Control Engineering
Dr. Ramakrishna Pasumarthy
Department of Electrical Engineering
Indian Institute of Technology, Madras

Module – 02
Lecture – 01
Laplace Transforms

Hello everybody. In this series of lecture or in this module we will essentially revisit what we had learnt in our basics signals system course. So, in order to make the course as self content will quickly run through these concepts especially of Laplace transforms and the several properties associated to it. We will take a side d to because at the moment we not do linearization of non-linear systems. So, we will do that slightly later when we will deal with straight space.

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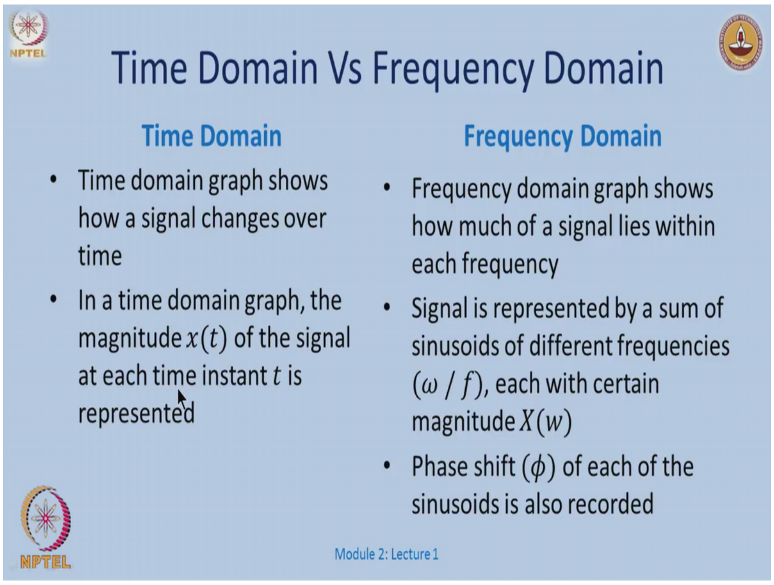

So, starting with basic definition of signals or basic classification of signals, what we would learn in basic systems course would be to essentially differentiate between time domain and frequency domain.

So, what is the time domain do the time domain shows how a signal changes over time, similarly the frequency domain graph shows how much of the signal you have within each frequency for example, even when you do things like filters like low pass high pass band pass and all pass filters and so on. So, in time domain the magnitude of the graph as each instance is represented when you draw a graph, I will show you examples of this very short very briefly when we do this when we do when we learn of what this actually mean. In similarly in the frequency domain this signal is represented by some of sinusoids of different frequencies with certain magnitude, and also in some cases a certain amount of phase shift.

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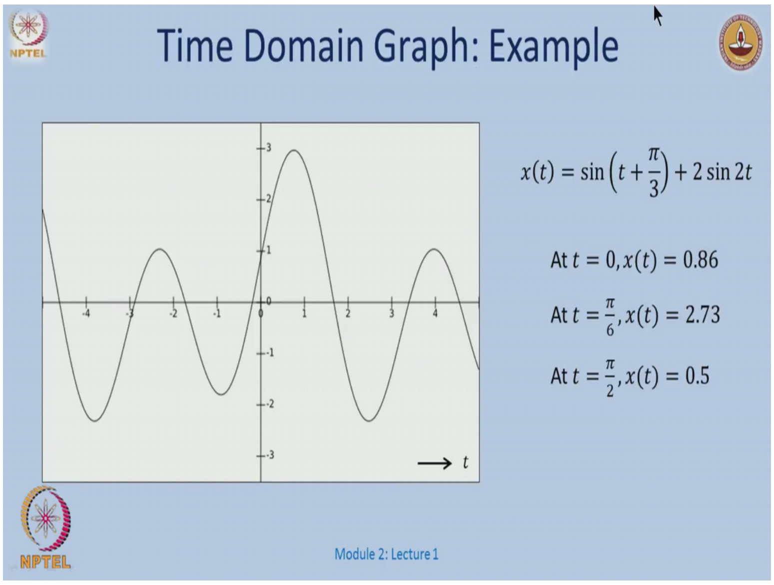

So, what are this mean? So, let us take a very simple looking signal, signal x of t which is a composition or addition of 2 signals sin t plus pi over 3, plus 2 sin 2 t. So, at t equal to 0 if I compute the value, so this is I goes to 0 I am looking at sin pi over 3 and I get a certain value. At t equal to pi over 6 the value becomes 2.7 at t equal to pi over 2 and so on and I can get this nice beautiful looking graph over here right.

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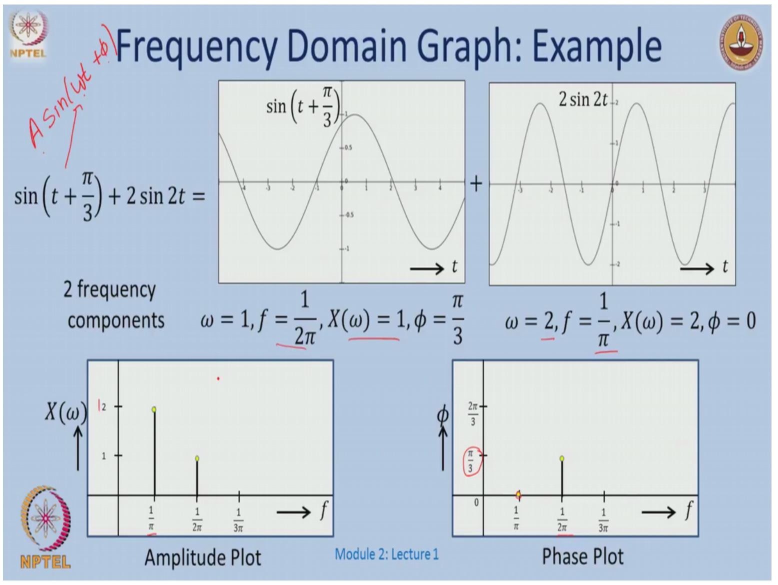

So, what happens in the frequency domain again? So, here the signal is again composed of 2 signals sin t plus pi over 3 the graph of it looks like this and sin 2 t looks like this.

So, typically a signal when we represent we write it or at least a sinusoidal signal as A sin omega t plus phi, with the amplitude the frequency and what you also called as the phase shift right. So, here if you take these 2 signals t plus pi over 3, and 2 sin 2 t well they have 2 frequency components what do I mean; now look at this signals, sin t plus pi over 3 and if I compare with this expression over here. So, here the amplitude is 1 right. So, the if I denote the amplitude as x the amplitude would be 1, omega would be 1 and the frequency would in turn be 1 over 2 pi or the time period is 2 pi therefore, the frequency is 1 over 2 pi and phi the phase shift would be pi over 3 right so this signal as omega as 1, frequency as 1 over 2 pi magnitude as 1 phi as pi over 3.

Now, look at the second signal 2 sin 2 t; 2 sin 2 t has. So, this is with the 2 here. So, omega would be 2 frequency would therefore, be 1 over pi or even it is a time period if you look at. So, this as twice the time period as this guy, the magnitude is 2 and there is no phase shift. Now if I want to represent this 2 frequency components in a graph I look at the amplitude plot right. So, this signal the first one sin t plus pi over 3 right exists only at the frequency of 1 over 2 pi; at frequency of 1 over 2 pi and with the magnitude of one right. So, this is its frequency. So, this is its frequency and the magnitude. So, look at the second signal 2 sin 2 t which exists at a frequency of 1 over pi with the magnitude of 2 in the graph it could be represent here 1 over pi and magnitude of 2. Now what about the phase plot; well the first signal sin t plus pi over 3 has a phase shift of pi over 3 right. So, it this is corresponded to frequency 1 over 2 pi, I have a phase shift of pi over 3 and this guy dose not have any phase shift. So, at 1 over pi I just have 0.

So, this is the frequency domain representation of these 2 signals which are represented in time domain as a sin t plus pi over 3 plus 2 sin 2 t. So, more on this will come later, but this is a nice explanation of what is happening in the transition between time and frequency domain.

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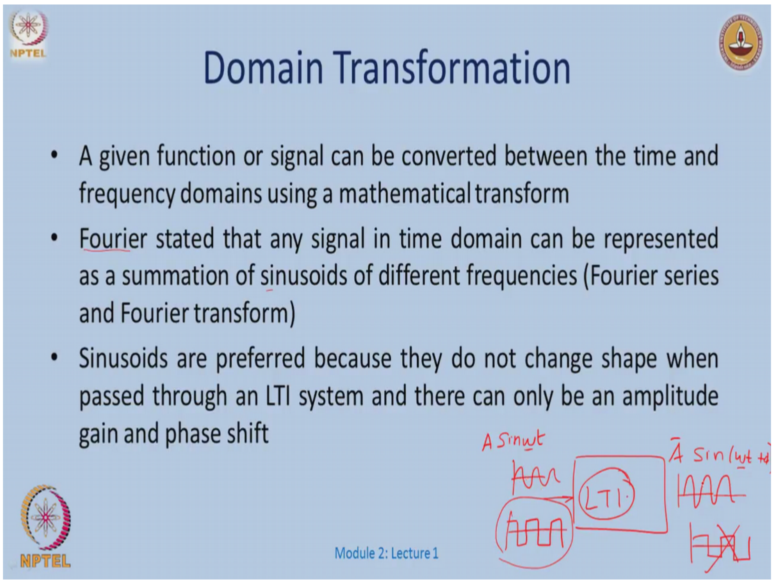

So, given function or a signal what we see here is we can actually switch between the time and the frequency domain; and this can be easily done using some kind of a mathematical transform right and what is the transform well Fourier already told us that any signal in the time domain can be represented as summation of sinusoids of different frequencies, we have this infinite series and what we call as the Fourier series and also resulting in the Fourier transform and why are sinusoids so useful to us.

So, if I take a system which is LTI which means linear time in variant system and as the input I give some sine wave right, and then the output would be some sin wave may be of different magnitude right and some phase shift. So, sinusoids are preferred because they do not change shape when pass through an LTI system, there can only be an amplitude gain and phase shift everything else remains the same. On the other hand if I send square wave right the output will not be a square wave and therefore, analysing sinusoids is important or is useful because if I send the sine wave have a sin wave over here. Only thing is if this is sin omega t with a magnitude A, this guy could be with some other magnitude sin omega t plus some phi that depends on what is sitting inside here, and this A bar phi depends on what is sitting inside this LTI system, but the frequency remain the same this does not change; and of course, I can I could analyse this signals based on the Fourier series that you know that any signal can be represented as summation of sinusoids.

And therefore, I could even analyse these kinds of signals by making use of this transformation at the sin wave if here transforms very beautifully to this side.

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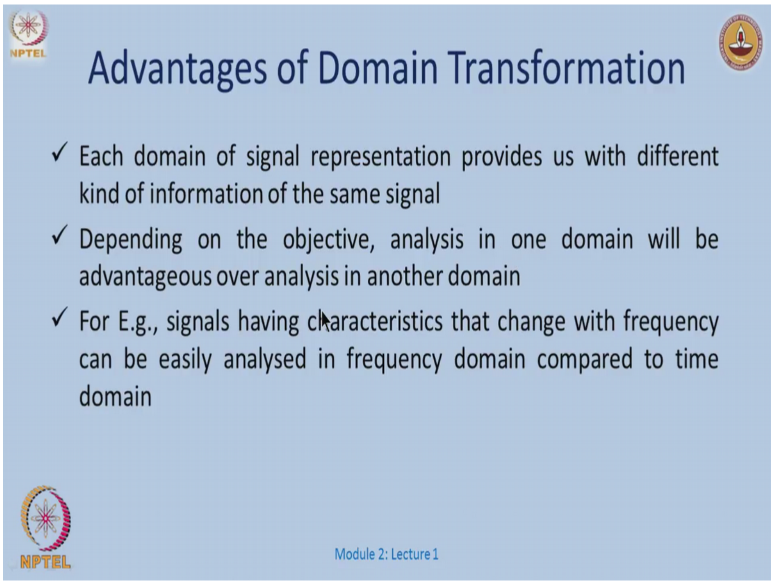

So, are there any advantages or disadvantages of writing things in the time or the or the frequency domain; well each domain has or provides us with its own information that it depends on the objective what kind of information we are looking for in the system. So, an analysis done in one domain could be in some times advantages over analysis in other domain that we will investigate slowly as the course progress.

For example, a signal having characteristics that change with frequency can be easily analysed in frequency domain compared to the time domain alright. So, here in this signal in this things if you see the here the frequency domain is more or less not very informative right that there is where as there is some kind of different things happening as the signal progress is over time, also possibly here may be again depending on the objective we could either use the time domain or the frequency domain analysis.

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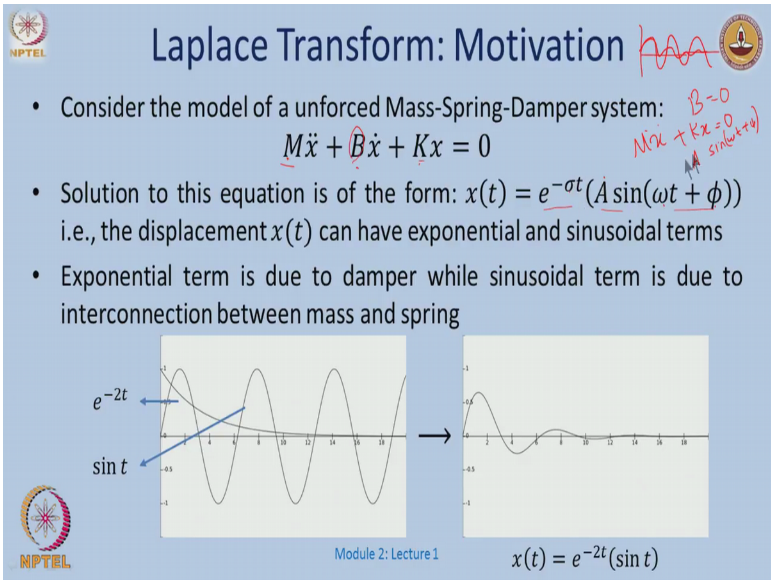

So, what is the relation between what we have learned earlier and then what we are trying to do now right. So, we all know or we discussed this extensively of modelling a mass spring damper system or a second order system with Mx double dot plus Bx plus Kx equal to 0, and if I aim to solve this equation it would have a e power minus sigma t term and some sinusoidal term right it may be a first course on differential equations will tell you this, and this number sigma a omega phi will depend on this parameters M B and K.

So, will not at the moment not interested in what is exactly the sigma what is exactly the A or the omega or the phi; just we are interested in the form of the solution and the form of the solution has an exponent term which is well possibly decaying, in this case it is decaying and then a sinusoidal term right.

So, the solution to the equations has an exponential term and a sinusoidal term. The exponential term is usually because of the damping term here. So, if B goes to 0 then I will have a system which is Mx double dot plus Kx equal to 0 which we know has a solution of the form A sin omega t possibly with the phase shift or not right you will just be. So, the solutions you will just look like a periodic once right this once right and this as a nice physical interpretation also right that if there is no damping in the system then my system will just keep on oscillating and the energy will keep transforming just between the mass element and the spring element here, and this once I have B I have some kind of a exponentially decaying term which the response would then look like something like this right that you know there is a responsible eventually go to 0 will do this analysis little later right.

So, the there the exponential term is you to the damper whereas, the sinusoidal term is due to the interconnection between the mass and the spring, if there is no b they will just keep on exchanging energy if there is a B the energy will slow exponentially d k to 0 right; so this how the how they look like. So, sin of t is just a nice periodic signal for all times t e of minus 2 t for example, just goes down to 0 go exponentially, and the combination of these 2 signals e power minus 2 t minus sin t is just looks something like this right these are also called as damped oscillations.

So, for what we observed is that solutions could have a combination of exponential terms and sinusoidal terms or either just have exponential term are also only a sinusoidal term as we saw in this thing over here.

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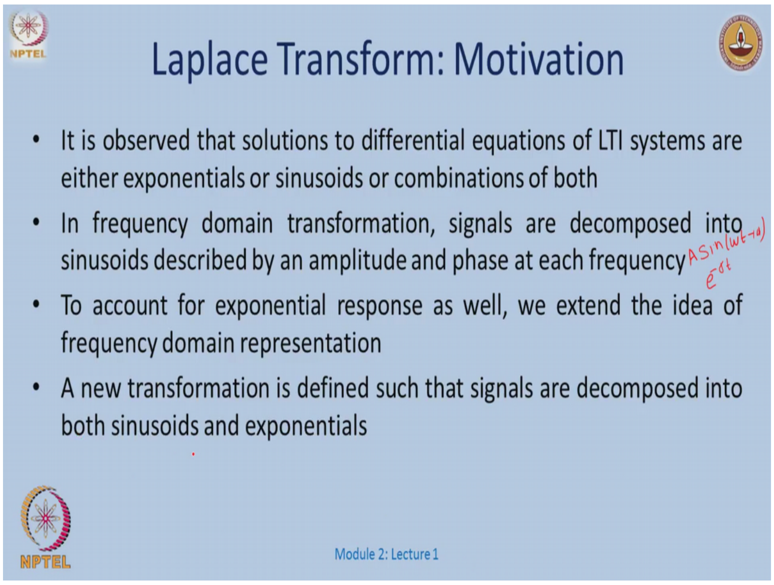

So, in the frequency domain transformation the signals are decomposed into sinusoids which are described by again an amplitude and phase at each frequency.

As we saw in the earlier examples and in addition to this now what we have to do is also to account for instead of just sin omega t plus phi possibly with the magnitude a, we also have to account for the exponential response which was e power minus sigma t in the previous slide right. So, can we combine these 2 into a single idea or the single transformation that will help us analyse the systems better that is what we need is a new transformation, such that signals are decomposed into both sinusoids and exponentials; why because a solutions essentially consisted of these 2 components, at the exponential component and some terms which was just sinusoids.

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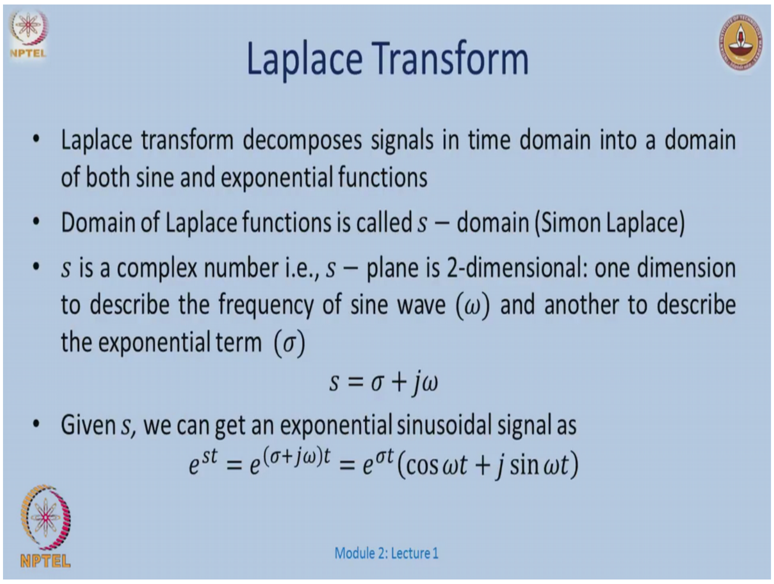

And that is where the Laplace transform comes into picture; what does the Laplace transformer do the transform decomposes signals in time domain into a domain of both sine and exponential functions. So, here sin is not only sine, but it goes include cosine sine terms so on.

So, this domain is called the s domain or the lap lacing domain and invented by Simon Laplace, and I think the name as the s comes from the Simon. So, s is essentially a complex number right s is usually sigma plus j omega that we learn in our signal course is 2 dimensional the first dimension this correspondence to the frequency of the sinusoidal component, and second describes the exponent. See if I just write down the signal of this form e power or a component of this form e power s t. So, s is decomposed into the exponential term and the sinusoidal term, and this expands as e power sigma t cos omega t plus j plus sin omega t.

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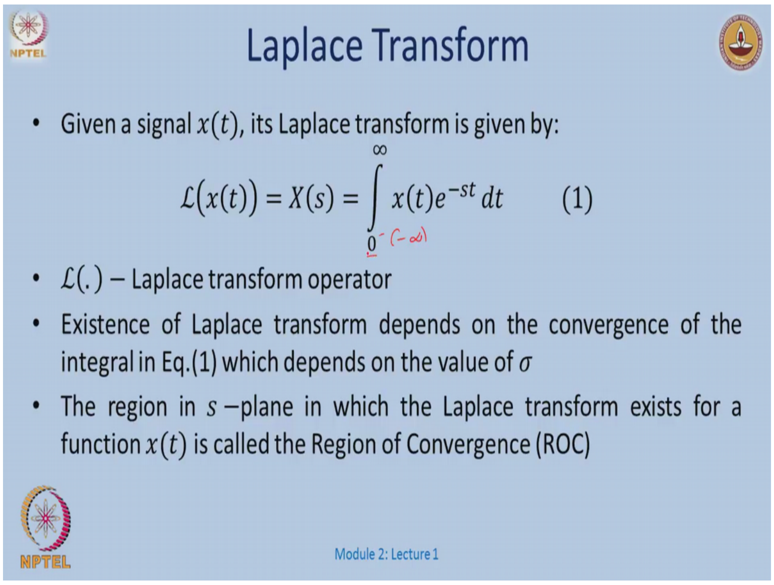

So, by definition how do we find the Laplace transform? So, given a signal x of t right that Laplace transform is given by L x of t I call it as X of s because I go from the t domain to the s domain usually computed as 0 to infinity, x of t e power minus s t d t. In most texts you would see this as 0 minus that would account to the question of what if there is an impulse at the origin or at when t equal to 0; we just start at t minus.

But for all purposes we will stick to the notion of rotation of just having the rotation of 0 here, right and this L is the Laplace transform operator which transform signal from the time domain to the laplacian domain via this expression. So now, the question would come will can I will solve this integral, does the Laplace transformer always exists. So, we will try to answer those questions; existence depends on the convergence of integral does this integral exists right, and in the region in the s plane for which this integral will exists is called the region of convergence, we will see these things through help of some examples.

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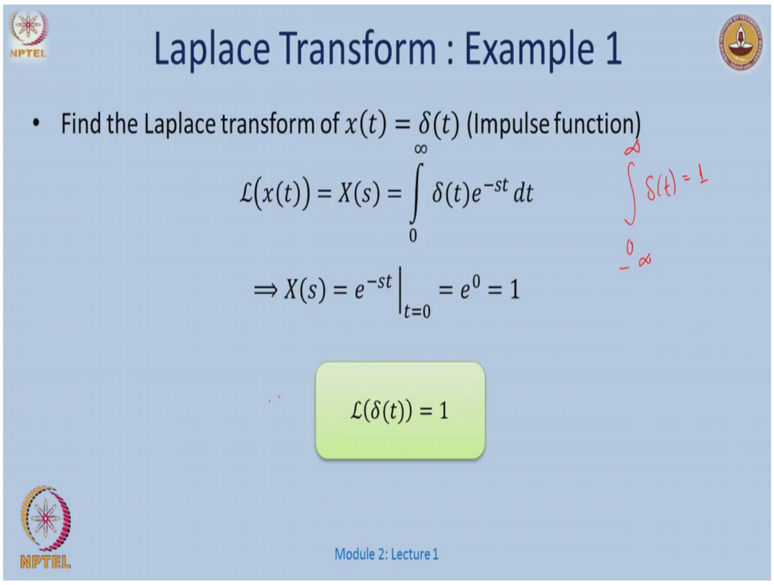

So, let us take the very basic of the signals the impulse function right and I say will find me the impulse find me the Laplace transform of this the impulse function right. So, which is usually denoted by delta of t and I just use the definition here 0 to t, x t e power minus s t d t replacing x with delta am left with this expression.

Now, I can just compute x of s to be 1, where I just use the property of the impulse function where the impulse is defined as 0 to infinity delta t is 1 right and again most of the analysis here we will do is again from 0, even though there exists something called the bilateral Laplace transform which may which actually go from even the 0 replaced by minus infinity, but since throughout this course we are interested only in causal signals or causal systems we restrict our analysis to just this one right things starting from zero. So, making use of this property that 0 to infinity or even say it is good also go from minus infinity to plus infinity the delta t equal to 1 I have x of s equal to 1. So, the Laplace transform of an impulse signal is just one.

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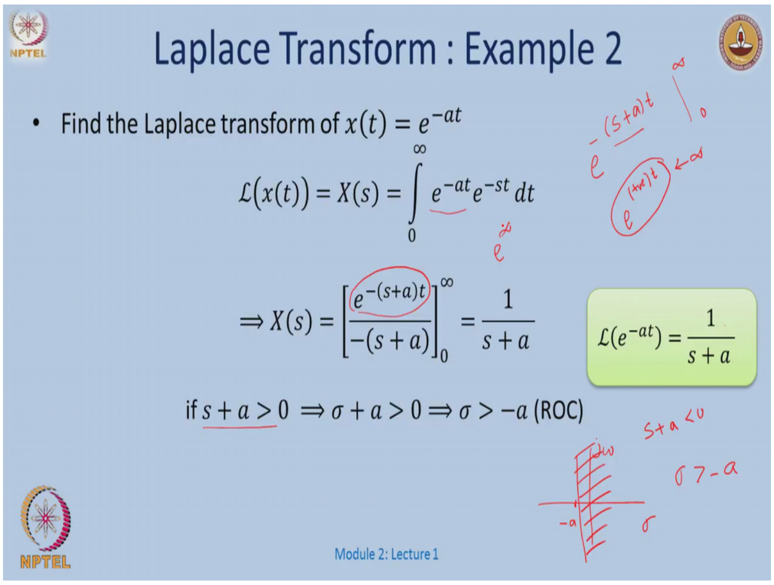

Now, let us take an exponential signal, Laplace transform of e power minus a t. I substitute into the definition of x, x of t here is e power e t minus s t d t and I use the basic formula for integration and what I see is X of s, where x of t is e power a t, the Laplace transform is given by 1 over s plus a; now is this is this enough does it exists from the time.

Well now let us take the case here where of here I am just evaluating this things right e power minus s plus a times t right and I say well do this from 0 to infinity. Now what happens if this guy is positive right if s plus a is less than 0 then I will just have a term e power some number which is a positive number here times t, and this as t goes to infinity does not exists right it also goes to infinity therefore, all these things or the Laplace transform of L of e power minus a t equal to 1 over s plus a exists only in this region where s plus a equal to 0 right.

So, if I just draw it on the s plane spilt as sigma and j omega right. So, the real part that sigma if I put is this here I call this minus a. So, this guy will only exists for this reasons right. So, this sigma is greater than minus a for all other sigmas this limit here will not exists thus I will have something like loosely speaking e raise to infinity and I want to avoid those things, and that is the notion of region of convergence which we are talked about in the earlier slide right. The region in the s plane where the Laplace transform exists is called the region of convergence and this is precisely the region reason why we need to define the region of convergence right just because of this one and this one.

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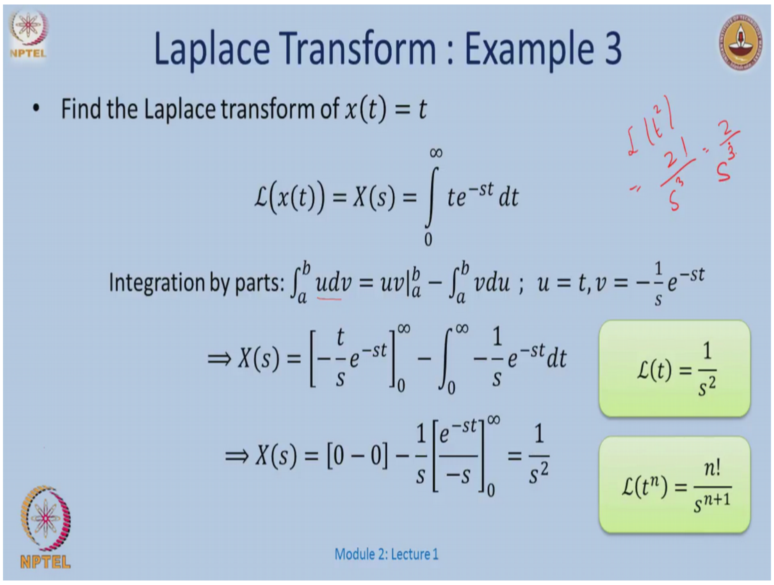

So, let us quickly run through just computing Laplace transforms of few very basic signals if x of t equal to t, I can compute this just by standard integration by parts. So, there I just use this one of integral from the u d v formula as we famously call it, I just keep through this steps, but we will see what it means. So, Laplace transform of t is simply written as 1 over s square.

Similarly, if I have t square or t to the power n, I can just generalise this to an expression like this at L of t power n is factorial n s power n plus 1. See if I have to write Laplace transform of t square that is factorial 2 over s 3 that will simply be 2 over s cube and so on right. So, this is kind of straight forward to compute.

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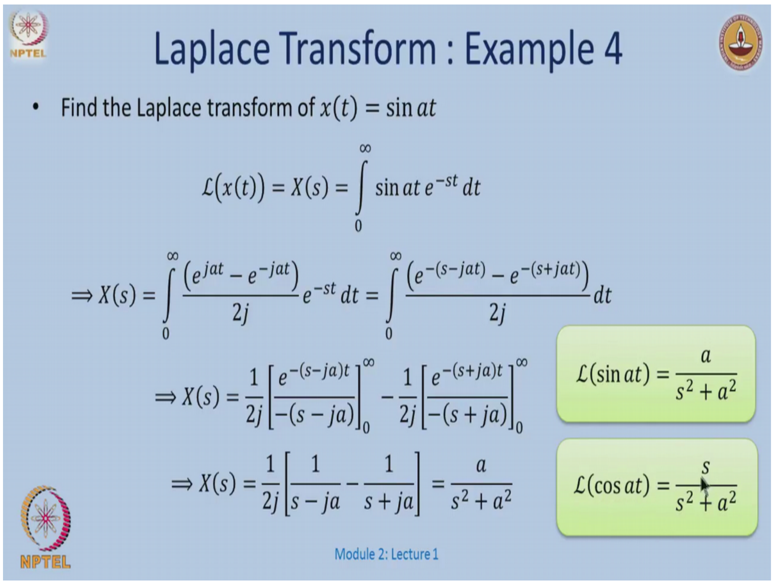

Let us take sinusoids what happens to the Laplace transform of sin of a t where I just put into the formula, and what I get is the Laplace transform of sin of a t is a over s square plus a square, and I can we will this follow this little computational process here where I can write sin as something to do some of some exponential signals. Similarly I could even do for cosine signals the cos Laplace transform of cos of a t is s over s square plus a square.

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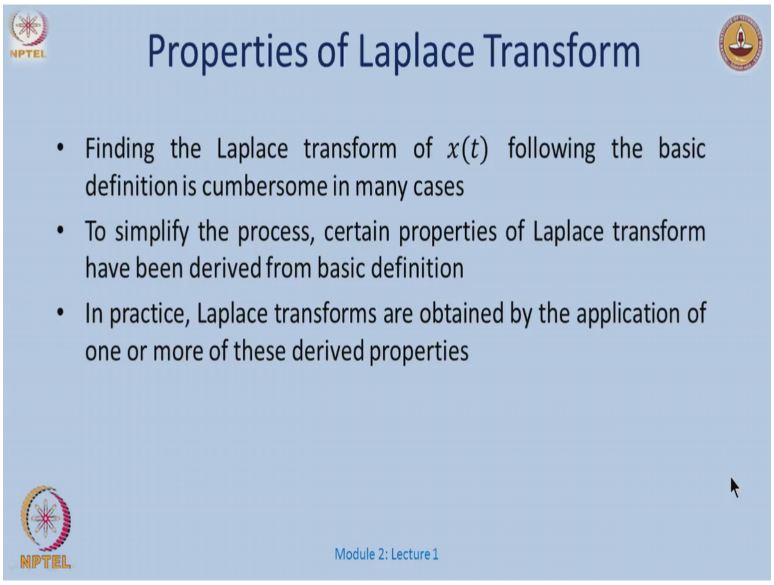

Now, why are this important or do this guys come with beautiful properties well we will investigate them one by one. Sometimes it may happen that I may not be able to compute the Laplace transform just by definition just because I need to solve some complicated integrals right. So, to simplify this process we learn certain basic properties of Laplace transform, and this which basic properties have been derived again from some basic definitions; and by making use of this basic definitions we will the basic definitions plus this properties we can then go on to get Laplace transforms more kind of more complicated looking signals.

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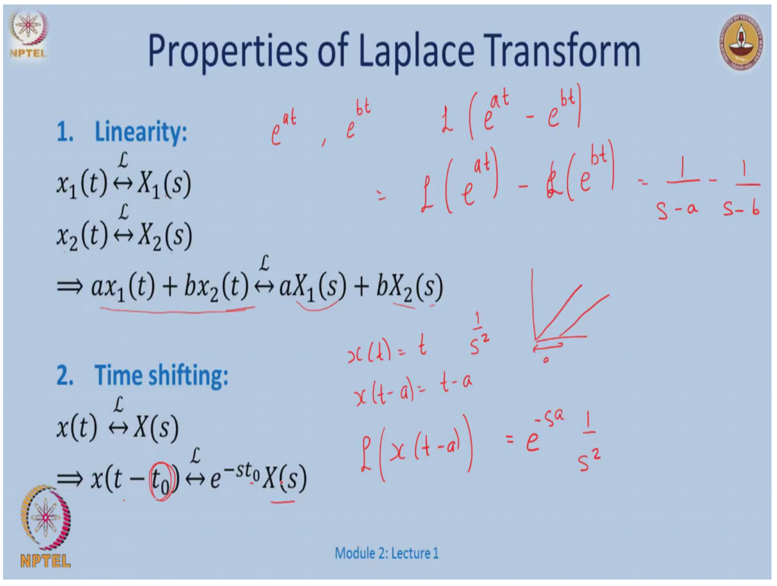

So, let us investigate these properties slowly and one by one. First is the linearity property right. So, if I take 2 signals x 1 with the Laplace transforms x 1 s, x 2 with the Laplace transform x 2 s and I define a new signal which is a combination of these 2 a times x 1 plus b times x 2. So, the linearity condition or the linearity property says that the Laplace transform of this signal is simply a right with its equivalent Laplace transform, plus b with its equivalent Laplace transform.

Let us take a very simple example. So, I have 2 signals e power a t as my x 1, and say e power b t as my x 2, and I want to find Laplace transform of a signal which looks like this e power a t minus e power b t. So, what is this? This is simply according to the property is Laplace transform of e power a t minus these the Laplace transform of e power b t, and that is simply computed to be 1 over s minus a minus 1 over s minus b. So, I am just individually computing and then adding up.

The next property would be the time shifting property, where if I have a signal x t right now I have t shifted by some number t not, the Laplace transform under the time shifting thing would just be e power minus s with the shifted amount of time times x of s. So, this is kind of also straight forward to verify. Say if I have signals that x of t equal to t right which essentially looks like this and then I have another signal which is. So, this x I call them this x shifted by t minus some number a is t minus a which is like this. So, the Laplace transform of this guy I know is 1 over x square, now what is a Laplace transform of x t minus a this would simply be well I still have this x of s which is 1 over s square and in addition I have e power minus s with the amount of time t 0 which is the signals has shifted; so this guy a. So, this is e power s times a over 1 over x square. So, this is just how it looks likes.

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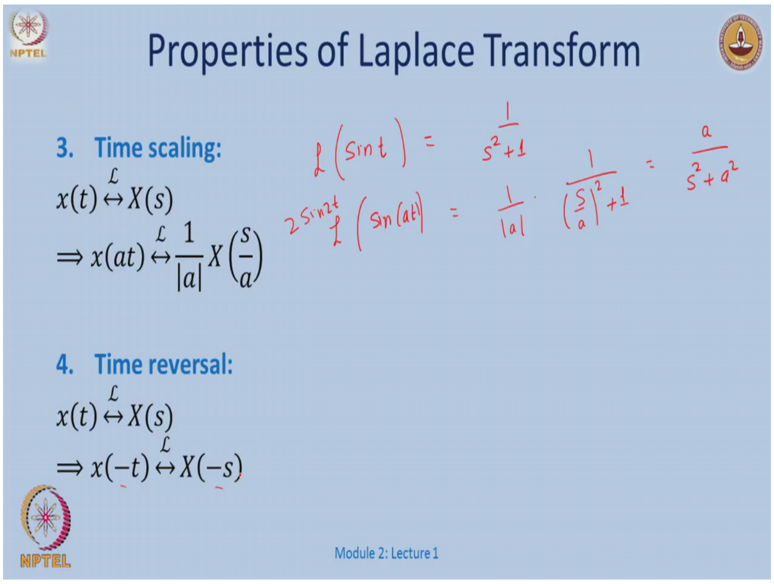

So, the third property is the time scaling property where I have a signal again x of t with an equivalent Laplace transform x of s, I have x of a times t would just be 1 over magnitude of a s x of s over a. So, let us see this in terms of say a simple say sinusoidal signal so.

So, from the earlier derived formulas let us say Laplace transform of sin of t would be 1 over s square plus 1. Now if I instead of t I have to find what is the Laplace transform of sin of a times t right the time is scaled with the factor of a like in one of our examples we had the signals 2 sin 2 t right in the one second slide. So, what would this be used by this property this would be 1 over the magnitude of a, with the Laplace transform with of the same signal with s replaced by s over a. So, I have this one s over a square plus one. Now I write this down and expand this and what I have is now this guy would be e over s square plus a square this is what we exactly derived earlier.

Now the next property is time reversal is kind of straight forward it is just replace the minus t with the with the minus s, we not do any examples from this.

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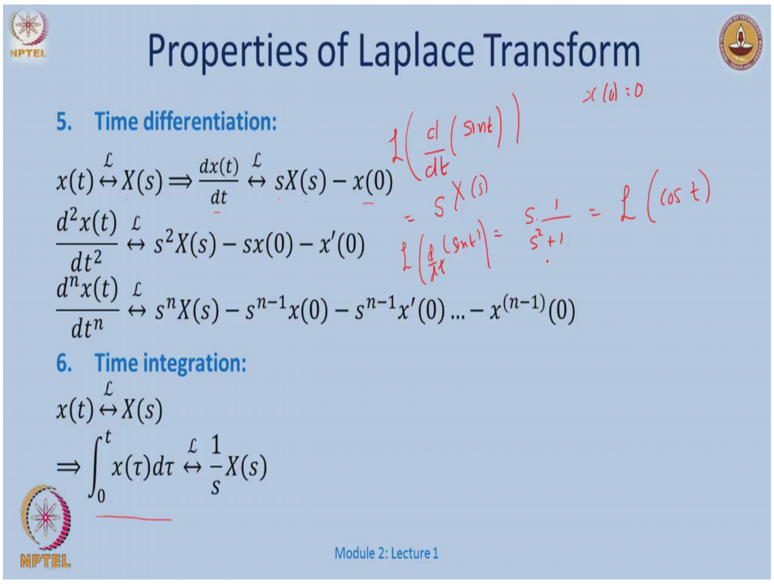

Now, the next important property is that of time differentiation. So, I have a signal x of t with its equivalent Laplace transform x of s. So, what the property tells we is a d x, t of d t the Laplace transform of this would be s times x of s minus x of 0 well this is initial condition of the signal. And similarly I can even do for around derivatives. So, let us see if you could like to do an example with this. So, let us say I have a signal say sort with sin of t for which I know that the Laplace transform is 1 over s square plus 1.

Now Laplace transform of d of d t of sin o sin t with this formula assume that you know x of 0 equal to 0 for simplicity, that d of d t of sin t the Laplace transform of this guy with this property would simply be s times X of s right. Now what is x of s this is 1 over s square plus one and therefore, Laplace transform of d of d t of sin of t would be s multiplied by the Laplace transformer signals are and this is again nothing, but the

Laplace transform of cos of t as we had verified earlier ok. Similarly to with the time integration right so if I have a signal I integrate with 0 to t it is just as replacing it by 1 over s if we just do the reverse of this example and that will be kind of (Refer Time: 27:56).

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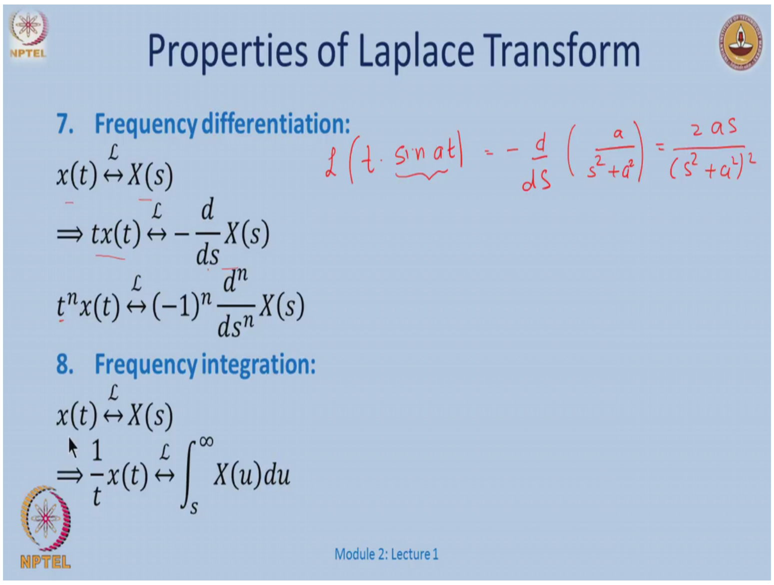

Next would be frequency differentiation. So, if I take a signal x of t right and I multiply it. So, x of t has an equivalent Laplace transform of x of s, I multiply this by t this would amount to just differentiating the Laplace transform of this original signal by s and adding an negative sign. So, let us say I have to do this Laplace transform of t of some signal sin of a t.

So, this would by this property just be minus d over d s right this guy times the Laplace transform of only this guy right x of s that is a over s square plus a square and this becomes now a straight forward thing for me for compute right and then just taking the formula then you know multiplying this by e power minus s t and so on. So, this just becomes twice a s over s square plus a square whole square right. So, this is kind of a very useful property for me and I can just send this to when I am just multiplying it with t n times.

Similarly, I have the property of frequency integration that if I have x of t with the equivalent Laplace transform X of s 1 over t x of t would be just be this integral s to infinity X u d u.

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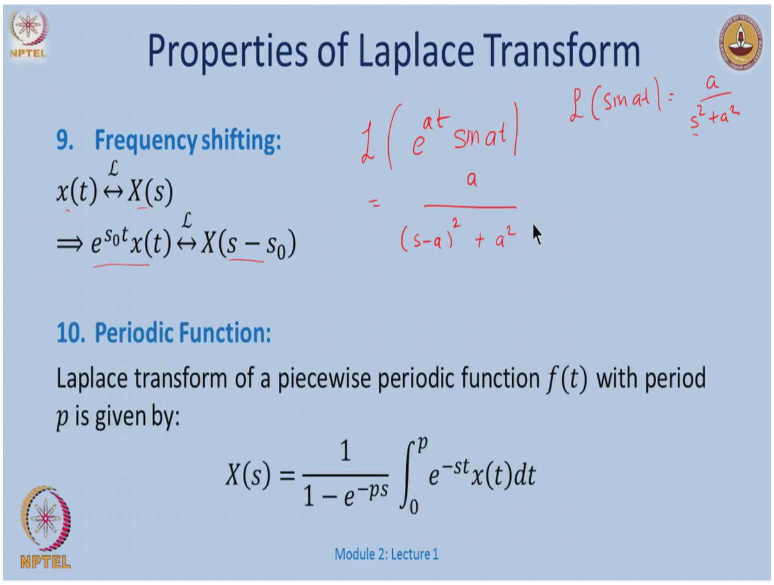

Frequency shifting right if I have again the signal, signal x of t with the equivalent Laplace transform x of s I am multiply this signal with e power s not t it would just be amount to substituting in the original Laplace transform of the signal s with s minus s not. So, let us say I want to do this something like this Laplace transform of my original signal is sin of a t and I say I just multiply this to the left by e power a t.

So, this would simply be what is this. So, the Laplace transform of sin of a t what is this Laplace transform of sin of a t is a over s square plus a square. Now multiplying this by e o e power a t would just means substituting s with s minus s 0 in this case s 0 is a. So, this would just be a over s minus a whole square plus a square and similarly if I have periodic function with some period p the Laplace transform is simply given by this formula, that is why I do not really need to go from 0 to infinity, but I can just make use of substitute.

We not derive each of this we may have learn this in our earlier courses but I am just doing a bit of quick recap of this things. Now some other important properties; so first is called the Initial Value Theorem.

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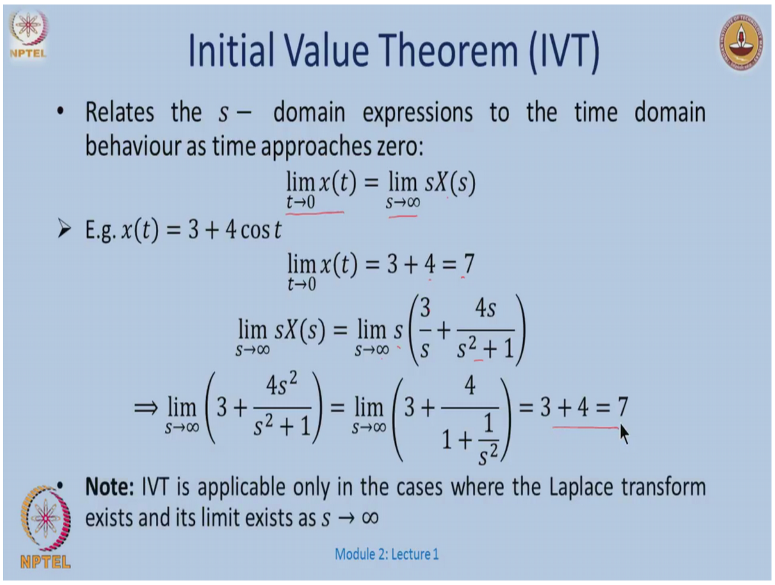

So, what is initial value theorem do it relates again the signal to the in the time domain to the one in the s domain as time approach is zero. So, limit t goes to 0 of x of t would be limit x tends to infinity of s times x of s. So, let us take this little example. So, the signal x of t goes as 3 plus 4 cos t, if I just do the limit 3 stays as it is the cos of 0 becomes one. So, I have a four here. So, the limit goes to 7.

So, in the s domain I just use this formula that limit s tends to infinity as x of s is limit s tends to 0 and multiplied with s, and then the Laplace transform of x of s 3 is like a step of size 3 that will be the Laplace transform would be 3 over s, and 4 times cos t and we in our earlier slides had computed the Laplace transform for cos of t. So, I just do all the computations take the limit and I come at with this number seven. So, only thing which we need to be careful here is that the Laplace transform should exist I will give you counter example of this shortly.

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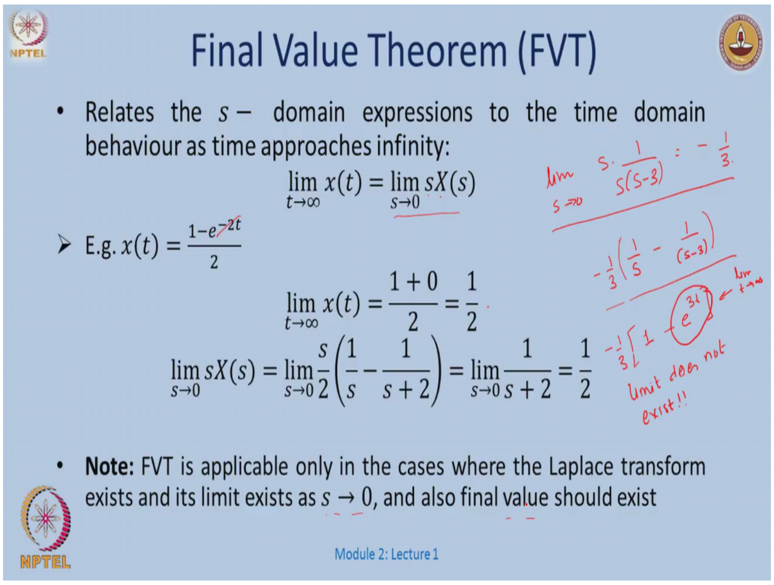

Similarly, with the final values theorem what happens to my signal as time goes to infinity? So, the final value theorem says that limit t going to infinity x of t is equivalent to limit s going to 0 s times s of x. So, let us take an example.

So, I have this example x of t is 1 minus e power minus 2 t by 2, I do the simple computations and I get the final value that this signal convergence to 1 over 2 as e power minus t goes to infinity as t goes to infinity. So, this term will dispute. So, what is how does it look in the laplacian domain? Well 1 becomes 1 over s, e power minus 2 t becomes 1 over s plus 2, I apply the limits and I get 1 over s plus 2. Again we have to make sure that the limit actually exists. So, let us do something very quickly of a case when the limit may not exists all the time. Let us take the case of a signal 1 over s, s minus 3 and I say I just blindly apply this final value theorem at limit s going to 0, s times this would give me a value of minus 1 over 3 now is this is this correct.

So, let us me. So, let me do the; also do the time domain over is like a do it over here. So, this signal can be written as 1 over s minus 1 over s minus 3 we have a s minus 3 this going away and minus 1 over 3 here, and if I write on the equivalent time domain representation of this. So, what I will have here is 1 over 3, and 1 over s corresponds to one in the time domain like Laplace transform of one is 1 over s minus e power 3 t. If I take the limit of this signal as t going to infinity, I see that this term actually grows up right the limit does not exist and therefore, I just do not blindly apply I should make sure that first the limit actually exists. So, this final value theorem and also the initial value theorem is applicable only in the cases where Laplace transform exits and it limit exists as s tends to 0, and also the final value should exists that is what we should careful of when we apply the final value theorem.

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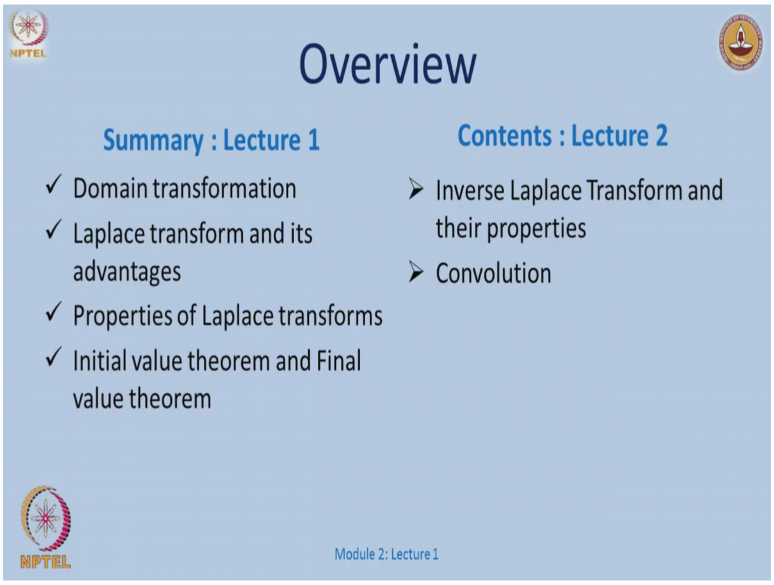

So, what we have done so for is had a quick recap of Laplace transforms basic properties and how we use those properties to compute Laplace transform for some little complicated looking signals, and then the initial value theorem, the final value theorem and what we should be careful of while applying those theorems. Next we will discuss the inverse of Laplace transform. So, if I can go from the time domain to the s domain can I actually come back and another important property called the convolution which will be very important for us in the control setting.

Thanks for your attention.