Control Engineering
Dr. Ramakrishna Pasumarthy
Department of Electrical Engineering
Indian Institute of Technology, Madras

Module - 02
Lecture - 02
Inverse Laplace Transforms

In this lecture we will continue with what the properties which we have learnt with the Laplace transform. Now we will do the inverse Laplace transform say if I can go from t to s, can I come back and is a transformation unique; what are the properties that are preserved in this transformation going from the s domain to back to the t domain. So, as I mentioned earlier what we will deal with essentially are causes sickness and therefore, the integration we compute is essentially from 0 minus or 0 to infinity, in that case the transformation or the inverse Laplace transform tends to be unique.

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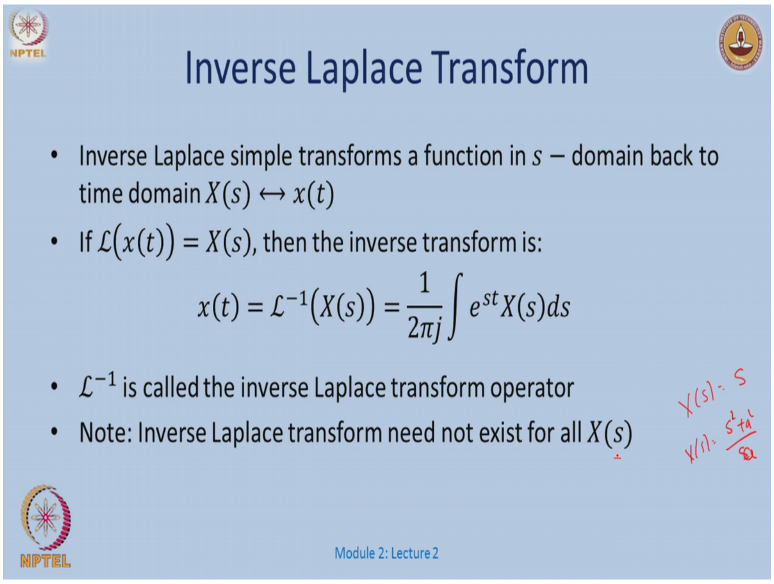

So, I said earlier the inverse Laplace transform helps me to go from the s domain back to the time domain, and there is a reason why we are do actually doing this which will be obvious few slides from now. So, how do I compute this if given X of s the inverse Laplace transform is computed by this formula L inverse of X of s is 1 over 2 pi j integral e power st x of s ds; and this l inverse is called the inverse Laplace operator. Now, one could be tempted to believe that I write any expression in s and they will exists a inverse Laplace transform while that is not true. Inverse Laplace transform for example, does not exist when x of s equal to s or even when x of s is s square plus a square by s for example, or may be this s may be replaced by a for this is has special kind of transform this we will deal with later.

But just to such as that anything written as a function of s does not necessarily have an equivalent time domain representation.

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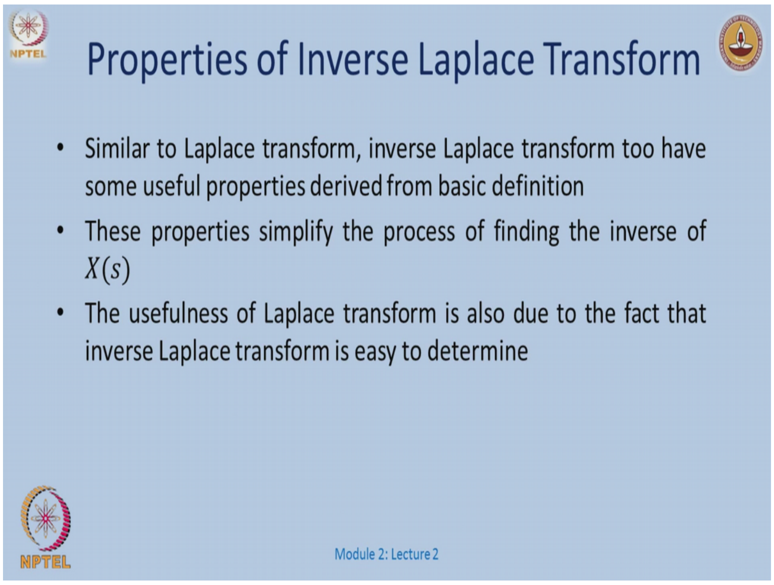

So, similar to the Laplace transforms there are also some good properties of the inverse Laplace transforms right, and then these properties well this simplify the process of finding inverse Laplace transforms for a complicated looking expressions right, and this is the one of the usefulness of Laplace transform is also due to the fact that the inverse is most of the times easy to determine and therefore, we can start from the time domain go to the s domain do the computations and come back again to the time domain.

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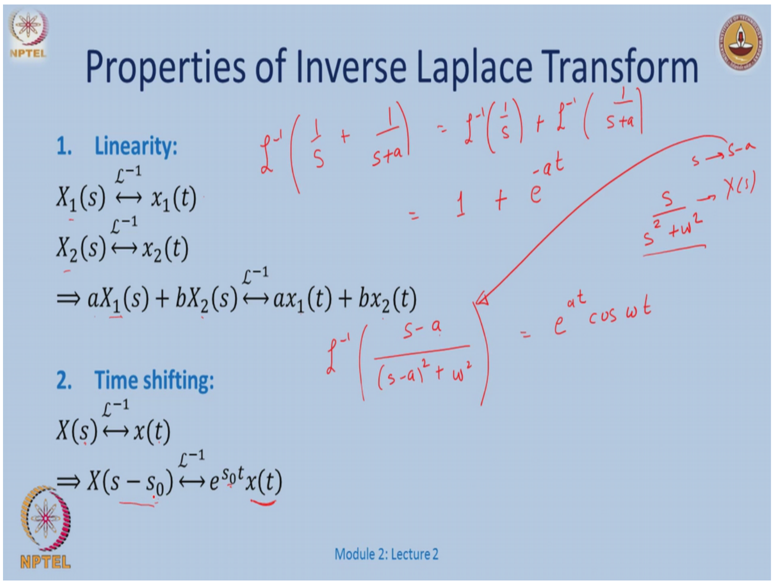

So, back to all these properties which we have learnt earlier they also exist here. The first thing is the linearity property say I have 2 signals in the Laplace domain denoted as x 1 s x 2 s with their equivalent inverses being x 1 t and x 2 t. Now if I have a combination of these signals that I have a times x 1 s plus b times x 2 s the inverse of this entire sum would be a times the inverse of x 1 s this is x 1 t, plus b times the inverse of x 2 s which is x 2 t. So, which means let us do a example I want to compute l inverse of a signal which looks like this 1 over s plus 1 over s plus a.

So, what I do I just do because of the linearity property I compute l over L inverse of 1 over s plus L inverse of 1 over s plus a, and I know that the l inverse of 1 over s is 1 and this will be 1 over s plus a would be e power minus a t. So, that is just a straight forward property.

Next I have the time shifting property this again was a which we had in the transformation from x of t to x of s. So, suppose I has x of s with an inverse of x of t and now I replace s with s minus s naught, then the equivalent domain signal would be of course, x of t would remain as it and I have an extra signal e, e power t multiplied by s naught this is the shifting in time here.

So, let us take a signal which looks like this and I want to compute l inverse of s minus a over s minus a square plus omega square right. So, here I have s shifted by this number a s minus a over s minus a square plus omega square. So, this could be written as. So, if I just look at independently of this how x of t or x of s would look like. So, this is like s over s square plus omega square right and with s being transformed to s minus a, this looks like this. So, this means that I can use this property of time shifting that my signal in the time domain would now just look e power at.

Now, I know the inverse of this guy this is my x of s, the inverse Laplace of this guy is simply cos of omega t. So, I used this very nice beautiful property s minus x 0 would be the original signal plus are multiplied by e power s naught t s naught is the amount of shifting.

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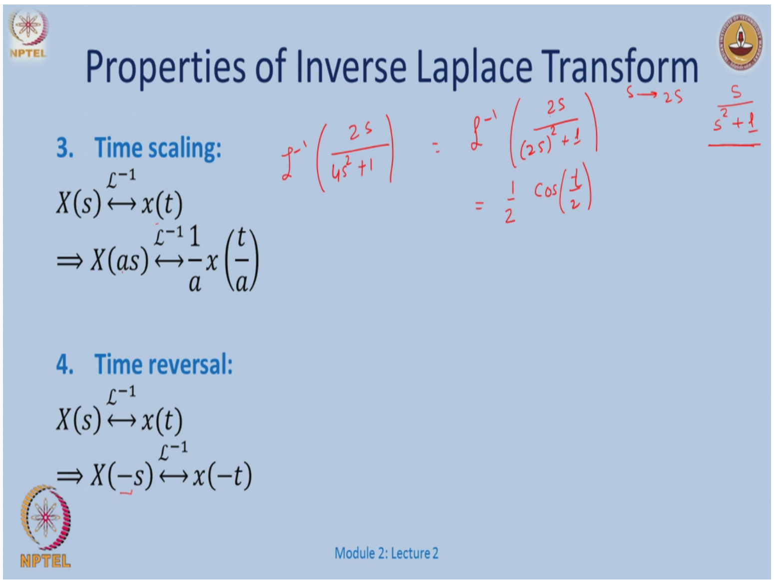

Similarly, with time scaling; so I start with the signal again x of s with an inverse x of t and I multiply s by a then the inverse is just given by 1 over a with x, the t now replaced by t over a. So, let us again do some example and these are pretty simple looking examples 4 s square plus 1 this is inverse of 2 s, 2 s square plus 1 this is almost like s replaced with 2 of s. So, s replaced with 2 of s if I just write down the signal without the time scaling it will look as s square plus 1 again the cosine square kind of signal. So, this will simply be 1 over a, what is a? A here is 2 1 over 2.

Now, this is the inverse of the original signal cosine of t now with t replaced by t over 2 this is a simple way of computing this. Time reversal is is very straight forward x of s goes to s x of t and therefore, x of minus x goes to x of minus t no need of an for example, here.

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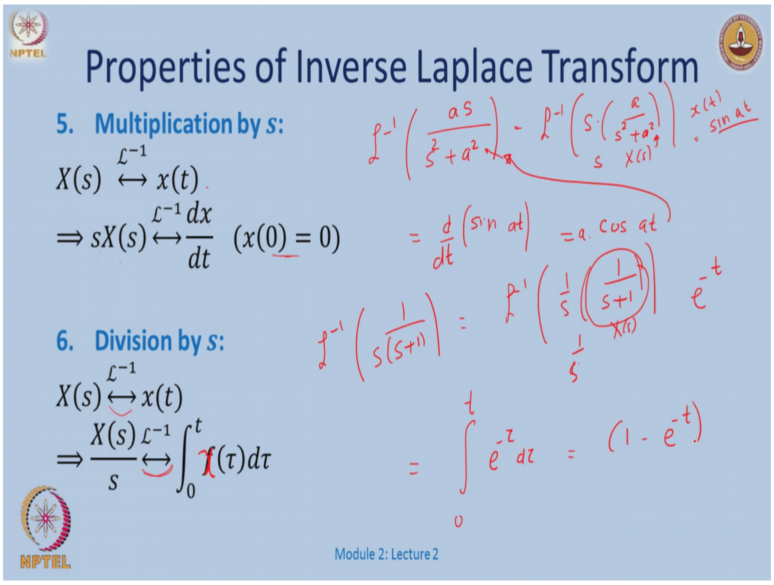

Next is multiplication by s. So, take again a signal x of s with an inverse in time domain being x of t, then s times x of s the inverse would be again I start with the signal x and this is differentiated; well assume that dx at the initial condition is zero. So, again simple nice looking example say take a s over s square plus a square, now this is inverse Laplace of this signal s multiplied by a over a square plus a square this is my s this is my x of s and this looks very familiar now right x of s being a over s square plus a square what is the equivalent x of t. X of t here is simply sin of a t.

Now, going by this one; so x s times x of s would simply be the differentiation of the original signal. So, the inverse Laplace transform of this guy would be d over dt right. So, this why d over dt of the original signal what is original signal this is sin of at now this can be computed to be cos of a t with a. Now you go take the Laplace of this and you back to here what is inside this bracket here. Similarly division by s would give you the integral of it x, x of s with the equivalent of x of t would be. So, therefore, x of s by s the inverse Laplace would be this one.

So, you see the observe that we are actually using this arrows both ways; that means, I can go from here to here and also hereto here. So, well and thing we did in a example let us quickly do this. L inverse of 1 over s, s plus 1 I can write this as L inverse I have 1 over s I have 1 over s plus 1 this is looks like this is my x of s, this is 1 over s. Now this can now be equivalently written as integral from 0 to t now the original function. So, this should be x this. So, the original signal now look at this is my original signal x of s, the original signal x of s the original signal would therefore, be e power minus t. So, I am just integrating from 0 to t e power minus tau d tau, and that will be just be 1 minus e power minus t. Now you can just do the Laplace transform of this and recover the original signal.

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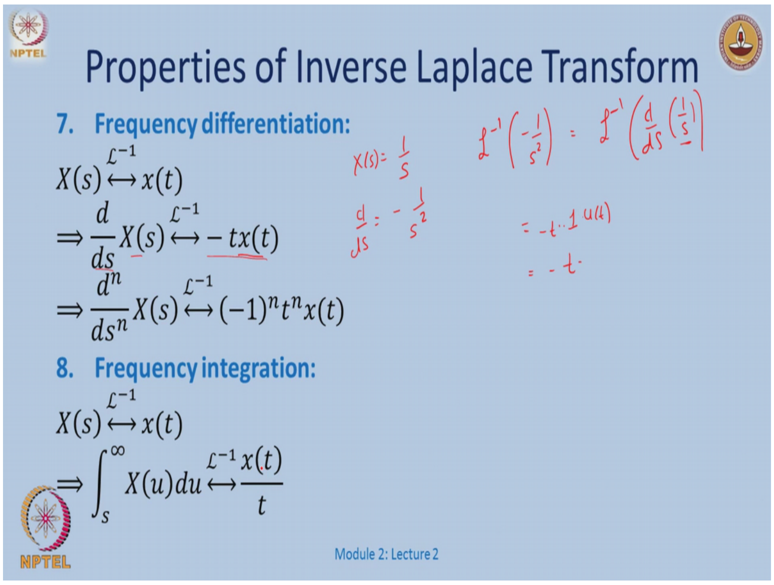

Now frequency differentiation says if I in my Laplace domain differentiate my signal d over ds that is simply here amounts to multiplying it with a minus t.

Let me do a simple example here let say I have signal let say this start with X of s is 1 over s, d over ds is minus 1 over s square. Now the inverse of this s square is inverse of d over d s of 1 over s. Now what is the original signal x of s going to x of t that is 1 of s right 1 over s that in the equivalent time domain would just be the unit step or else say 1 dot u of t, this multiply by minus t because it simply be minus t because inverse of minus 1 over x square is sorry if the inverse of this signal minus 1 over x square is simply minus t and similarly with a frequency integration. So, x of s again going to x of t the integration s to infinity x of u will again go back to a signal like this and then I will not do an example for this, but it its straightforward to a to refine once we know step number is a property number 7, 8 is easy to verify.

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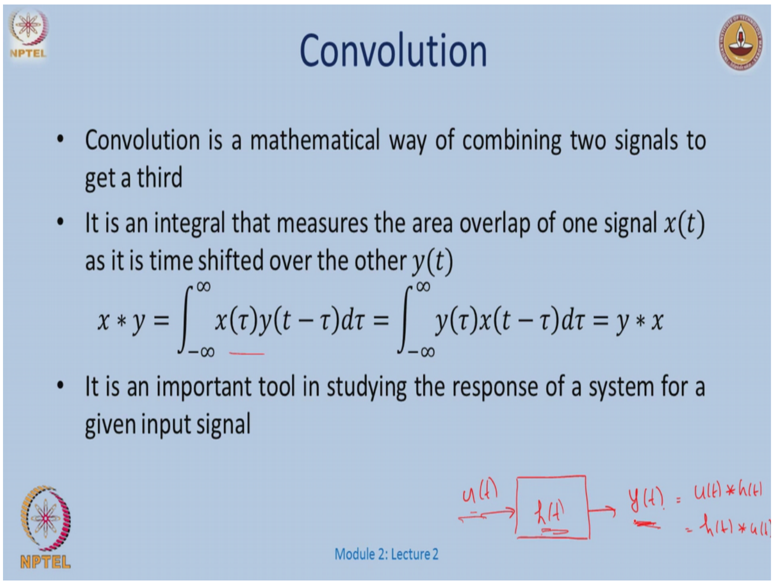

The important property is what we learn again is signals is the property of convolution, and why we do this is essential because in a control terminology or in a control system is essentially what we saw in envelope of our first lectures is I have input which goes through system could be some something here on fa and then I want to measure the output Y of t. Now this Y of t is in some sense some kind of a combination of ut and ht right and in signal strength this would be it is a convolution of u times h, this is the convolution is defined by this property and this is also community commutative h times t convolved with ut. So, convolution is a mathematical way of combining 2 signals so here my input together with the system to get my output of the system.

And this is how we will view things in control. So, it is an integral that measures the area overlap of one signal as it is time shifted over the other, we will see a little graphical way representation of this. And why this is important for us because throughout this course it is important for us to study the response of a system response is Y of t the system is h of t for a given input signal right.

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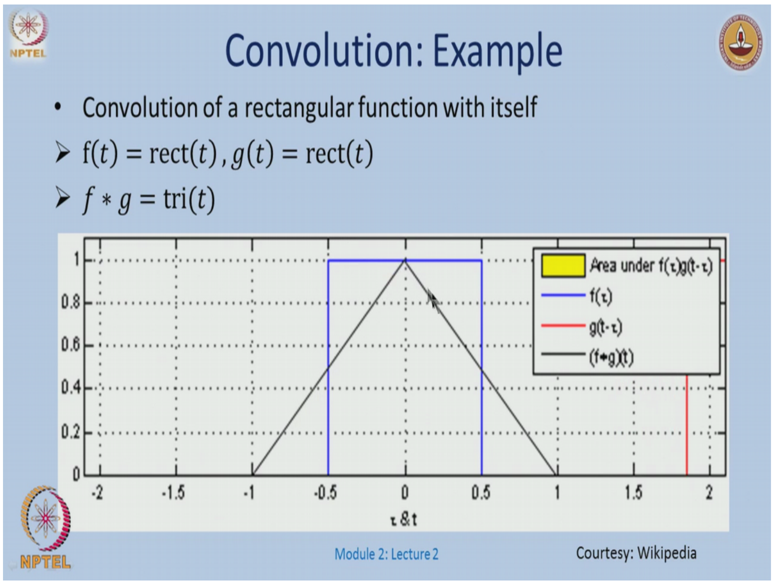

So, this is not simply multiplication that u tie u of t times h of t will give me y of t, so little more complicated than that. So, let see that I have 2 functions a rectangular function and g of t is also a rectangular function, and we will see how the convolution of these 2 gives me a triangular signal.

So, let us nothing I will hope it will play again. So, this is some the videos directly taken from Wikipedia. So, thanks for Wikipedia to make a life little easier for us you see this is 1 rectangular signal convolving with this red guy, and you see that it just the yellow one or the area. So, this is the area this is computed here. So, this goes and then the area reaches a term maximum here and then again the yellow line keeps that the yellow area keeps decreasing and goes to zero. So, the convolution of a rectangular signal with another rectangular signal is just a triangular wave the black signal here.

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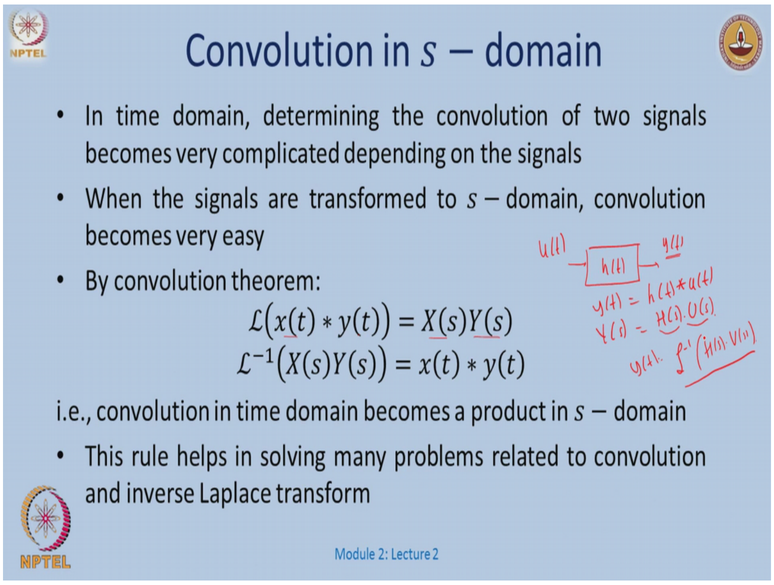

So, this actually looks quite complicated and if you look at even the expression that actually I have to actually compute the you know x of tau, y of T minus tau d tau and so on to get signals which are the output of those, and this actually becomes to too complicated sometimes in doing the integration. But then when they are transformed into the laplacian domain the convolution thing becomes very straight forward. So, what is the convolution theorem says say if I have a signal in time domain x of t, and I want to see its convolution with another signal y of t it just becomes a simple multiplication of their individual Laplace transforms.

So, the convolution of x of t with y of t is this is a very simple multiplication of x of s and y of s. So therefore, if I now look at in terms of control what I had written earlier that I had a u of t here, I had h t and I had a y of t here right. So, my y of t which earlier was h of t convolved with u of t, now I can write this simply as y of s is some h of s times u of s as simple as that. So, convolution becomes a very simple multiplication out of product and this actually helps now to solve lots of problems. So, if I want to see how this signal looks in the time domain given u t and h t and first go to the laplacian domain I compute h, I compute u, I do the multiplication and now y of t can be easily computed as the L inverse of h of s times u of s, this tricks we know how to compute right with several properties and therefore, starting convolution becomes a very easy property just by a multiplication and the inverse Laplace transform.

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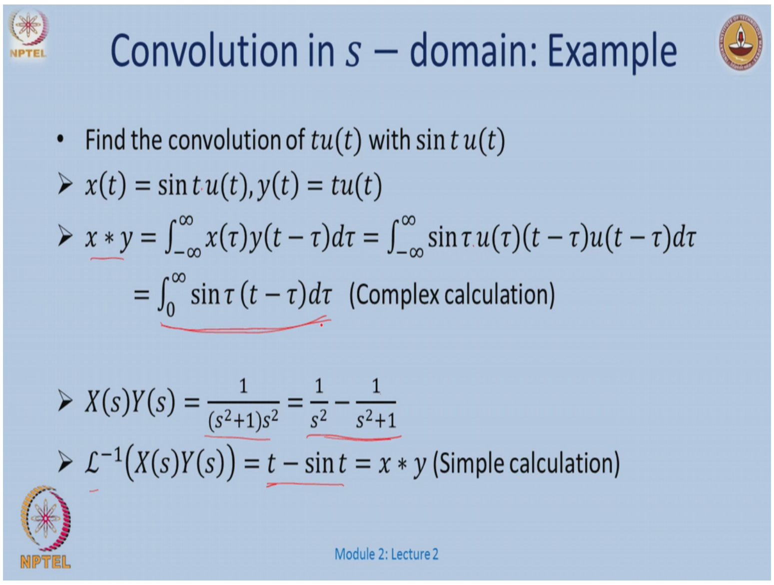

So, as an example right; so I want to find the convolution of a signal t t u of t with sin t u of t this is like the unit step actually like t and t. So, x of t is sin t u of t, and y of t is t times u of t if I just go by the formula x times y going from minus infinity this will just x of tau, y t minus tau t tau I will just substitute for each of the signals, and since this is you have t is the sub unit standing at t equal to 0, I need to compute this kind of complicated integral right. Instead what I could do is I can just go to the Laplace domain or the s domain where I compute the Laplace of sin of t times ut that is this guy plus why this t sin of t would be 1 over s square plus 1, Laplace transform of t would be 1 over s square.

And I can write this as sum of these 2 Laplace transforms and I know that the inverse of 1 over x square is t and the inverse of 1 over s square plus 1 is sin of t. So, the inverse of the convolution x with y in the time domain simply looks like this, and I do not have to go through this complex process of computing this integral.

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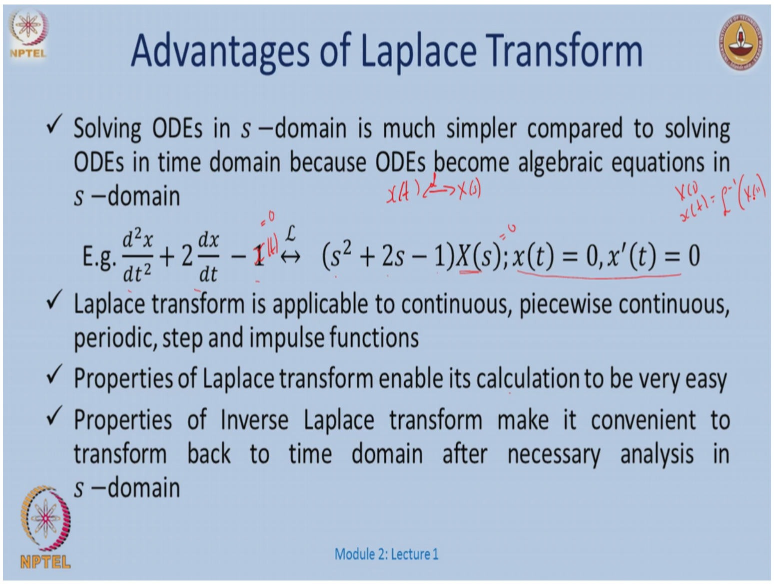

So, one advantage we saw so far is that the Laplace transform easily helps us in computing the convolution, where convolution is just a multiplication. Second advantage is in solving ordinary differential equations. So, solving an ode in the s domain is much simpler because. So, let us see this little example. So, I have this example d 2 x by dt square, plus twice dx by dt minus 1, this is my differential equation may be this is equate to 0 on the right side. So, now, what does this expression become well this expression on the right side becomes by using those formula for differentiation s square, this will be 2 s minus 1 all multiplied by x of s well; this is under the assumption well we just make it for simplicity that initial conditions all are 0.

So, a differential equation is now transformed to a linear equations, I know very well how to solve linear equations, and this is Laplace transform is applicable to lots of signals continuous signals piecewise continuous periodic step and impulse functions yes.

Student: That has to be minus x of t, s square x of t square as told you (Refer Time: 19:52) minus x square.

So this little question that this should be actually be minus x of t. So, this x of t transforms via L to x of s, thanks out pointing out. So, d 2 x by d t square transforms to s square times x of s with a initial conditions being 0, 2 dx by dt transforms to 2 times s of x s and x of t simply transforms to x of s. So, this should actually be a x t here thanks for a pointing that out.

So this transformation from a differential equation to a linear equations would make computations very easy, and we also know so if I determine x of s from this like my equating this to 0, and I can then find what is x of t is simply the inverse of x of s ok.

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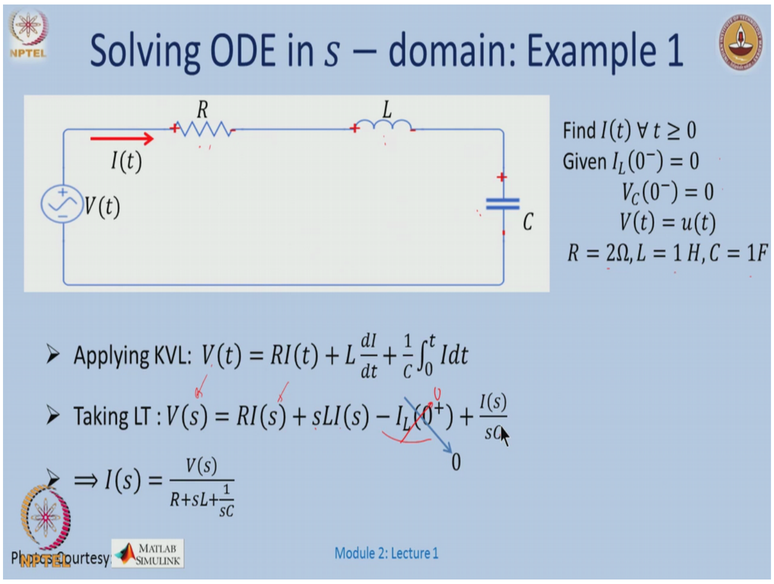

So, we will just see how we could do that. So, let us consider a simple RLC circuit. So, I have a voltage source, I have a resistor inductor and a capacitor and let say all initial conditions are at 0 and V is simply a unit step.

And for computational purposes we will assume that R is 2, L is 1 and C is 1. So, first thing I would know is well I will just apply a KVL to get this dynamics of the system right. So, V t is ri plus L di by dt plus 1 over C integral Idt right. Now the Laplace transform V transforms to V of s, I transforms to I of s, a differentiation of I would multiply it by s with I of s the L remaining as it is and with some initial conditions which are assumed to be 0 and we will see that that this guy also eventually will go to 0, because a inductor does not allow a rapid change in current and then the integral property right integral of I d t would be I of s over s with the c remaining as it is.

So, the current if I were to solve for the current the current would be I of s is V of s over R plus s L plus 1 over s c right just this differential equation now transforms to a very nice linear equation over here.

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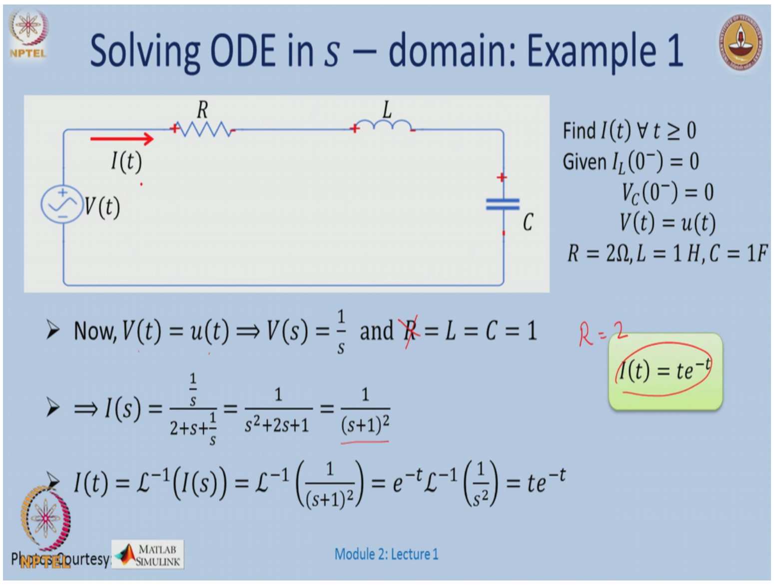

So, let us a try to find out how the solutions look like this for these values. So, I have R plus s L plus 1 over s c a substitute values, u of t is a unit step, V of t is a unit step therefore, V of s is one over s and for values of R L and C I think this should correspond to R equal to 2 right. R equal to 2 would be 1 over s and all these equations and I just write this to be I of s is 1 over s plus 1 square, and now I have this it in the Laplacian form I can easily compute what is the time equivalent of that.

The time equivalent of that is a inverse of this guy, Laplacian form inverse of 1 over s plus 1 whole square and I can use the properties of my inverse Laplace transforms to just get that the signal in the time domain is just t e power minus t, which is the solution which I am looking for right. So, this differential equation or that the dynamic initially were differential equations here I convert that into a linear equation to get an expression for the current in the s domain, and I do the inverse Laplace to get the expression in the time domain.

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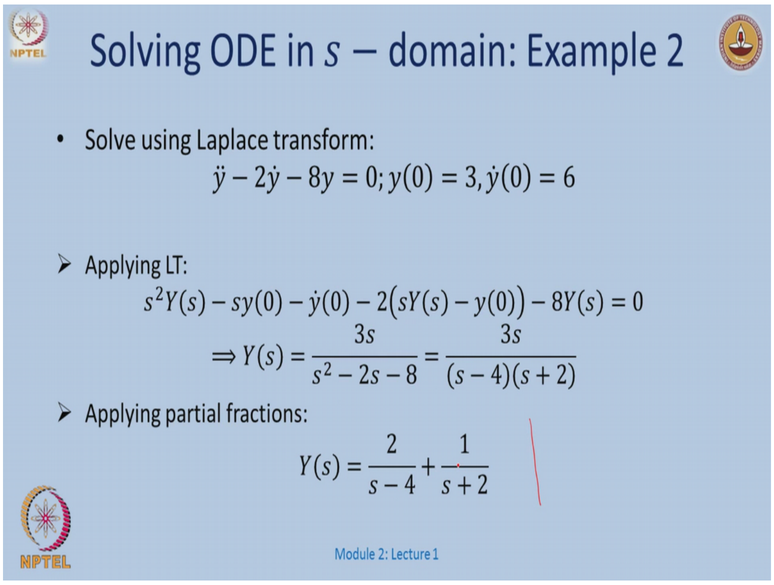

Now, I will do some other example with non zero some initial conditions, y double dot minus 2 y dot minus 8 y is 0, y 8 0 is 3 and y dot at 0 is 6.

So, the first step would be to write this equation in the Laplace domain. So, this would be s square y of s, minus s y 8 0, minus y dot 0 again with the second term to s y of s minus y of 0, and the third term minus 8 y of s is 0. So, this would mean that y of s could be written just in term of a expression in s like this, 3 s over s square minus 2 s minus 8 or just that the denominator could be written as a product of 2 of these terms s minus 4 and s plus 2. So, applying the rule of partial fractions which we would have learnt while dealing with solutions of equations, I can write y of s as composition of 2 signals now 2 over s minus 4, 1 over s plus 2.

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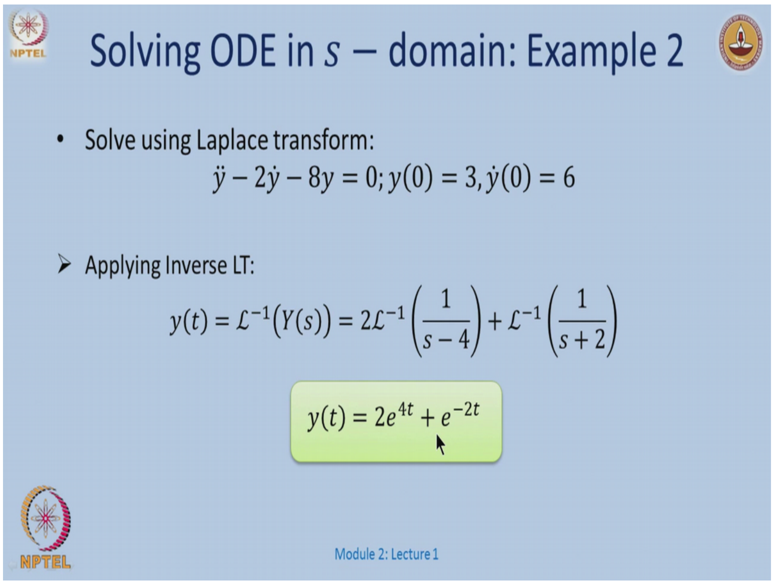

And I can now easily do the inverse Laplace to find out that the inverse of 2 over s minus 1 over s minus 4, plus the inverse of 1 over s plus 2 gives me the following expression for y right. So, again the process is becomes very straightforward I am just now solving for linear equations in the s domain, and then I am applying the inverse Laplace transform to get the solutions to equations right.

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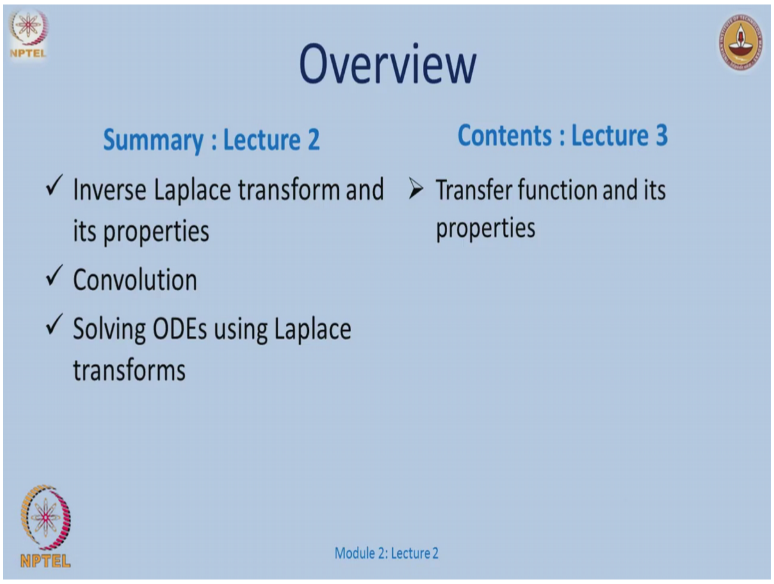

So, to summarize what we have learnt in this lecture is inverse Laplace transforms and its properties, a very beautiful property of convolution where the convolution test translates now into a very simple multiplication in the s domain, and solving of ode s using Laplace transforms. In the coming lecture what we will look at is to write down system or a given set of equations as a transfer function and then explore its properties, we will try to do stability analysis its various its response to various kinds of signals and so on that will be in the coming lectures.

Thank you.