**Control Engineering**
Dr. Ramakrishna Pasumarthy
Department of Electrical Engineering
Indian Institute of Technology, Madras

Module - 02
Tutorial
Lecture - 09
Solving Problems on
Laplace Transforms and Transfer Functions

During this outsource we will just do some problems related to Laplace transforms inverse Laplace transforms and also some simple examples on calculating the transfer functions.

(Refer Slide Time: 00:34)
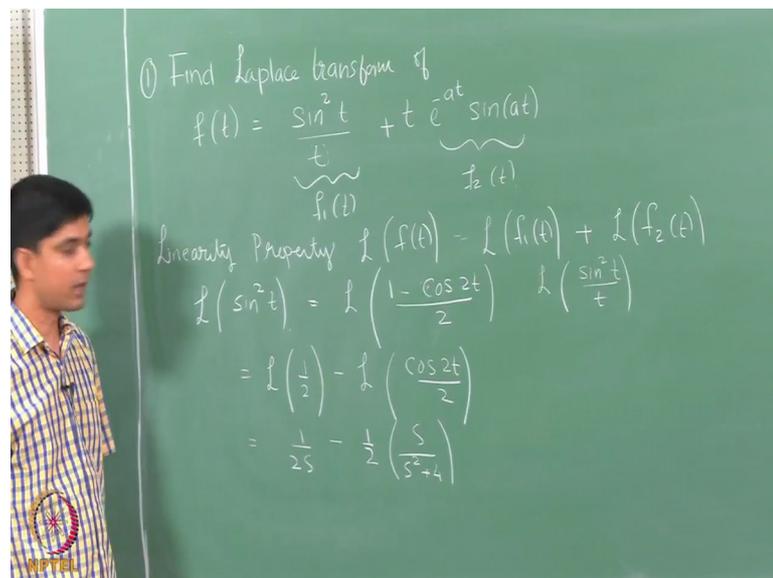
So, let us start by finding Laplace transform of signal which looks like this say in time domain is sin square t over t, plus t e power minus a t sin of a t, and during the computation of this we will use lots of properties of Laplace transforms which we learned earlier. So, this is a combination of 2 signals f 1 of t and f 2 of t right and using the linearity property on the Laplace transform of f of t can be individually computed as the Laplace transform of f of t plus the Laplace transform of; so f 1 of t and f 2 of t.

So, let us start with this thing first finding of Laplace transform of sin square t. So, do not worry where the t has disappeared I will tell you a magic of how to take care of that t. So, this would be sin square t can be equally be written as 1 minus cos of 2 t over 2 here

I can further use the linearity property this would be Laplace transform of 1 over 2 minus the Laplace transform of cos 2 t over 2 now I just recollect the formulas. So, this would be 1 over 2 s minus 1 over 2 cos 2 t would be s over s square plus 4 right my original signal is of the form sin square t by t.

(Refer Slide Time: 03:17)
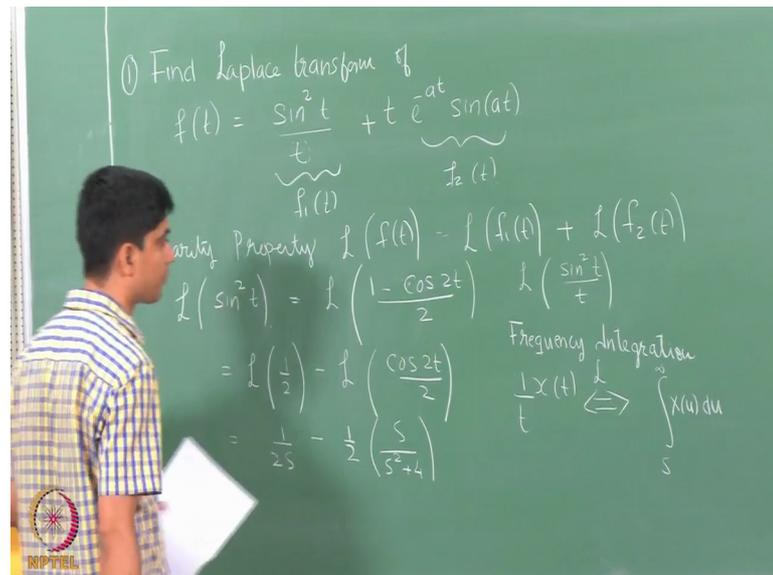
I need to found find the Laplace transform of sin square t over t. So, so far I found what is the Laplace transform of sin square t? Is there any property which helps me in finding the Laplace transform when I just divide a signal by t? So, just recollect the frequency integration property.

So, what is the frequency integration property tell me is had if I have a signal in time domain x of t, 1 over t then this Laplace would look something like this s to infinity X of u d of u.
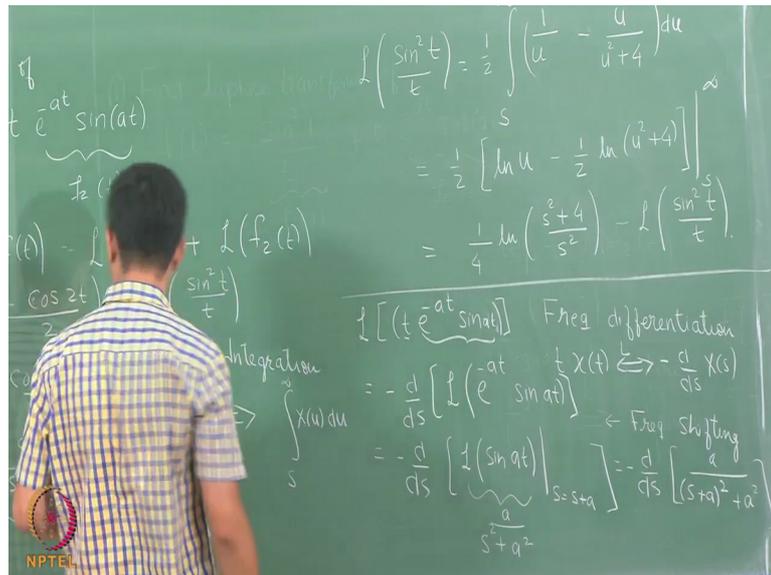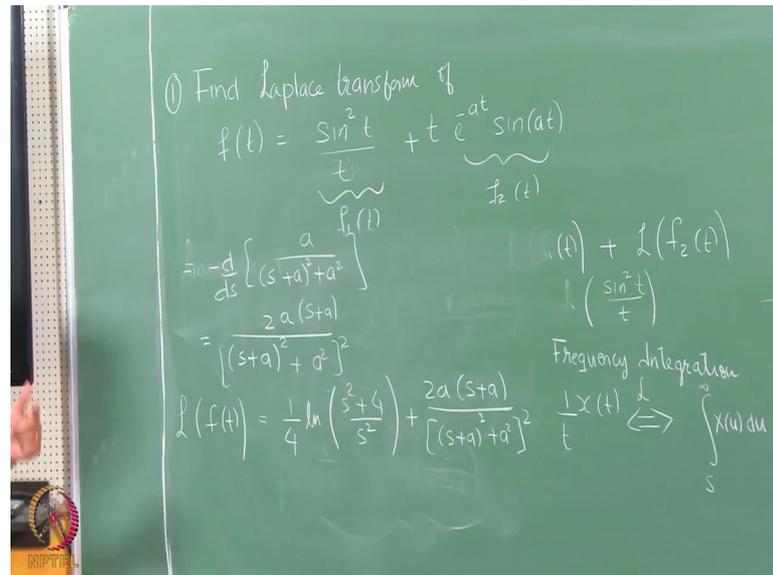
Therefore the Laplace transform of sin square t over t given that the Laplace transform of sin square t is this, I just used the frequency integration formula this would become 1 over 2 integral of s to infinity and using this formula should be 1 over u it is a minus u over u square plus 4 d u.

So, I just. So, what would this p half, half of u minus half log of u square plus 4 and everything going from s to infinity. So, this simplified would looks something like this 1 over 4, s square plus 4 over s square. So, this is the Laplace transform of sin square t over t.

So next we need to compute the Laplace transform of the second signal this guy Laplace transform of t e power minus a t sin of t. Now I use equivalently the frequency differentiation formula sorry sin of a t. Now I just use the frequency differentiation formula here, what does the formula say that t x t the Laplace of this would be minus d over d s x of s right here I have t multiplied by this signal. So, this would be minus d over d s the Laplace transform of this guy e power minus a t, sin of a t. Now next I use the frequency shifting formula right, but time wave e power minus a t. So, this would be minus d over d s the Laplace transform of sin of a t this. So, this would be now why no; the Laplace transform of sin of a t, d over d s over. So, Laplace transform of sin of a t would be a over s square plus a square, frequency shifting would give me s replaced by s plus a plus a square.

(Refer Slide Time: 08:21)

And that d over d s of a over sorry s plus a whole square plus a square would be twice a over s plus a square plus a square of whole square right and therefore, the Laplace transform of our signal f of t is a combination of sin square t over t plus t e, e power minus a t sin of a t would be is this one, the first I the this 1 over here 1 over 4 log of s square plus 4 a square plus. So, this is be s plus a here if I differentiate the denominator I have 2 s plus a, thus twice a, s plus a over s plus a whole square plus s. So, this is illustrates that you are using the frequency integration, the frequency differentiation, frequency shifting and of course, the linearity property.

So, next if I have a signal let us see f, f of t looks something like this right it is 1, for t between 0 and 2 its minus 1 for t between 0 and 4 together with the property that f of t plus 4 is f of t, it is like a periodic signal with the period of 4.

So, how can this plot the signal. So, this is f has a value of 1 from t going from 0 to 2. So, its looks something this right and at 2 it takes a value of minus 1. So, this until it reaches 4 and keeps on doing this. So, this I have 6 here and 8 and so on. Now this is a periodic function. So, what is the Laplace transform for a periodic function L of a given function f of t is 1 over 1 minus e power minus p s integral 0 is integrated the time period p, e power minus s t f of t d t right this guy p is the time period, which in our case is 4.

So, this would be simply now translates to 1 over 1 minus e power minus 4 s and this signal from 0 to 4 has 2 components from 0 to 2 this component, and from 2 to 4 this component. So, I will integrate each individually 0 to 2 e power minus s t, d t plus 2 to 4 it is a minus 1 there it will minus e power minus s t, d t. Then I just keep all the computations this is a very straight forward integration which you learn very earlier in your in your high school. So, this would simply be 1 minus e power minus 2 s over s 1 plus e power minus 2 s, this quite straight forward right once you just plug in this formula these are easy to compute.

So, next we look at solving non homogeneous differential equations or also called 4 differential equations. So, just looks like this, say y dot or say y prime plus y is some sin of t, with the initial condition y of 0 1. Now I write I take the Laplace transform on both sides. So, I have s times Y of s with this initial condition Y of 0 plus Y of s write inside I have sin of t the Laplace transform of sin of t is 1 over s plus 1. So, I substitute s plus 1 times Y of s, Y of 0 is 1.

So, is 1 over s square plus 1 plus 1 now just take this guy over here, has should give me Y of s is s square plus 2 over s plus 1, s square plus 1 right just we arranging terms. So, I can rewrite this as 3 over 2, 1 over s plus 1 plus 1 over 2 square plus 1 minus 1 over 2 as s square plus 1 and this just by applying the method of partial fractions of solving this kind of equations. Now y of t would simply be the inverse Laplace of Y of s. So, the first inverse of s plus 1 would be e power minus t, plus why this inverse is a sinusoidal signal sin of t and the last one is s over s square is 1 is cos of t as simple as. So, just this step here is a crucial one of identifying this numbers here and to write this in terms of fractions.

(Refer Slide Time: 16:21)
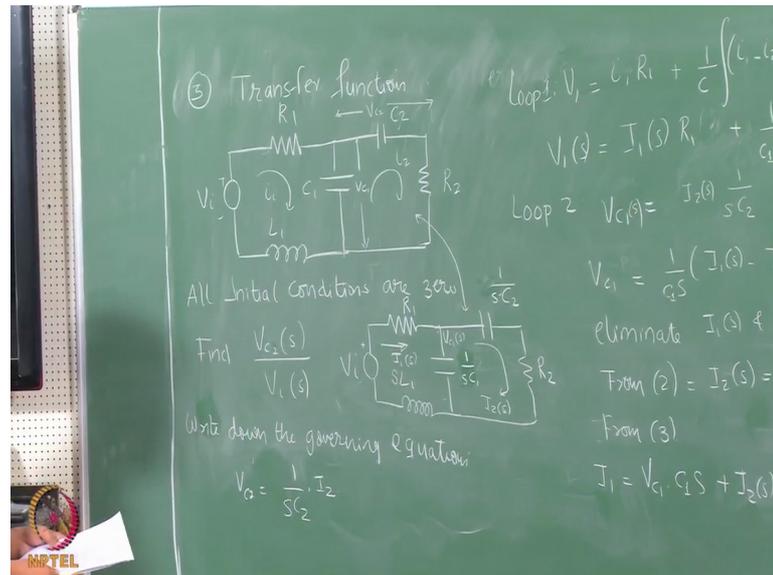
So next you will look at deriving transfer functions from given simple circuit models. So, let us say I have circuit with the input voltage V i, r here. So, I will call this as R 1 the capacitor C 1 L 1 this would be C 2 and this is R 2. So, I need to find what is. So, all initial conditions are 0 this is the assumptions I will make right the first is find the transfer function. So, this is capacitor C 1 let me call this voltage as V c 1, find the transfer function V c 1 of S over V I of S.

So, let us step by step follow what we had learned earlier how to determine my transfer function. So, first is well, write down the equations right like or the differential equations which model my system. So, let me call this current as I 1 as I would do it in any circuit analysis and I 2.
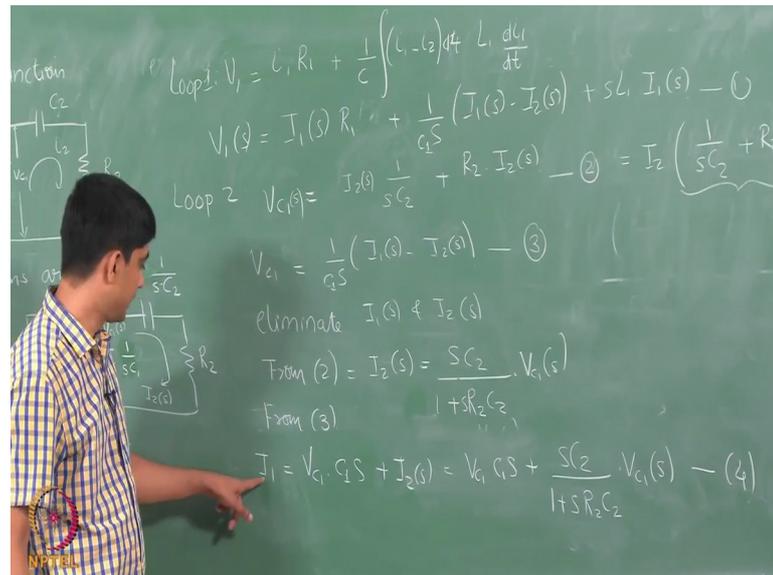
So, the first equation for loop 1 would look like this rights that V 1 is I 1 R 1 plus 1 over C, I have I 1 minus I 2 plus I have this guy L 1 d I 1 by d t.

Student: (Refer Time: 19:06).

Sorry 1 over C integration. So, there is also another way of doing this, now since we are used to writing things in linear equations I will just write this entire circuit in Laplace representation right. So, I will just try draw this here, this a capacitor here a V i, this guy the inductance here capacitance R 2, this is C 2 this is my C 1 L 1 and R 1. So, in the laplacian domain I could write this as S L 1 capacitor I could rewrite as 1 over s times C 1, this guy would equivalently become 1 over s times C 2 right and this would be I 1 of s the current in this loop, the current in this loop would be I 2 of s. So, let us go back here and if I were to write these equations in the Laplace domain.

So, that will be I 1 of s, R 1 s plus is this R 1 right there will be no s c. So, it will be same this would be I 1 of s, minus I 2 of s plus s L 1 I 1 of s right. So, I can even write it directly from here right. So, I have V 1 is I 1 R 1, I can write I do not even need to bother about writing the integral equation about I have 1 over s c 1 I can just look upon this now as a linear element 1 over s c 1 the current flowing through this is I 1 minus I 2 plus the current here I 1 times s L 1 right. So, I can just write down the equations without even going through this step, if I can transform from this circuit to something like this right and for the loop 2

So, loop 2 I have V c 1 equal to. So, I just use this circuit now this is my V c 1 here V c 1 of s then we do the convention like this V c 1 is current through here is I 2 times s C 2 plus R 2 all this will be minus R 2 I 2 of s. Now I have equation 1, I have equation 2. So, now, what do I need to find? I need to find out V c 1 by V 1. So, just take this one right the voltage across this capacitor V c 1 is also well what is this write this voltage if I look at this equation here this simply this one write here this one or this one; this is one over c s, I 1 of s minus I 2 of s we call this as equation number 3. So, what do I need to do right? So, I need to do to eliminate I have three equations this, this, this, and this and it eliminate I 1 and I 2 right. So, from 2 what I have that I 2 of s can be written as sorry just write this equivalently as I 2 times 1 over s C 2 plus r 2.

So, I 2 of s would be V c 1 divided by this entire thing. So, this would be s C 2 over 1 plus R 2 s C 2 times this one, V c 1 of s. So, all this will be s. So, now, I eliminated this from here. So, I know I 2 in terms of V c 1. So, we have eliminated I 2 from equation 2 and from equation three now what we have is I 1 is V c 1 times C 1 of S, this goes here plus I 2 of S. So, this is V c 1 times C 1 of S plus what is I 2 of s this entire thing it is c 2 1 plus S R 2 C 2 times V c 1 of s. So, I will call this 4. So, my equation 4 I eliminated I 1. So, I 1 could now be written in terms of V c 1, I 2 can also can be written in terms of V c 1 from this expression here and. So, I can mix or this guys here to get my final expression in the following form.

(Refer Slide Time: 25:44)
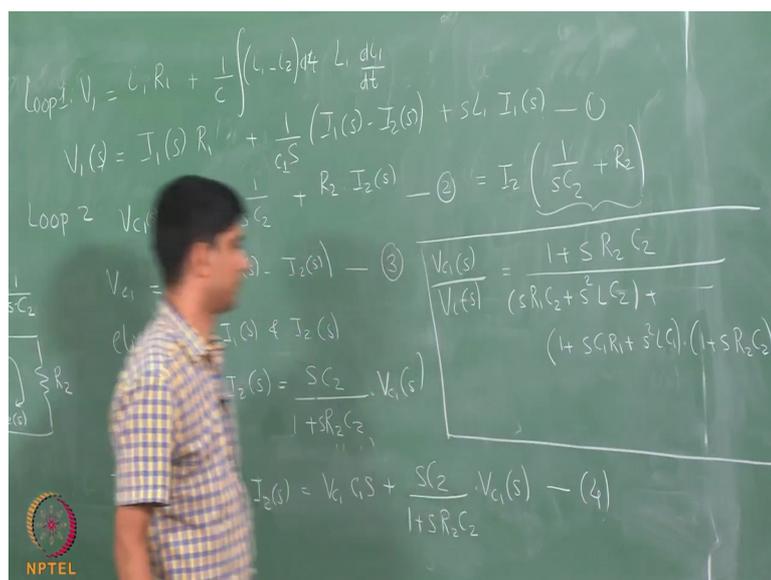
V c 1 s over V I of s is will be computations or they should be straight forward to do and s R 2 C 2 over s, R 1 C 2 plus S square L C 2 plus 1 plus S C 1, R 1 s square s L C 1 this multiplied by 1 plus s R 2 C 2. Yes all this as in the denominator this multiply here right. So, what I have done right again. So, first is write down the individual loop equations and try to eliminate each of the variables, to get an expression which will give us the ratio of V c 1 by V 1. So, similarly I could also compute the transfer function from V 1 or V I to V c 2 right nothing change; so all this.

So, instead of this is my V c 2 the voltage across my capacitor C 2, I can equivalently then find my transfer function as this 1 right. So, equivalently what is V c 2. V c 2 is 1 over s C 2 times I 2. So, you could do the many thing as an exercise to find out what is V c 2 by V i given that V c 2 can be written as an expression like this right it is just the same steps that we will follow.

(Refer Slide Time: 27:50)
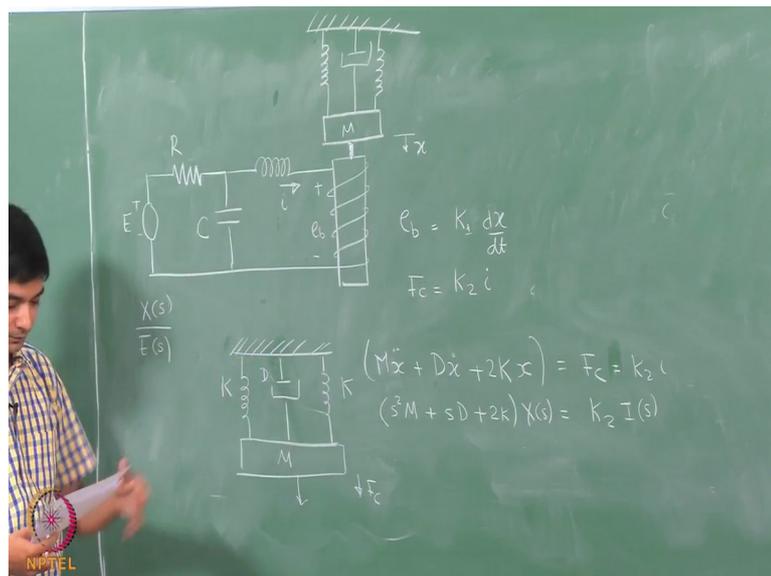
So the last problem for today; so again to in the last problem for today we are required to find the transfer function X of s over E of s. So, what is this electro mechanical system do right. So, I have a e which generates a current here, this current results in a back e m a 4 here and these also force which the system exerts from this one right. So, what are the constitutive relations here right? So, the back e m a e b is K d x by d t and the force is proportional to the current K 2 times i right the force which is experienced by this mechanical system over here. So, we are required to find out what is the transfer function

starting from this input e resulting in an output here. So, these are 2 subsystems right. So, one is this electrical component, and one is the mechanical component. So, let us analyse each of them individually.

So, the mechanical component for M here which possibly this experiences the force in this direction like this. So, the K here there is a K here and there is a D here right. So, well I can directly write down the equations for this because I already know this right; so M x double dot plus D x dot plus K times x for the spring. So, you have 2 spring that will be 2 K x, D is the is the dumping and k r the spring constants this guy would be equal to the force and what is the force the coupling between this mechanical system and this electrical systems comes this via current I right; so this F as i. So, I can write down as s square M plus s times D plus twice k X of s is sorry I have made a mistake here this is K 2 times i. So, this would be K 2 times I of s that is why that is this part here over here.

(Refer Slide Time: 32:09)
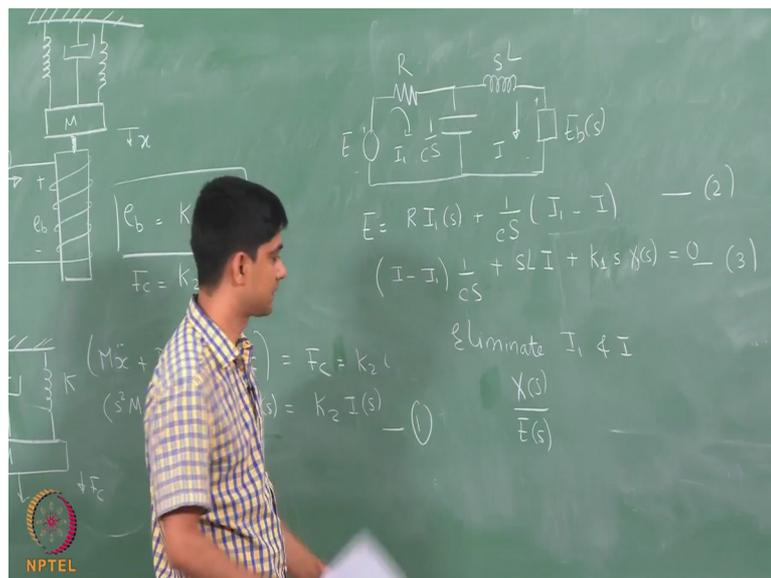
Now, the electrical part right I can write down this as equivalently in the s domain directly. So, I have a 1 over C S R here, S times L and I have a E and I have some I just draw block here called the e b or the back e m f and this is the current I which I am interested it, and let me call this current over here as I 1 right. So, the first equation would say E is R I 1 of s plus 1 over c S I 1 minus I, and the second expression would be I minus I 1, 1 over c S plus S L times I again plus E b is 0. So, what is this e b? E b is K 1

d x by d t. So, I could simply write this as K1, s, x of s is 0. So, I have this equation relating X and I, I have these 2 equations relating the currents and the input voltage.

So, I can just make use of these three equations to eliminate I 1 and I and ultimately find the transfer function X of s over E of s right. And I will leave the steps for you, but they should be very straightforward right. So, take this equation you can eliminate one of the current variables here, one of the other current variables from here and here you can directly eliminate possibly write I directly as something on X 2 substitute that I over here you could eliminate I 1 you already know I in terms of X, do all the computation and this get the equivalent expression I will just use the computations again for you as an exercise.

So, this was some simple examples in you know of course, of solving Laplace transforms solving ordinary differential equations, and starting from system configuration how to arrive at the equivalent transfer function again via all the you know everything in the Laplacian domain. So, for future purposes what would be easier is to equivalently draw this circuits in this form right then you just have to treat them just as here impedances and then right the equations directly as linear equations, you might have done this in some of your circuits course circuits on networks course. So, just a little re correct of that would help you write on the equations in a much compact and much easier way.

Thank you.