Control Engineering
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Module - 12
Lecture – 03
Controllability and Observability

So, welcome to this last lecture on this course on control engineering and to continue on what we had learned in state space techniques. So, we started off by writing down state space equations, then we analyze stability by looking at linearization of a non-linear system, then we looked at stability in terms of the Eigen values of the matrix A. So, we will go a little further and see; what else can we do with the state space analysis. So, we will keep the analysis to a very minimal, right. So, I will not do proofs of what I am going to talk about I just give you a very general idea for a very maybe just small 2 by 2 systems.

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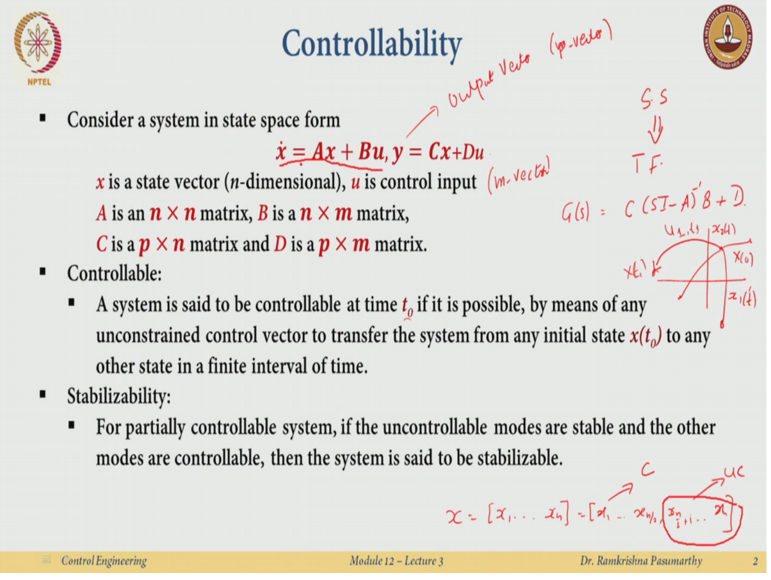

So, the first thing which we will learn is the notion of controllability. So, we start with a general state space system x dot is A x plus B u and y is C x plus D u where as usual x is an n dimensional state vector A is the system matrix which is of dimension n; n cross n, u is a control input could be an m dimensional input, it could also just be a 1 dimensional input or just a scalar input C is a P cross n matrix given that y is a set of n m outputs, I just missed it, I just write it down here. So, y is the output vector and could be an mm sorry; a P vector and the input is a m vector. So, what else can we do now? So, just given a system x dot equal to A x, I know how to find out stability, I also know how to go from the state space representation to the transfer function this we had learned in the earlier lecture.

So, I will just write down the formula for you again G of s is C S I minus A inverse B plus D. So, this has all been good so far. Now a very important notion, so far when we were doing the entire transfer function based analysis of designing compensators in via root locus where the frequency domain bode plots steady state errors and so on, we never asked our self; is the system controllable, right, we just say. So, given this system of given a certain transfer function as long as it is stable or stabilizable as long as I could pull the root locus to the left half plane and things like that I even knew in the Nyquist plot that I could start with the open loop unstable system end up with a closed loop stable system I never asked is it controllable before I even start controlling.

Now, even before I answer is it controllable or not first we need to formalize the notion of controllability. So, the definition says that a system is controllable at some time t 0, if it is possible by means of any unconstrained control vector u to transfer the system from any initial state to any other state in a finite interval of time, let us say just if I just were to draw it graphically, let us say I have a 2 dimensional state space x 1 and x 2 both t. Now I say given any initial state here, can I transfer to any other state, it should be in the entire R 2 entire of this 2 dimensional space can I go for example, from here till here by application of some control say u one in some finite time t 1 and that is true for all points from here till here, here, here and so on, right.

So, is it possible by means of any unconstrained control I do not say, I do not have a limit on the amount of control effort I put it, I say well can I at least control maybe the effort is really large unconstrained control vector to transfer from any initial state to any other final state in a finite interval of time now this finite interval of time is important. Now there could be systems which are just partially controllable again, I will not go to the details of this, but I will just; so, if I have say systems with say some n states. So, which means x is a vector x 1 till x n and I say that the first n over 2 states are controllable and the remaining n over 2 states are uncontrollable.

So, these are controllable states these are uncontrollable states, now what can I do with these systems is I just ask a question are these systems stabilizable, it is a little weaker notion of controllability. So, I can do anything good with the system as long as the uncontrollable modes are stable and if these are stable then I can do something with these control modes here. So, for partially controllable systems partially controllable here means I can control only half the states if the uncontrollable modes are stable these are the uncontrollable modes and the other modes are controllable then the system is stabilizable. So, what could be a very vague example of this? So, let us say; I am in a small classroom and I want to control the classroom control essentially would mean here I just want to not make no, no, not the students should not have any crosstalk they should not make any noise for example, like a union in a high school.

So, if nobody is there, there is no control input; there is no teacher in the class, there is no control input. So, the entire class is in a miss everybody shouting talking throwing paper balls and stuff like that now say I enter the classroom say the classroom has 10 people and it is more or less possible that I control I could control the 10 guys, I could just maybe you have warned them by whatever means I could control them, I could certain them, I could say I will give you to mark less or I will give you a chocolate or things like that and I could control.

Now, say if the size of the class is say 20 I could possibly still manage, if it is say 100, then there could be 2 people who could control the classroom, right, if there are say 500 and it is very likely that only 2 people may not be able to control the classroom. So, I can split those class that classroom into well mischievous guys and good guys and I assume that the mischievous guys that that the entire you know all these guys are good guys and they are controllable. So, if I say they say the amount of mischievous guys is 100 and the good guys are say 400 that they are just quiet by themselves.

Now, I ask my question is and I actually do not have control over the forehead say the first 100 are sitting in the first couple of rows and then the 400 guys are really far that I will even if I shout I yell at them, they will not be able to listen, but they are all quite guys, they will naturally be quiet, right. So, here it says that the system is partially controllable; I only have access to the first 100 guys. So, this entire classroom is stabilizable I would say or its or it can be made quiet only if the other 400 guys who might do not have any control of are quiet by themselves I could control the first 100 guys, right.

So, this is a very and I am just thinking of an example and the best well known engineering example I could come up was this one. So, these are partially controllable systems when I say I could do something I can make the class quieter only if the guys who might do not have access to are quiet by themselves now how to test controllability. So, what am I given I am given this equation x dot is A x plus B u and I say; can I start from x an initial condition x naught if I may call it. So, just let me denote this as x naught and I will go to some say state x at t one some finite time t and with some unconstraint control.

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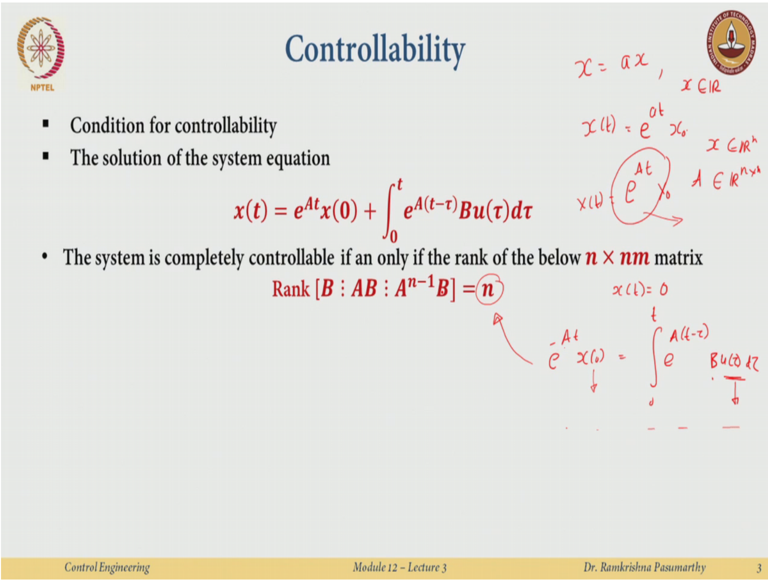

So, can I do something with this differential equation? So, the solution to this differential equation would look like this I have A e power 80, let us not really worry about how the exponential of a matrix looks like that is very detailed topic by itself, but just assume that if I this is a very very crude and not a very smart way of looking at it, but if I have say a scalar differential equation x dot equal to small a and x where x is just a one dimensional vector it is in R, then I say that the solution is x of t is e power A t x naught. Now similarly if a this is a vector x is in R n as in our case a is n cross n matrix, then this exponential would be replaced by e power A t on the left hand side, I will have a vector of dimension n and my initial condition will also be a vector of dimension n right.

So, I should not really write this a multiplied by t write it A is. So, not A of t, but I should write it simply a multiplied by t. So, how to compute this is not really important, but this is also called the state transition matrix. So, it means that given a certain x naught, how will my system the system is defined by A will take this initial condition over time to some other x t which is the evolution of an initial condition. So, if I have the input I will have something like this. So, what is unknown in this? So, if I just little write down this say I have let me for examples for simply assume that x of 0 is some number and say x of t, I want to go to the origin right from any initial condition. So, this would look like minus x, sorry. So, on the left hand side, I will have 0, I get these guys here.

So, this would be x naught e power minus a t would be integral of all this entire thing e power A t minus tau B u tau d tau. So, essentially here I am the answer or the problem here is to what are what are known to me the initial condition is known to me the final condition is the origin A is known to me, B is known to me and this is looks like an equation right. So, given a B the initial condition can I say that I can go from an initial condition any non 0 to A 0 final condition? So, the only unknown is this 1 u of t, right. So, does there exist a solution u of u of t such that this equation is solvable. So, that is the entire controllability problem, again I will not go into the details of computing how the solution goes, but I will just directly introduce to you to the way of checking this one.

So, the system is completely controllable if and only if the rank of this matrix n cross n times n cross m yes this would be is if you look at here V is an n cross m matrix and I keep on doing this that this matrix should be of rank n and I say that the system is then completely controllable this one. So, let us do something else now, right. So, to really understand these concepts you know in a different way I will use this after a very long time. So, let me start with a simple looking example let me say I have a circuit which looks like this R; this R 1 C 1 R 2 C 2 and input I call it V you see some, you see it is the current source.

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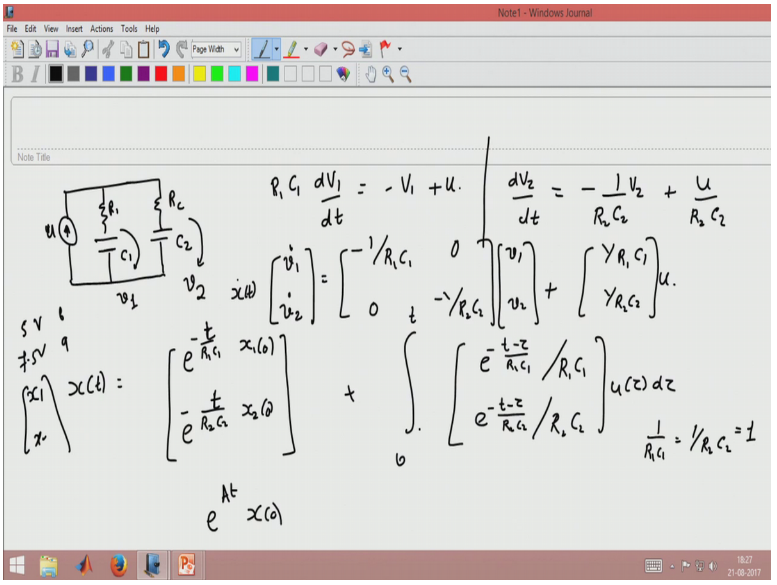

So, this is how it looks like. So, first is what are the states well the statesif I could call the voltages across these had states. So, I will call this as the voltage across the capacitor C 1 as V 1 and sorry as V 1 and then the voltage across the capacitor is C 2 as V 2. So, I can write down equations what is the; so, this is a current source the total currents add up you can say that R 1 C 1 d V 1 by d t is minus of V 1 plus u, similarly for d V 2 by d t. So, that is our say d V 2 by d t is just get this in the denominator minus one over R 2 C 2 V 2 plus u over R 2 C 2.

So, I just write down this equations now say in the state space form V 1 dot V 2 dot is minus 1 over R 1 C 1 0 0 minus 1 over R 2 C 2 with V 1 V 2 plus mod and I know is 1 over R 1 C 1 1 over R 2 C 2 times u. So, what does control mean in this case. So, if I look at this in terms of solving my differential equation for example, given an input current here can I from starting from in any initial condition can I get say C 1 to be at 5 volts and C 2 to be at say 7.5 volts or either ways it could be six here it could be a 9 here and so on all possible combinations can I stabilize them to some arbitrary voltages. So, how will my solution look like solution if I say? So, let me just call this in the standard notation this is x dot of t. So, x of t here would be e power minus 1 over R 1 C 1 x 1 of 0 x 1 is V 1 it. So, I am just calling this V is to be x and e power minus one over R 2 C 2 x 2 of 0 plus 0 to t although all the other term sets which will have the B u t minus tau d tau.

So, this will be e power minus t minus tau over R 1 C 1 and this entire guy divided R 1 C 1 and it is e power minus t minus tau over R 2 C 2 with the entire guy divided R 2 C 2 and I have u tau d tau. Now let me take a special case where the time constants are the same where 1 over R 1 C 1 is 1 over R 2 C 2.

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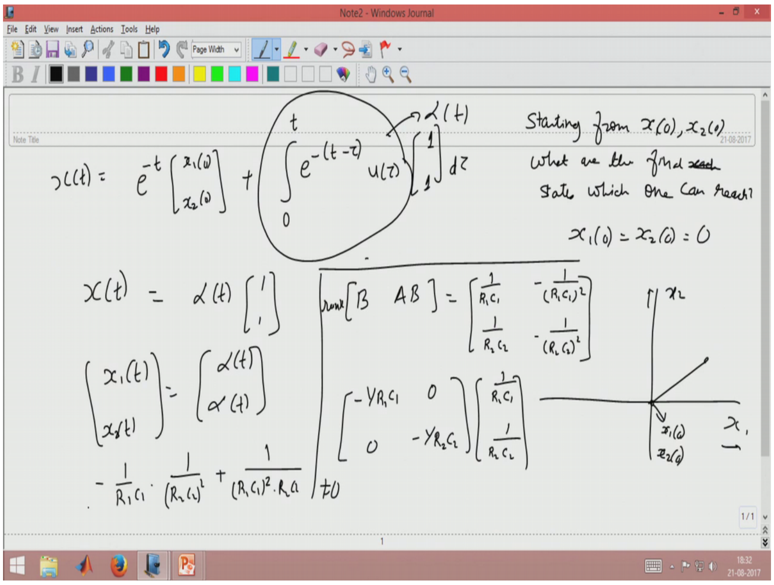

So, what happens in that case is my x of t would be e power minus t you can have the 2 initial state right x 1 naught x 2 naught plus integral. So, if I go back to the previous one here, right. So, this all these are the same R 1 C 1 equal to R 2 C 2, I let me even call this as 1 right, for simplicity and therefore, say all this will be one. So, x 1 if this is; so, this vector essentially is x 1 and x 2. So, x 1 is e power minus R my e power minus what is it e. So, from the first one, yeah; so, I just missed t here. So, this is like the e power a t x naught. So, I just miss the t here.

So, if I write down these 2 equations separately or I just maybe drop them into 1, so, how will the overall solution look like. So, this is integral. So, everything all the parameters are set to 1 0 to t e power minus t minus tau u tau all this multiplied by 1 and A 1 and A d tau because they are the same rate. So, this e power minus t minus tau this is one this is one this is one this is one. So, both are the same numbers essentially both of the same numbers multiplied by u t u u tau d tau. So, look at this and I will are starting from. So, this is our question right starting from x 1 0 x 2 0 what are the states final states I can reach final states which one can reach or starting from here what all points I can control my system too.

So, look at this thing, right. So, this entire thing here for a particular input would be just a just a constant, right. So, my x of t and let me also assume for simplicity that x 1 of 0 is x 2 of 0 is the origin does not really matter, right. So, in a linear system, I am here, if I can go from here to here, it also would mean I can come back here as a simple proof, but just assume that we know this or we will try to believe this. So, my solution all possible places where I can go starting from this, this is my initial condition x 1 naught x 2 naught. So, x of t is some function let me call this some alpha t alpha t 1 1, if I just write down the vector thing this would be x 1 of t x 2 of t is. So, both are at the same values alpha t alpha t and I started my problem by posing the question, can I get capacitor 1 2 to 5 volts capacitor 2 to 7.5 volts, well, the answer in this case is that I cannot; if this is; if C 1, if V 1 goes to 5, V 2 should also go to 5.

Now, this is happening in this special case only when R 1 C 1 and R 2 C 2 1 both have the same values for all others other values I can see that the system is controllable. So, I could just compute the B matrix A B and check for its rank this would be some matrix B is 1 over R 1 C 1 over 2 C 2 A times B, what is the center A is 1 over R 1 C 1 0 0 minus 1 over R 2 C 2 and I multiply this will be what is B this is R 1 C 1 R 2 C 2. So, this essentially is. So, I will have minus 1 over R 1 C 1 square and here I will have one over with a minus sign R 2 C 2 square, right.

So, the rank of this should be should be 2 which means I look at the determinant and it says minus one over R 1 C 1 1 over R 2 C 2 square this plus 1 over R 1 C 1 square R 2 C 2, this should not be 0, then this full tank and you see the special case where everything becomes 1, it goes to 0. So, it loses controllability. So, for all other values apart from this one, I can get to a variation like this and I can this can go V 1 can go at 5 V 2 to 7.5 and so on, but for some particular value of R 1, C 1, R 2, C 2, the system is not controllable well, let us also see what happens when the system is not controllable what does it mean in terms of transfer functions.

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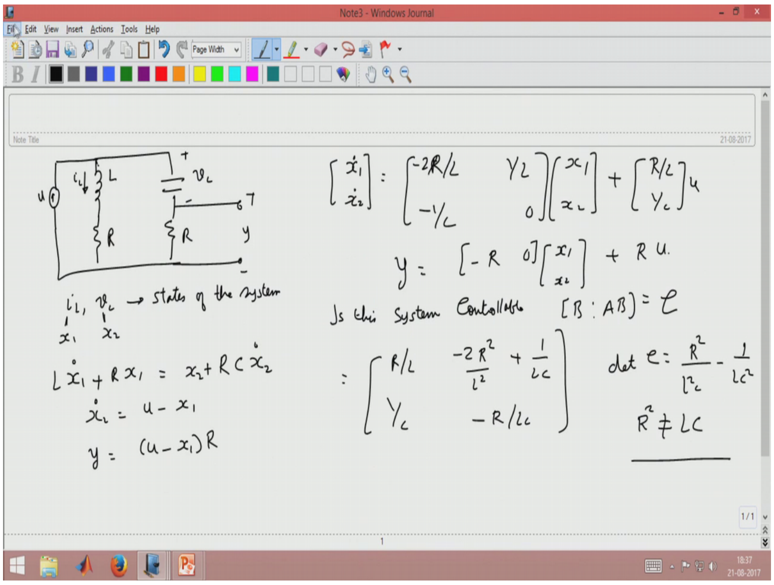

Let me instead consider a circuit which looks like this u L R, this is R this is R, I measure the output as the voltage here plus minus y this is plus minus V C the current here is i L and it is quite natural to choose i L and V c as the states of the system. So, let us first write down the state equations for this. So, I will well; how do you start with. So, you have; so, let me call this as x 1, I will call this as x 2, I write down the equations L x 1 dot plus R x 1, this is x 1 is i L from u. So, this will be x 2 plus R C x 2 dot. So, from here over the plus here and similarly x 2 dot is u minus x 1 and the final equation y is u minus x 1 times R which results in the state space equation which is like this x 1 dot x 2 dot is minus 2 times R over L, I have 1 over L minus 1 over C 0, I have x 1 x 2 plus R over L 1 over C times u and the output y in terms of C x plus d u is minus R 0 x 1 x 2 plus R times u.

Now, first is this system controllable well the idea would be to check the rank of B A B. So, this is usually also referred to as the controllability matrix. So, this C turns out to be in our case as R over L 1 over C and so this the B and then this is 2 R square over L square plus 1 over L C and minus R over L C and then you see that the determinant of this C is R square by L square C minus 1 over L C square or the system is controllable as long as R square is not equal to L c. So, for all other values the system is controllable what does this now mean in terms of transfer functions. So, we know that from the transfer function the relation between the state space and transfer function.

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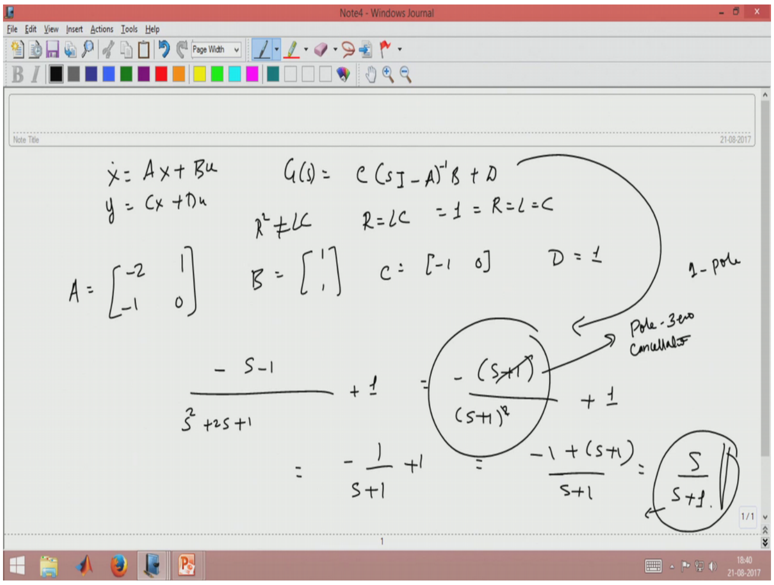

So, given x dot is A x plus B u and y equal to C x plus d u the transfer function G of s is simply C s I minus A inverse B plus d.

Now, let us choose a value or just use values of this the circuit parameters such as the system is not controllable. So, what it is said here is that R square should not be equal to L C should not be equal to L C. So, let us be and say well what happens if R equal to L C and say everything is equal to one you know a one is R 1 is L 1 is C. So, the system violates the controllability condition in that case my system matrix A becomes minus 2 1 minus 1 0 B is 1, 1 C is minus 1 0 and d is 1.

Now if I just do the formula, they substitute this here. So, what I get is C. So, this entire term C s, my minus A inverse B plus d is minus s minus 1 over s square plus 2 s plus 1 plus 1. This is also minus of s plus 1; what is on the denominator, this is s plus 1 whole square plus 1. So, this cancels out and what I am left with is minus 1 over s plus 1 plus 1 and this is what minus 1 plus s plus 1 over s plus 1 is s over s plus 1. So, what is happening, right? So, how many poles do I have in the system, well, if I just look at this transfer function after I blindly do all these computations will say the system has only one pole and therefore, we may say it has only one energy storing element well that is not true the original circuit had 2 states this guy i L and V c and one of the one of those has disappeared now.

So, what is what is happening when the system is uncontrollable is what we see here is a pole 0 cancellation. So, whenever there are uncontrollable modes of the system they get cancelled or there they encounter a pole 0 cancellation and what we see as the remaining transfer function is now the transfer function of only the controllable part of the system. So, when we talk about transfer functions, we do all the computations the simplifications what we end up with is the transfer function of only the controllable part of the system there is a proof for that, but we will not do that. So, this where this is would like to show you an evidence that well when there is a loss of controllability you see a pole 0 cancellation and what is left is the transfer function of only the controllable part.

So, let us go back here. So, so this is the condition to check given a system configuration with A matrix B and A, I can always find out with this rank condition if whether or not the system is controllable once I know that the system is controllable what could I do with that.

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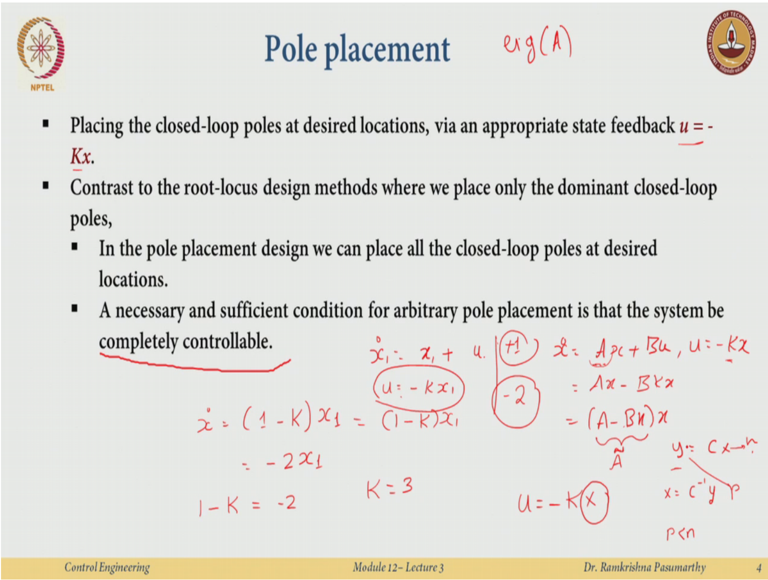

So, while we were doing the root locus analysis what we were essentially doing is does the root locus passed through the desired pole locations the desired dominant pole locations that was what we were interested in right and then we said well we designed a lead compensator or lie compensator and pull or push the root locus such that the root locus of the of the compensated system passes through the desired close loop locations. So, here also I can ask an appropriate questions, right.

So, pole placement here means placing the closed loop poles these are also what are the poles the poles are also the Eigen values of the system matrix A. So, can I by choose off by choice of an appropriate controller u equal to minus k x place all the Eigen values. So, what does this mean? So, I have x dot is A x plus B u and I say u equal to minus k of x is my control law and if I substitute this here I have a x minus B k x or in other words I have A minus B k times x. So, these are the Eigen values of my original system, the uncontrolled system now why are the control my new system matrix has changed to this new a tilde if I can call this. Now can I change or can I place the poles or the Eigen values of the closed loop system where this choice. So, this shows that well something is changing via B and k. So, this technique is called the pole placement techniques.

So, what is what is the contrast with root locus well in the root locus methods we were placing only the dominant closed loop poles whereas, in the pole placement design we can place all the closed loop poles at desired locations this is not surprising again this is a good mathematical proof, but we will not do this we will just learn this with some very small examples that are necessary and sufficient conditions for arbitrary pole placement what is arbitrary, if I have n poles here or n Eigen values can I place all Eigen values can I shift all the Eigen values all the n Eigen values to desired locations. So, this is arbitrary pole placements. So, when can I do this I can do this only if the system is completely controllable, right.

So, if I just write down some equation let us say just say I have x 1 dot is say x plus some k times u. So, what is the Eigen value of the un-control system the Eigen value of the uncontrolled system is at plus one now if I say can I place this at say some value minus 2 well does there exist a k such that I can place this at minus two. So, how should the; my system which is controlled the control system look like the closed loop system. So, does there exist x dot or say does there exist u sorry you equal to this is also x 1 by the way minus k x 1 such that the closed loop system closed loop system is what x dot is a minus sorry, let us get this k also this looks nicer now. So, x dot equal to x 1 plus you can I use a controller like this in such a way that the closed loop system is a minus k times x 1, the closed loop system looks like this minus 2 times x 1.

Now, what is known to me I know this a here. So, this A is this one. So, this is one minus k x 1. So, this A is just one over here directly A is 1. So, I just am solving a linear equation, alright. So, 1 minus k is minus 2. So, this means that k is three. So, put k equal to three here you just arrive at this one and this has starting with the open loop pole which was at plus 1 with why are you equal to k x 1, I could place it at minus 2 right. So, this is essentially the problem of pole placement, right and what I said a necessary and sufficient condition for arbitrary pole placement is that the entire system be controllable.

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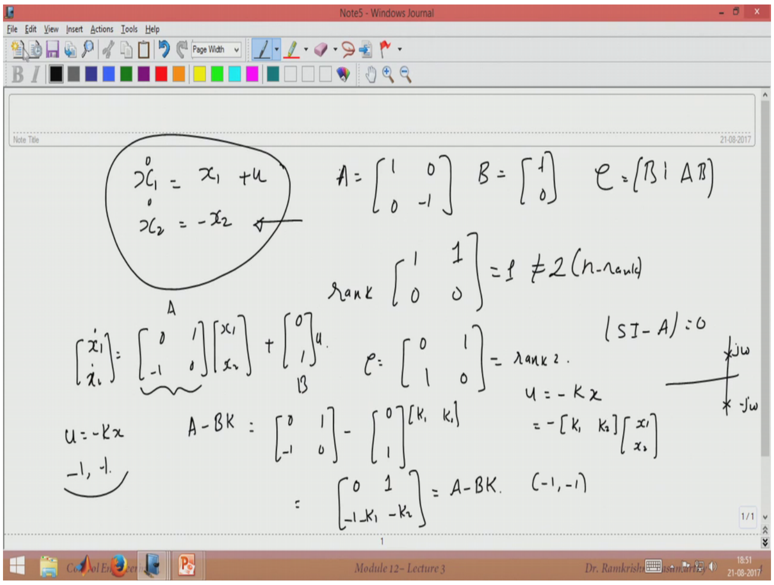

So, let us do a little more some more example, let us say continue the same one say x 1 dot is x 1 plus u and say I put another state x 2 dot is say minus x 2.

Now, can I do anything with the poles of x 2 well the answer is no because there is no control input influencing in the x 2. Now look at the controllability matrix. So, this system has a as 1 0 0 minus 1 B is 1 and 0. So, what is B what is a controllability matrix C equal to B A B. So, that is B is 1 0 and A times B is what 1 and 0 and this has rank of 1 not equal to 2, it is not full rank this is this is the n rank which in this case is 2. So, I can see here that the system loses rank it is not completely controllable therefore, my claim that for arbitrary pole placement the system should be completely controllable well this is 1 example of that this is not a bit proof anyways of that. So, just go back to what we said here that a necessary and sufficient condition for arbitrary pole placement is that the system is completely controllable.

So, let us do 1 last example here. So, let us say, I have a equation which is like this x 1 dot x 2 dot, this is my system 0 1 minus 1 0 x 1 x 2 plus 0 1 u. So, this is essentially the un-damped pendulum, the state space equation you could have seen this in the previous lectures on state space. So, my question here is can I place the poles of the closed loop system why are u equal to minus k x at minus 1 and minus 1. So, how do I go about doing this first I check is the system controllable well, I guess I can check the rank of B and A B. So, this is my B this is A. So, the controllability matrix C is 0 1 what is A B? This is 1 and this is 1 times 0, this is 0 and this is of rank 2, right. So, well what I can say here is that I can actually do this.

Now, the system is completely controllable there can I therefore, I can place both the poles at minus 1 minus 1, well where are the poles of the open loop system that is just you look at the solution of the characteristic equation s i minus a equal to 0 and you find that the poles or the Eigen values are just yet plus minus j omega. Now can I choose u equal to minus k x or well, in this case since k this is a system of 2 variables as a k 1 k 2 x 1 x 2 such that this holds. So, how will we solve the closed loop system look like? So, how will A minus B k look like? This is A is 0 1 minus 1 0 minus well what is B? B is 0 1 and k looks like k 1 k 2 and this is what; this is 0; this is 0 here and what will I have here? I will have A 1, here nothing changes in the first part, here I have minus 1 minus k 1 and here I will have minus k 2 this is my closed loop system matrix A minus B k.

Now, I want the Eigen values of this system to be at minus 1 minus 1. So, what do I do? So, what is desired is the closed loop system to have Eigen values of minus 1 minus 1.

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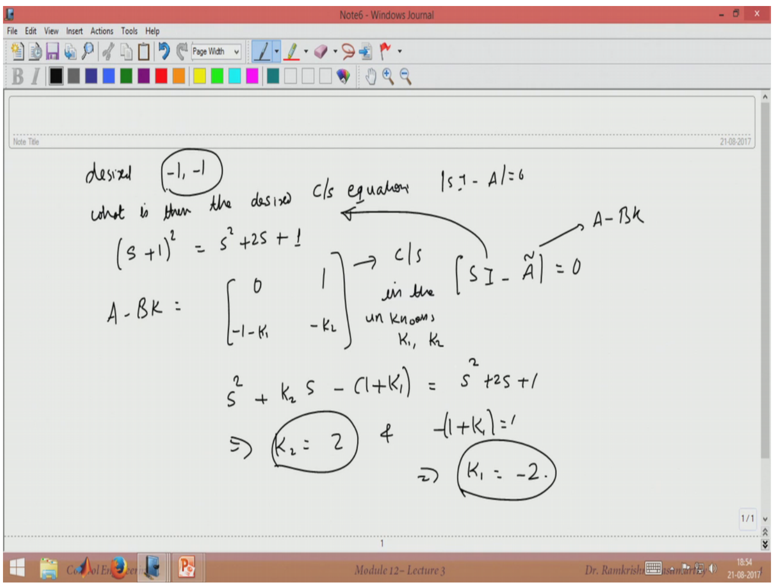

Which means well what is then the desired characteristic equation s i minus A. So, that is; so, I know that well since the poles are should be processed at minus 1, it should be something like this, s square plus 2 s plus 1. Now what do I have with me well my A minus B k now looks like this, this is 0 1 minus 1 minus k 1 minus k 2.

Now, where are these roots? So, what is the characteristic equation from this, I just want to find s i minus A equal to 0, right, oh, this is the characteristic equation, this characteristic equation should just match this one this is a desired one, right, this is what I want this is the characteristic equation in the unknowns once k 1 and k 2. So, this is actually this A tilde, right. So, just a little A tilde is actually A minus B k. So, if I write down the characteristic equation. So, this without after all the computations the characteristic equation will look something like this s square plus k 2 times s minus 1 plus k 1 and this should be equal to the desired one that is s square plus 2 s plus 1 which means that k 2 is 2 and minus 1 plus k 1 is 1 which is actually means that k 1 is minus 2.

So, with this controller, I can place my closed loop poles to be at the desired location and how do I check; if I can do this or not it is just where the system being controllable or not. So, this is about controllability now if you look at what was this based on this was essentially based on listener. This equation; this controller u equal to minus k x which then assumes that I know the entire of x at all xs are measurable, then I can feedback these states, but usually the states are not up for measurement. The states are not available for measurements what is available instead is the output y and this y, let us simply assume for the moment that this d is 0. So, so what is available for measurement is y equal to C times x and I can ask myself a question given this measure values, right can I get what is x, well you can say an answer that then x is C inverse y, this is true when well C is invertible or if y; the dimension of y is the same as the dimension of x.

But usually this is not the case maybe there are n inputs and P outputs, right. So, this is P vector, this is an n vector and usually P is less than n, then if I were to do this state feedback control law, based on these measurements, I should somehow be able to get an estimate of the states from these P measurements estimate of these n states via P measurements.

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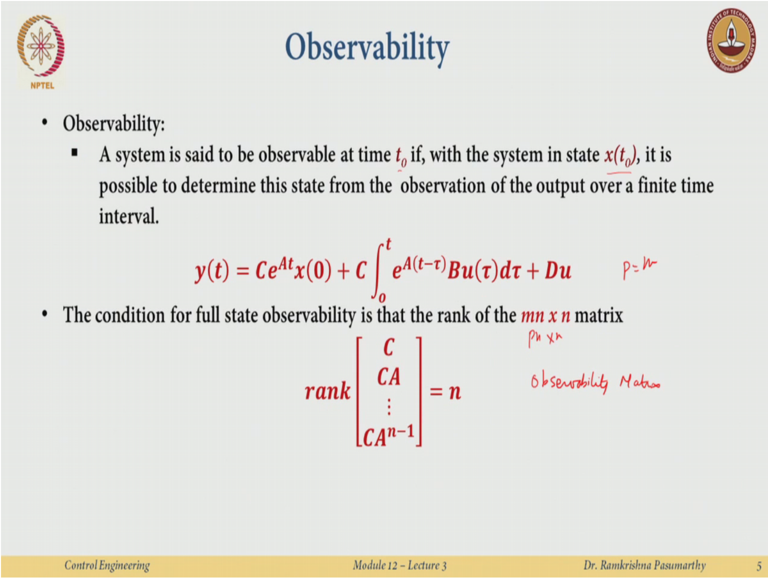

And that leads to me to define what is observability, well again, I am not really go into details of anything of this, just to see what is happening; A system is said to be observable at time t 0, if the system starting with x t 0; it is possible to determine this state from observations of the output over a finite interval of time from 0 to t, right, essentially the problem is given x of t. So, given set of observations can I construct x t naught, well it turns out that again. So, this actually is this measurement here, I have measurements y of t from 0 to t, can I consider x t at 0, this turns out that if the rank of this matrix will; here I assumed that P equal to m and the number of outputs is equal to the number of inputs which is generally should not be the case otherwise it just be a P n cross n matrix.

So, nothing this is a loss of generality here by the way. So, this matrix which is called the observability matrix is of rank n. So, once this is of rank n, then I can get an estimate of the state via as measurements of the output y and once I know what the states are; I can always design a state feedback control law to place the poles of my closed loop system at the appropriate locations. So, I will not do any of this proof. So, why this rank condition is true or why the complete controllability condition is true for placing the poles at arbitrary locations, but just to give a general overview of what is the concept of controllability and observability.

Similarly, to what happened in the case of controllability, observability also happens when there are poles 0 cancellations which you could again check with the example which we had done. So, that really brings us to the end of this course and well I had fun recording these videos, I had fun interacting with you over the discussion forum, sometimes over the hangouts and I hope it would be useful to you in some form or the other for gate exams, for your regular university exams, people who are teaching elsewhere and any kind of feedback is welcome. You could write to us; what you would like to improve in the course; what you liked, what you did not like, what you would like us to elaborate a little more, but overall, I hope it was it was a good experience for you.

Thank you very much and see you soon.