equal to m?

Student: (Refer Time: 03:44).

It will be 0; obviously.

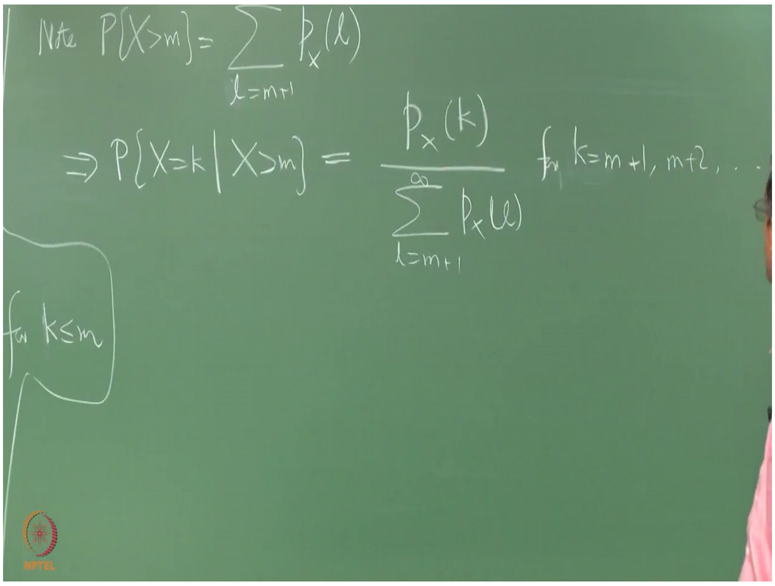
If you are given the fact that X is greater than m, X cannot be smaller than m or even equal to m. So, clearly this is going to be 0.

(Refer Slide Time: 03:58)



For, let me say less than equal to m and this is just an example, this is not right; this is also an example. So, what about the case that k is greater than m, in that case it turns out that this is a subset of this, X equal to k is a subset of X greater than m, if you take k equal to m plus 1, m plus 2 whatever, you are only taking one value of k at a time. X equal to k is you are considering only one possible value. So, that is, obviously, the subset of the event A. The event X equal to k is a subset of the event X greater than m. So, this will be what? This should be simply probability of X equal to k divided by the probability that X is greater than m, for k greater than m. And these we want to write this conditional probability in terms of the pmf of X, that is our goal. So, when I say omega X, I also assume that the pmf is given.

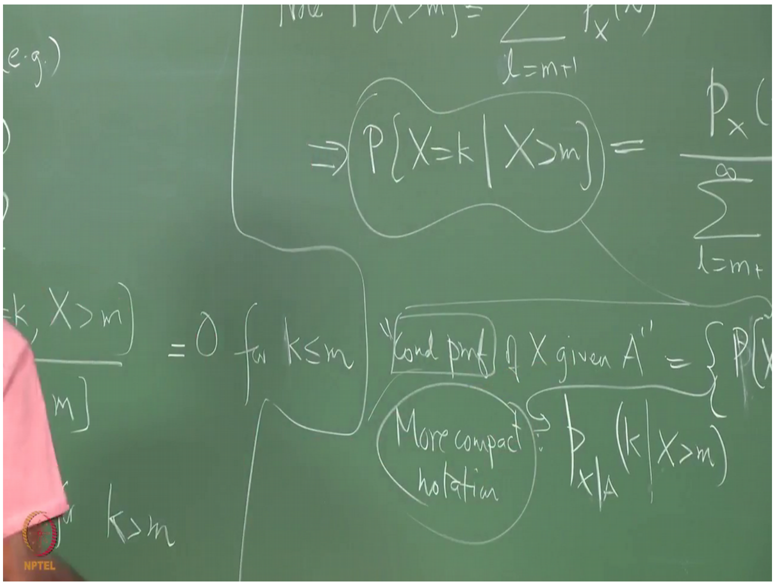
(Refer Slide Time: 05:46)



So what happens here now or note that X, what is the probability that X is greater than m. This is simply the denominator there. This is greater than, this denominator term is what? This is going to be in terms of the pmf, the summation from. So, in general you can go up to infinity of p X of, well, let me not use k here. I am going to use some other dummy variables of summation, I should use say l or whatever, could use I, anything. And the numerator is just the unconditional pmf or whatever you started off with.

So, this probability of X equal to k, given X is greater than m basically p X of k divided by this and it is 0 for k smaller than or up to m. In the geometric case what happens? No, before that let me introduce some notation here.

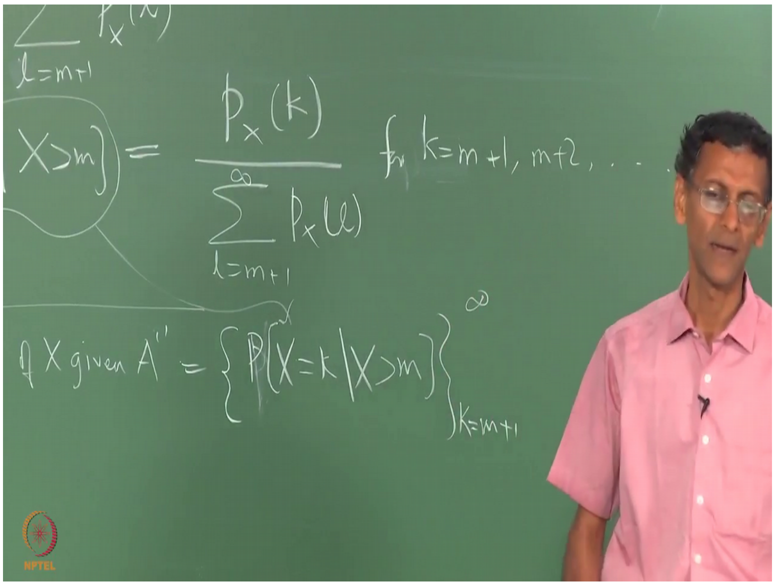
(Refer Slide Time: 07:34)



The notation we need to use is, if you look at this, so, this is a function of the index k or they vary the parameter k. So, this if you consider the collection of all these conditional probabilities, it is called the conditional pmf just like these guys are the basic unconditional pmf it is charged with the collection of all these probabilities from l going from whatever 0 to infinity or 1 to infinity as the case may be. So, here these probabilities are conditional probabilities and the whole set of the m is called the conditional pmf.

So, the conditional pmf of X given A, basically the collection of now we need to write some note let me write it like this p of X equal to I will rewrite this, just take this exact thing in put it here.

(Refer Slide Time: 08:43)



So, k oh whatever k equal to, this k is say m plus 1 to infinity for this it has to be only that. Since this is X is greater than m of said X is omega X is set of integers or some non negative integers this has to be from m plus 1 to infinity.

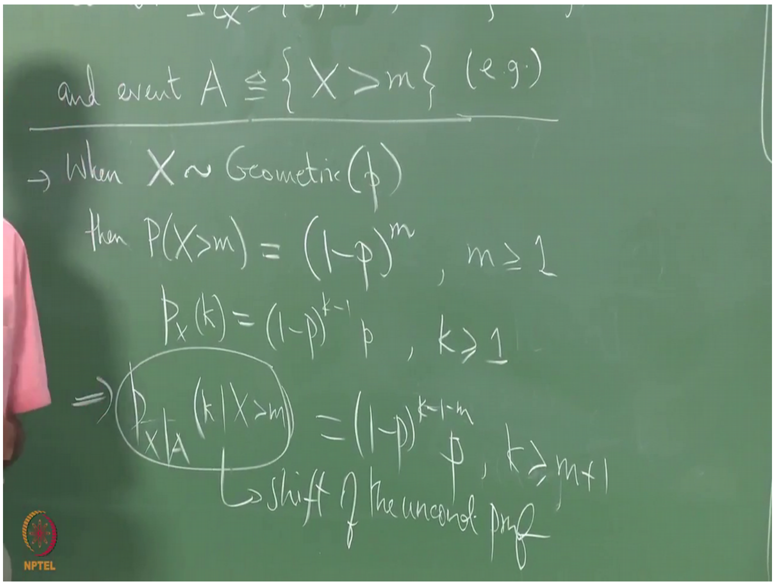
The understanding of course is that, it is this you can also extend it; you know put it from 0 to infinity with your understanding, it is 0 for k less than m, anyway. So, but we need some more compact notation. The more compact notation that we will use in this course is for this guy you write it as p of X given small p of X given A, at this thing is written as a subscript. X vertical bar A then you put the argument k just like you had this l out here and then you can optionally you can indicate the event which is useful to do. Because you do not know what the event A is unless you go and look at the context.

But you since there is space here typically it is always you know this is indicate in more often than not the event A is in fact, indicated this way with a vertical bar after the k and so, please use only vertical bars here no slanting bars, no up and down kind of things all in the same line. This is a subscript line and this is the argument line. So, what is the notation here? Let me write here more compact notation is this. Is this clear? X given A is a subscript. So, for people that want to try your hand at lay taking the subscripts and so on, it is very easy to do. All you have to do is put this whole thing is as one subscript string and then this is the argument string.

So, this is true for the general case, for the geometry case something very interesting happens. So, we can proceed to that, so I think this is fairly clear. There is nothing, there is no deep stuff happening here, just conditional probabilities, just collect it together and called the conditional pmf. That is what we have done.

For the geometric case, for that is, assume these will not change; of course, the 0 may or may not be there, that is not a big issue. What happens to the geometric case?

(Refer Slide Time: 11:48)



Now, this is where I am going to introduce some notation, extra I mean notation in the sense I need something to indicate the pmf of a random variable and to meet that requirement it is written like this. You write the name of the pmf in some form we can expand it in full geometric p and you use this tilde notation to indicate that this random variable X has this pmf or later on pdf whatever and this geometric I am writing out in full because it seems to be no universally used or universally accepted shortening of geometric or Poisson or binomial. So, you might as well write out the full word. Only negative binomial is a little on the long side, but if we ever need to do this when I need a binomial we can think about how to let us not worry about that situation right now.

So, in this case well and this p is a parameter of the geometric pmf. So, here, this tilde notation is going to come up a lot from now on. It is a very compact way of telling what is the distribution of X, of indicating the distribution of X. So, now, this when X is geometric P then what is the probability that X is greater than this we already did. This is 1 minus p to the power, what? This means that no success in the first I mean X starts only from 1, does not start from 0, does not matter. You can always say that 0 has 0 probability, it is not an issue.

But what is this? For any m, we have a nice closed form expression know, what is it? 1 minus p, come on we did this not last week or earlier part of this week, last week I guess. 1 minus p to the power of m; why you are giving me such blank looks? These are things you should immediately say and it is valid for all even for maybe I should say if m greater than 2 or is it m greater than 1? Can I say that it is a valid you know, in fact, it is true for m equal to 1 also, because X greater than 1 is perfectly valid; that means, that is the first toss or the first trial is failure, at m equal to 1. So, this is perfectly valid for m equal to 1, as well there is nothing wrong. It is valid for all values of m has given here.

Now, go back to that thing with p what is p X of k? This we already know as 1 minus p to the power k minus 1 times p, for again k greater than 1. So, what is the ratio p of X given A of k given X greater than m is what, please do it. What do you get? Tell me.

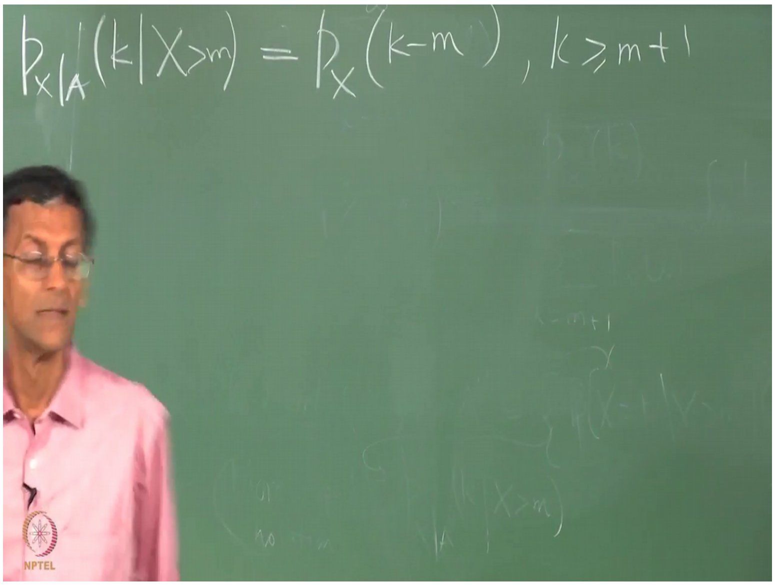
Student: 1 minus p to the power of k minus 1.

1 minus p to the k minus 1, minus m times p, for what values of k? k has to be greater than m or k equal to greater than equal to m plus 1, whichever way you want to say.

How do you compare this and this? This is the unconditional pmf, this is the conditional pmf for this particular event. What you can say about these 2? This is just a shifted version of this. This sequence is a shifted version of the sequence. You just moving it to the right by m samples. In other words the whole pmf is pretty much preserved intact, except that is just shifted. This is a shifted version; it is a whole thing is the shift of the unconditional pmf. This is the so called memory less property of the geometric distribution.

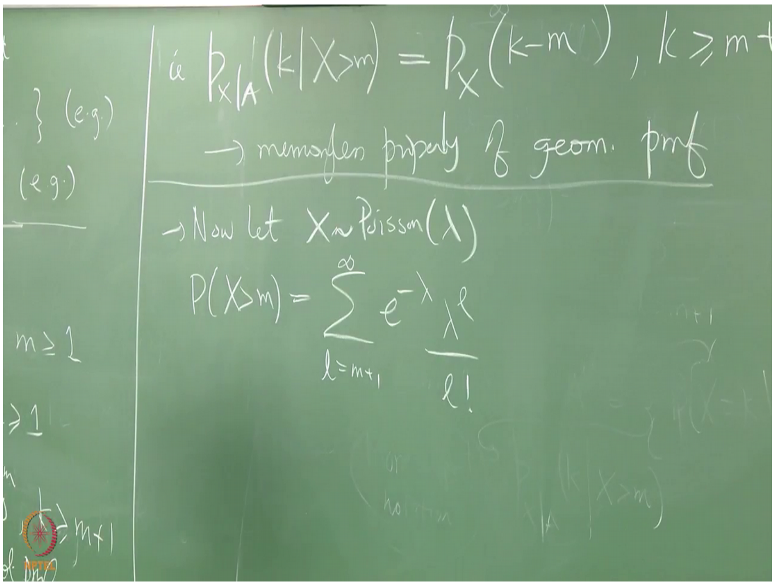
In otherwise, I can write it without using English, just write it in mathematical terms I get.

(Refer Slide Time: 17:15)



Now, I want to write p X, what would I write here? I would just write what would I put here, k minus m.

(Refer Slide Time: 17:42)



For, k greater than equal to m plus 1. So, that in one line encapsulates the memory less property of the geometric pmf. Why you call it memory less property? Because, its the fact that you didnt get any success in the first m trials, does not have any bearing on how the successes or failures will evolve after that. You are all back to the starting point pretty much. What happens in future is, has no connection to what happened in the past.

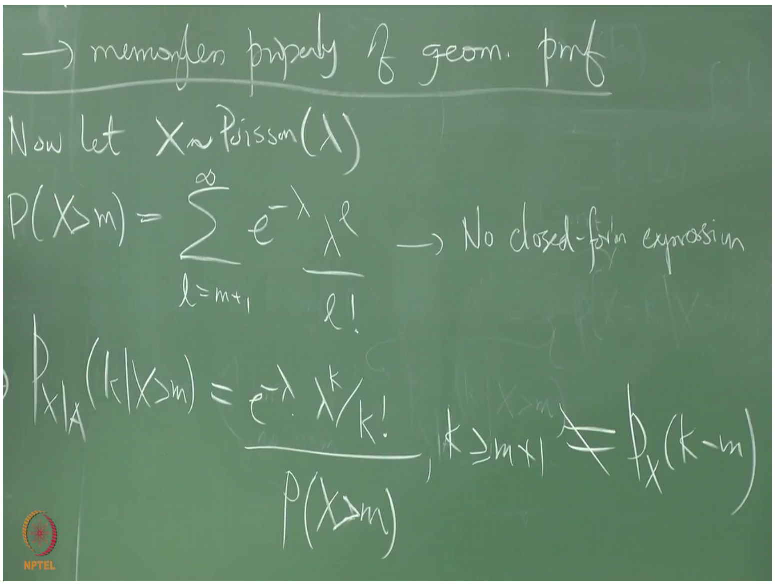
So, this is clear. So, just remember this one line tells you everything about this particular feature of the geometric pmf and it is not shared by other discrete pmf's. In particular Poisson, as an example we will look at that also. Supposing I use X tilde, X is Poisson with parameter lambda.

Now, it turns out that I do not have a closed form expression for p X greater than m in this case, but does not matter. I do not need a closed form now. One of the goals in this course especially for IIT students, not necessarily the online version, is that you people should use mathematical tools to evaluate these things. If necessary, if called upon to do so, you should be able to write a small program to evaluate given numerical values, although closed form expressions is not available here.

What is it closed form expression I am looking at, X p, X greater than m is what? Some summation once again it is the same whole thing; now, instead of writing p X of l, I will just put the exact expression this. What is p X of l in this case? Its e to the power of minus lambda power l divided by l factorial, l going from what? m plus 1 to infinity. All I know is that, this number is smaller than 1, its more than 0, smaller than 1. That is all I know.

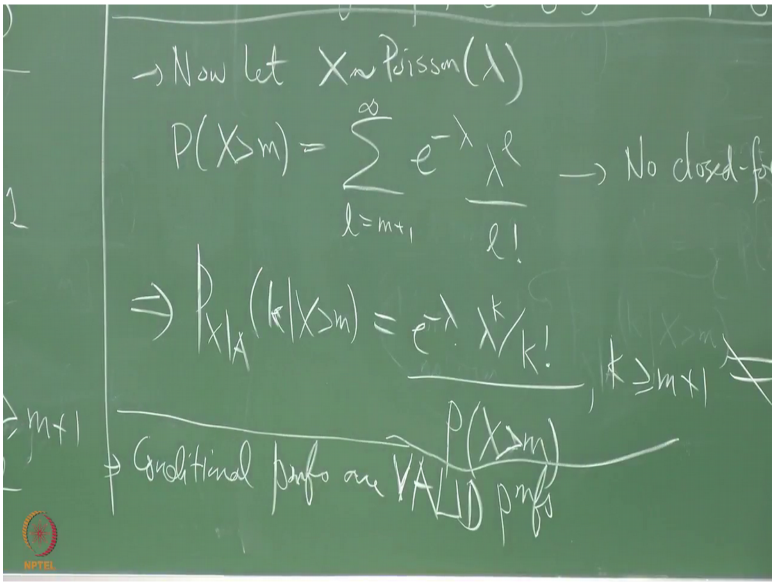
But I do not know what it is. So, this has no closed form expression.

(Refer Slide Time: 20:24)



Unlike the geometric case there is no closed form expression here, but does it mean that, one of the things that you have to remember, just because something does not have a closed form an integral or a sum does not mean that it does not exist. It very much exists. What is the meaning of any existence of a sum, means it converges; that is all or an integral. There is nothing more to it. Existence is just being finite which is and this is definitely finite, has been between 0 and 1.

(Refer Slide Time: 21:10)



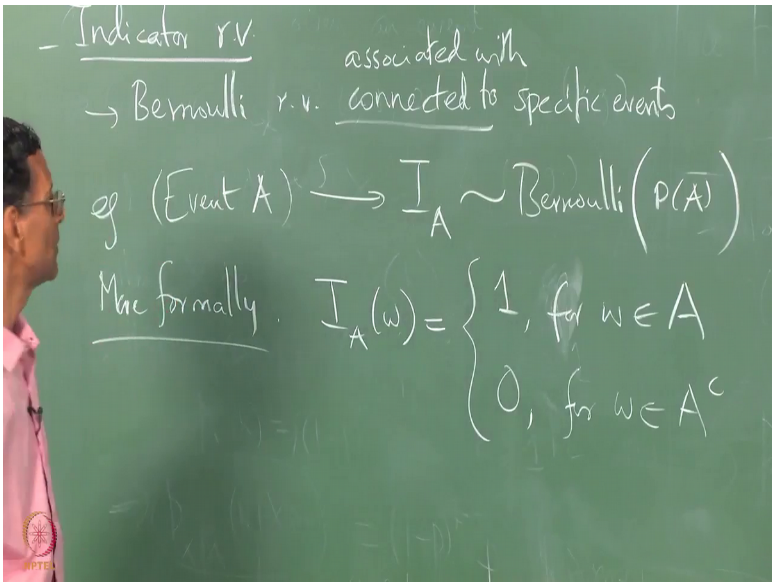
So, what is this? So, I can write it out. No I can write out this e power minus lambda, lambda power k divided by k factorial. This is a numerator isn't it and then in denominator there is all of this. Let me not copy this whole thing, I will just write p X greater than m. So, basically what will happen is the only simplification that is possible is e power minus lambda will cancel. That is the normalizing term otherwise nothing you would not get anything else. Any other possess simplification, because you should always put that. So, this is not equal to p X of k minus m by any stretch of the imagination.

So, do not think that this so just because it happened and in fact, it is turns out only the geometric pmf has that property, among the discrete distributions. So, the Poisson does not have that property, if you cut off the head of the Poisson distribution you cannot recover the rest of it, but if you cut off the head of the geometric distribution you can recover the rest of it. That is what that is saying. So, the geometric distribution the head of it is like the lizards tail; you cut it off, it re grows. So, like that all right this says that the head of the geometric distribution is there is nothing special about it you can recover whatever it is from the remaining bits of the sequence, Poisson that is not possible. Here, again you do the same thing it completely changes.

So, to summarize this whole thing I am not going to take any more examples because having this one example is good enough. The conditional pmf is as valid a pmf as any other pmf. It is a valid. If you add up all these probabilities from m plus 1 to infinity, you will get 1 again. Why will you get 1 again because, numerator and denominator will both be p X greater than m. So, it has to cancel, give you 1.

So, the bottom line, which I am going to which is really well and truly a bottom line here and that is coming at the bottom of the pair of the board right is that conditional pmf are valid pmf, but this is valid in big uppercase letters, but the domain of the pmf or the support of the pmf will change. It is not going to be the original 0 to infinity. It shifts. It changes. Supposing, I use a different conditioning event say the event A is that and I am going to condition on the event that X is between 1 and 10, then the conditional pmf will also exist only for 1 to 10. It would not exist outside that, because you have been told that, X is taking a value between 1 and 10, only thing is this event A here the conditioning event always has to have positive probability. You cannot condition on impossible events, for 0 probability. That is the only thing I have to watch out for, otherwise any kind of any other kind of conditioning is possible.

(Refer Slide Time: 25:11)



Indicator random variables are also very useful in many situations. So, basically they are Bernoulli random variables which are connected to events, that is all they are. To specific events, so, this is these allow you to convert an event into a random variable, essentially, whereas instead of saying the event A as occurred or not, you say well the indicator is 1 or 0. You can say they are associated with the specific events or connected to the specific. So, I can have another either way of understanding is ok; eg: supposing I consider an event A, on some probability space it does not matter, A can be any event in the any probability space. This will give me the indicator I A, usually you always use the letter capital I, uppercase I to indicate the indicator and the subscript A to show the event that it indicate. So, what do you want of this I A, you want probability of I A equal to 1. So, I A, you want what do you wanted to be? You wanted to be Bernoulli with what parameter? What is the success parameter of this Bernoulli?

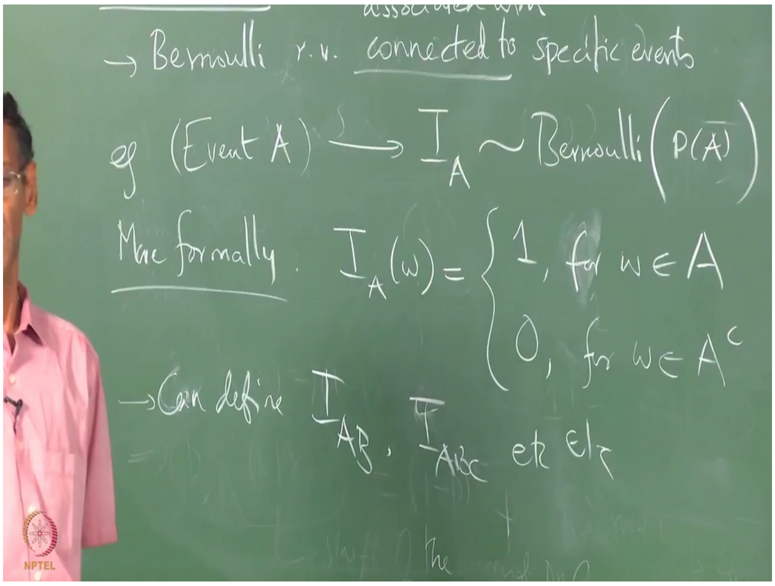
Student: P of A.

P of A. So, probability is that I A is 1, that is exactly P of A and probability of I A is 0 is 1 minus P A. More formally in terms of the sample space omega and capital omega and sample points small omega you say this that I A of omega, now, we are going back to the definition of a random variable as functions of points in the sample space when. So, I A of omega is just 1 or 0. So, for each point small omega and capital omega, this random variable I A, either it takes you to 1 or to 0. No other number. So, when does it when do you give the value 1, you give the value 1 if omega or let me say for omega in the 1 m in the set A, otherwise you give it the value 0 and there is no other possibility either omega is in A or in A compliment and A and A compliment are partition. So, there is no other they cover the space. So, you do not have some third possibility that is neither in A nor in A omega nor in A compliment and this is a complete definition. So, if you look at this clearly probability of I A equal 1 will be P of A itself.

So, these indicators are a bridge between events and you can always introduce a random variable to in place of an event essentially and they do have we will look at them definitely in more detail in the course. I thought this is a good time to introduce them because we are looking at all in many possible situations of discrete random variables and this is surely right place to talk about them.

So, you can do it for any you can define indicators for any event you want.

(Refer Slide Time: 29:19)



Can define I AB, anything, I ABC etcetera that is the power of the indicator. They can be defined for just about any event you wish to define them for and then there is a lot of manipulation you can do, because we will come to that later. Just note that they enable many calculations to be vastly simplified. That I will say for a later time, because we have not yet talked about manipulating random variables in a formal way. So, after we do that, we can look at how to use this indicator. For now, we just defining them.