

Control Engineering
Dr. Viswanath Talasila
Department of Telecommunication Engineering
Ramaiah Institute of Technology, Bengaluru

Module - 12
Lecture - 02
Part 2
Linearization of state space dynamics

Hello and welcome to the continuation of lecture 2 of module 12, which is on the linearization of state space dynamics.

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Previous lecture (module 12, Lecture 2)

$\dot{x} = f(x)$ *nonlinear system*
equilibrium points

↓
 $J_{n \times n}$ *Jacobian matrix*

↓ *evaluate the J at an eq point*
 $\dot{x} = Ax$

$\frac{dx}{dt} = \cos(x)$

$\frac{dx}{dt} = 0$

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So, if you remember what we have done in the previous lecture is part one of lecture 2. So, basically what we did was to consider a non-linear system. So, we model the non-linear system right then we computed the equilibrium points and we computed the Jacobean which is you considered all the first derivatives of the Taylor series and in general it is a n cross n matrix depending on the number of states that you have. So, you compute the Jacobean matrix, and then in order to compute the A matrix you basically evaluate the Jacobean all right at an equilibrium point right.

So, this is the basic theory which we saw in the previous class. Now before I go ahead with 2 very interesting examples for today I just want to tell make a small note about the equilibrium points itself. Now in the non-linear case which is what we are dealing with

in general you do not have a single equilibrium point, you can have multiple equilibria. So, if you consider a non-linear dynamical system it can have multiple equilibrium points I will give a small example to illustrate that let us say I have got a non-linear system of the following type all right.

Now, if you actually graph this. So, you have x axis and you have the cost of x on the y axis x of course, will be in degrees or radians; however, you like and. So, cost 0 is 1. So, you will actually get a graph like this right and. So, on now if you recall the definition of an equilibrium point it is basically where the derivative goes to 0 and equilibrium point is a point on your in your in the space of the independent variable. So, it could be any one of these points where the derivative goes to 0.

Now, you see that for the simple example of the cosine of x . So, if you look at the cosine of x and you try to compute the equilibria, you see that you have multiple locations the angles that is where the derivative of the dynamics actually goes to 0. So, this would be one place you have this you have this so on and so forth. So, for this you actually have an infinite number of equilibria depending on as long as x you are keeps on repeating. So, for a non-linear system it is not necessary that you have a single equilibrium point you can have multiple equilibria all right.

So, with that in mind let us actually go to 2 interesting examples for today.

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Predator-Prey Model





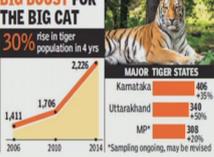
Tiger in the Bandipur forest



Male sambar deer in Bandipur forest

BIG BOOST FOR THE BIG CAT

30% rise in tiger population in 4 yrs



Year	Population
2006	1,411
2010	1,706
2014	2,224

*Sampling ongoing, may be revised

MAJOR TIGER STATES

Karnataka	406	+35%
Uttarakhand	340	+50%
MP*	308	+25%

Tiger is a symbol of wilderness and well-being of the ecosystem. By conserving and saving **tigers** the entire wilderness ecosystem is conserved

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The first example is a predator prey model and before going ahead with the mathematics of this the reason why we introduced you know tigers, deers and other kind of things you know in lecture in control systems or in control engineering is actually twofold. One is that the beauty of control systems is that it allows us to model in principle any physical system, and the physical system need not be a chemical process, it not be just a electrical system, it can even be an ecological problem as what I am going to present today it can even be a problem in economics right and so on that is one thing.

The second is that when we look at the predator prey model today, it is important to remember that the kind of dynamics which I will be talking about are also applicable when we model certain behavior in economics in social behavior and so on. So, with that in mind, let us go ahead with the predator prey model let us take the specific case of tigers and deers. So, these are 2 pictures which I have taken of the net and they show a tiger and the deer in the Bandipur forest and the male Sambar deer is native to the Bandipur forest and one of the reasons why people do a lot of modeling for these tigers or predators in forest is because the tiger is basically a symbol of the well being of the ecosystem.

The tiger depends on the prey the prey depends on say trees and grass and other kind of things they depend on other subspecies. So, if the tiger is actually conserved you are essentially conserving the entire ecosystem. So, that is why the predator plane models will hiss talk where historically develop. So, what is the predator prey model first let us look at the prey ok.

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The Prey Model

Let $H(t)$ represent the number of deers (prey)

$\frac{dH}{dt} = r_h H$

In absence of predators, the deer population keeps increasing

Handwritten notes: $r_h > 1$ and reproduction rate

Graph: A coordinate system with a vertical axis and a horizontal axis labeled $t \rightarrow \infty$. A red curve starts at a point on the vertical axis labeled '50' and increases exponentially as it moves to the right.

Logos: NPTEL (National Programme on Technology Enhanced Learning) and RAMAIAH Institute of Technology.

Page footer: Control Engineering, Module 12 - Lecture 2, Dr. Viswanath Talasila

So, let H of t represent the number of deers and it is a function of time. Now in this particular context of the predator prey model t is not in seconds, it is typically in months or in years ok, that we need to keep that in mind.

So, if H of t is a number of deers in the absence of predators we are going to make an assumption, that the deer population keeps on increasing which is an assumption I am making in my modeling and that essentially follows this particular equation. It basically says that the rate of change of the population of the deers as a function of time and time being in months or years is a positive rate of change where r_h is greater than 0, and h is basically the population of the deers are is r_h is actually greater than 1 to be more specific right and this particular parameter r_h times H , r_h essentially is the reproduction rate of the deers or the growth rate of that population a right.

So, as long as r_h is greater than one we have the population of deers going on increasing which is basically this what this equation talks about over here.

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The Prey Model

$$\frac{dH}{dt} = r_h H$$

In absence of predators, the deer population keeps increasing

$\rightarrow \infty$
K very large

$1 - \frac{H}{K} = 1 - 0 = 1$

$H \ll K$

$\frac{dH}{dt} = r_h H$

Obviously population cannot increase forever!

$$\frac{dH}{dt} = r_h H \left(1 - \frac{H}{K} \right)$$

≤ 1

K represents the maximum population of deers (in the absence of tigers and absence of natural death)

For the remainder of this talk we assume that $K \rightarrow \infty$ i.e. the forest has infinite resource capacity

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So, we know that this cannot be really true in general, that the population of the prey keeps on increasing because what this particular equation seems to be telling us is that as time goes to infinity or a very large time like our maybe 5 years 10 years and so on, the population of the deers which was say for example, 50 in the Bandipur forest, according to this equation would keep on increasing right which is not true because your forest ecosystem cannot support such a large population of deers right, and the way that we would model this constraint is to say that, let k be the maximum population of deers which this particular jungle or the ecosystem can actually support. And the number of deers cannot go more than k and the way you would model that is by this expression over here and what this expression basically tells us. So, this part is the same as what we have seen before the growth rate or the reproduction rates of the deers, then have a 1 minus H/K. H is generally less than k in the case where k is very large.

In the example where k is very large let us say even approaching infinity, this term goes to 0. So, 1 minus H by K becomes 1 minus 0 which is 1, and basically says exactly what we talked about before that the growth rate of the deers is constant and it keeps on increasing and it can reach a very large number possibly infinite. In general H will be much less than K because the forest cannot really support such large population of deers there is not enough food basically. And as long as H is less than K this expression over here is less than 1 all right and. So, that basically means that I do have a constraint on the total number of population of deers which my jungle can support.

However for the remainder of this stock without any loss in generality, for the remainder of this stock I will assume that k is very large meaning I have this hypothetical jungle which can support as many deers as possible it is not a correct assumption in reality, but we will go with that for now to just show the modeling capability of state space systems. So, this slide basically concludes with the equation which we are going to consider as dH by dt equal to r_h times H . So, this is the dynamics of the prey in the absence of the predators ok.

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The Prey Model

Enter the predators !!






In the presence of predators, the prey population can change dramatically

$$\frac{dH}{dt} = r_h H - aLH$$

a is an interaction parameter between the tigers and deers

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Now, in the presence of predators you see that this growth rate of the deers is now compensated by interaction say parameter between the deers and the tigers where L of t is the number of tigers in the jungle, H of t of course as we saw before was a number of deers, and the parameter a is the interaction parameter between the tigers and the deers. So, we can see that now this equation basically says yes I do have a growth rate in the deers because of reproduction and there is plenty of available food for them, but then I have a decay rate over here this negative term, this is an interaction when the tiger basically hunts the deer and the population of the deers go down all right.

So, this is the. So, this is our prey model. So, dH by dt equal to r_h times H minus aLH and it is actually an interesting exercise I would like you to try this, this parameter over here the product of the number of tigers and the number of deers is your non-linear term this is a perfectly linear term r_h is a positive constant H of course, is the population of

deers, this is the non-linear term over here I would like you to prove this on your own. So, that is the prey model, now what about the predator.

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The slide is titled "The Predator Model" and features the NPTEL logo on the left and the RAMAIAH Institute of Technology logo on the right. The text on the slide reads: "Let $L(t)$ represent the number of tigers (predator)". Below this, the differential equation is given as $\frac{dL}{dt} = -r_1 L$, with a red underline under the L in the denominator. To the right of the equation, it says "Predators die without any prey (exponentially starve to death)". At the bottom left, there is a URL: <http://www.math.psu.edu/tseng/class/Math251/Notes-Predator-Prey.pdf>. The footer of the slide contains the text "Control Engineering", "Module 12 - Lecture 2", and "Dr. Viswanath Talasila".

Now, the assumption that we make for the prey model is that there is always plenty of food available which is basically grass and other leaves which the deers eat, but for the predator it needs the prey right now let us say that the predator is for whatever reason it is unable to actually kill the prey and then have food, in which case you are actually going to see that the predators are going to starve to death. And let us assume this may not be always true let us assume that the rate at which they starve to death is an exponential factor an exponential decay, in which case we get this expression. So, when L of t is a number of tigers in the jungle. So, the rate of change of the population of the tigers is this negative growth rate over here ok.

So, this is in the absence of prey

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The Predator Model

Let $L(t)$ represent the number of tigers (predator)

$$\frac{dL}{dt} = -r_1 L$$

Predators die without any prey
(exponentially starve to death)

In the presence of prey, the predator population can recover

$$\frac{dL}{dt} = -r_1 L + dLH$$

product $L(t)$ & $H(t)$
non-linear

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In the presence of prey the population of the other tigers will; obviously, recover right now they get food and they can they are healthy they can reproduce and so on. So, every time I have an interaction between the tigers and the deers, the population of the tigers over a period of time once years will be a positive growth rate, this is good for the tigers not. So, good for the deers, but it is very good for the tigers; now, this is the equation the dynamics for the population of the predator, and again as in the previous case I would like you to see and try to prove on your own that this product term between 2 variables which are actually the states between 2 variables L of t and H of t is actually a non-linear term, this is actually what makes the dynamics non-linear very good.

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Predator-Prey Model



Two coupled differential equations

$$\frac{dH}{dt} = r_h H - aLH \quad \frac{dL}{dt} = -r_l L + aLH$$

In matrix form we have

$$\begin{bmatrix} \frac{dH}{dt} \\ \frac{dL}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} r_h & -aH \\ aL & -r_l \end{bmatrix}}_{f(\vec{x})} \begin{bmatrix} H \\ L \end{bmatrix}$$

$$\dot{\vec{x}} = f(\vec{x})$$
$$\vec{x} = \begin{bmatrix} H(t) \\ L(t) \end{bmatrix}$$

nonlinear dynamics

So, now we have 2 equations and it turns out these equations are coupled right. So, you have H which is a function of both H and L and then dl by dt which is a function of both H and L. So, if you just rewrite this in a matrix form we essentially have this expression over here this again your classic state space thing and this is not the a matrix as we call it in the linear case. So, this is basically \dot{x} equal to f of x , where my state vector x is basically H of t and L of t all right and this is the f of x bar now here very good.

So, what we have done so far now is to basically capture and to actually write down the non-linear dynamics of the predator prey problem right it is a first step. Now the second step would be to compute the equilibrium points and we go to the Jacobean and so on. So, let us see how we do that.

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Simulation and Analysis of the Predator-Prey equations

Two coupled differential equations

$$\frac{dH}{dt} = r_h H - aLH \quad \frac{dL}{dt} = -r_l L + aLH$$

- In general, every time an “interaction” takes place between tigers and deers, multiple things can happen
 - Deers usually escape (the hunt success is usually low)
 - When a “proper” interaction happens
 - the deer population **definitely** reduces
 - But the tiger population will **not** increase
- Thus, the interaction term cannot be the same for both
- Further, deers reproduce much faster than tigers !

Thus the new coupled differential equations are

$$\frac{dH}{dt} = r_h H - a_1 LH \quad \frac{dL}{dt} = -r_l L + a_2 LH$$

with $a_1 > a_2, r_h > r_l$

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Before we do that there is an important modeling assumption which we should make. So, if you see in this slide we assumed, that the interaction parameter between the tigers and deers is the same now this is not true in general and it is not true for an interesting reason because every time an interaction takes place between the tiger and deer. Typically the deers usually escape most hunts are not successful; however, when an interaction actually does take place meaning the tiger manages to make a kill, the deer population will definitely reduce immediately by at least one depending on how many the tigers kill, but this does not mean the tiger population is going to increase immediately right.

So, the population of deers decreases because one of the most killed, but the population of tigers will not increase immediately it takes a much longer time. So, this interaction parameter a it cannot be the same. For both and more specifically in our case the interaction parameter for the deers will always be greater than the interaction parameter for the tigers furthermore the growth rate of the deers is much larger than the death rate of the tigers again it is an assumption and these assumptions are usually proven to be valid in actual field studies. So, we take this basic model and with the assumptions which we have placed over here, we get the new model which is exactly it looks the same we made certain assumptions on the parameters right.

So, we basically assume that a_1 is larger than a_2 , where a_1 is the decay rate for the deers when the interaction happens and r_h and r_l are the respective parameters for each of them. So, now, we take this set of coupled differential equation and we will compute the equilibrium points.

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The Jacobian comes back...

$$\frac{dH}{dt} = r_h H - a_1 L H$$

$$\frac{dL}{dt} = -r_l L + a_2 L H$$

There are two nonlinear terms in this coupled state space system

- 1) We can compute the equilibrium points
- 2) Linearize around the equilibrium point

Handwritten notes:
 $\dot{x} = f(x)$
 x_0 is such that $f(x_0) = 0$
 $\dot{x} = 0$ at x_0

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Now the and then of course, linearize it and we go ahead with that. So, the way you couple. So, the way you compute the equilibrium points we have seen before for a system of the type $\dot{x} = f(x)$, and equilibrium point x_0 is such that when you evaluate the function f at x_0 it is equal to 0, which is basically saying that $\dot{x} = 0$ when evaluated at x_0 all right.

So, we are going to the same here, we are going to set each of these parameters to be equal to 0 and we will see what is the solution of these 2 equations and that is of course, fairly simple because in this case is just algebraic expressions over here.

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The Jacobian comes back...

$\frac{dH}{dt} = r_1 H - a_1 LH$
 $\frac{dL}{dt} = -r_2 L + a_2 LH$

There are two nonlinear terms in this coupled state space system

- 1) We can compute the equilibrium points
- 2) Linearize around the equilibrium point

Handwritten notes:

$L(t) = 0$
 $H(t) = 0$
 $x_0 = (0, 0)$
 trivial
~~not interesting~~

$\frac{dH}{dt} = 0 \Rightarrow r_1 H - a_1 LH = 0 \Rightarrow$ ①
 $\frac{dL}{dt} = 0 \Rightarrow -r_2 L + a_2 LH = 0 \rightarrow$ ②
 $L(t), H(t)$

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So, we said dH/dt equal to 0, we will see that this basically means $r_1 H$ minus $a_1 LH$ is equal to 0. So, this basically gives you let us just call it equation number one, we will come to the other one later and we do the same thing where dL/dt , we set this equal to 0 and this basically would mean that $r_2 L$ plus $a_2 LH$ equal to 0.

We have 2 equations and 2 unknowns, the unknowns are L and H and remember these are functions of time. So, we have 2 equations and 2 unknowns and the unknowns are L and H . The trivial solution of this guy being equal to 0 and this equation being equal to 0 the trivial solution is $L(t)$ is equal to 0 and $H(t)$ equal to 0. So, this is one of the is an equilibrium point and remember in the previous example which we gave we had a single I mean a one dimensional system. So, the equilibrium point was just one single scalar value now the equilibrium point x_0 is basically 0 comma 0 all right. That is one equilibrium point and this is the trivial equilibrium point this is not very interesting, but because it basically says that you start off with 0 tiger 0 deers in the jungle and of course, what will happen to your dynamics, they will always remain at 0 this is not it this is not interesting that is what.

Now, what about the other equilibrium point if we solve these 2 expression these 2 algebraic expressions.

coupling parameter in the previous example of the non-linear system we saw that we had a coupling parameter $a L H$.

So, this was a non-linear term now those terms have disappeared over here, when you evaluate the Jacobean at the equilibrium point. So, now, we have \dot{x} equal to Ax . So, this entire Jacobean at the equilibrium point is a and we would always first thing you would always do is to compute the Eigen values, and when you compute the Eigen values of this matrix A in this particular case you see that the Eigen values lie on the imaginary axis, and when you actually do the simulation of this which I will show you in a few minutes you will see that these Eigen values they produce a very interesting behavior that the equilibrium point which we have considered, that this one over here the equilibrium point actually behaves in a very unique way which is called as a stable center ok.

So, how would that look? We saw in one of the previous examples in the previous lecture that an equilibrium point one of the definitions is that, if you start off at an equilibrium point your dynamics always remains in that equilibrium point for all time in the future stable center, has a very unique understanding or notion of what the dynamics would mean, and it basically would say that if this is an equilibrium point. So, when I evaluate these 2 values right and I get this particular point over here. So, this would be my r_l by a 2 and r_h by a 1, and if you start at the equilibrium point as by the definition and we will also see in this in the simulation shortly the dynamics will tend to stay at this point only it does not go anywhere else.

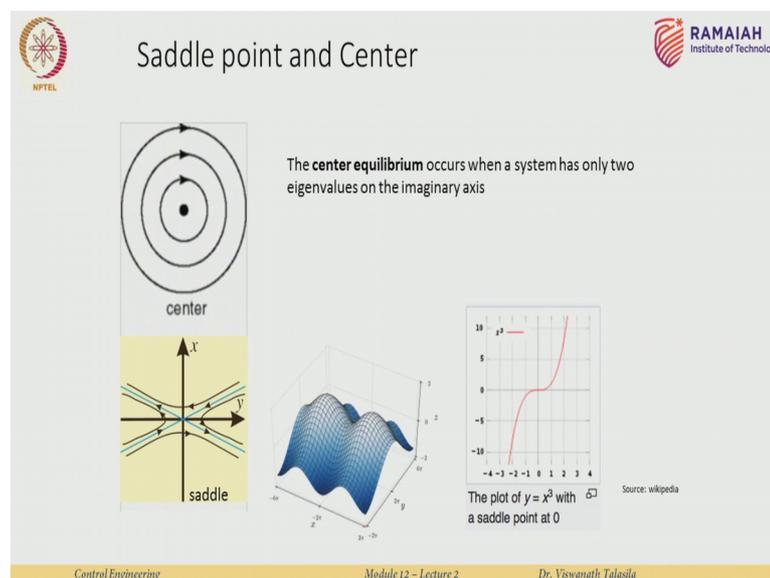
Now, because it is what is called as a stable center, it has a following really interesting behavior. If you start at any other initial condition any other initial condition, the final behavior of the dynamics when you plot H of t versus L of t , this behavior will always be a closed orbit or a closed response if you want to call it around the equilibrium point. Start at any other initial condition and it will go around that equilibrium point, start over here it will go around that equilibrium point. If you start somewhere over here it would actually tend into this part of the axis and then again create a new orbit and so on.

So, the concept of a stable center is that you have an equilibrium point as before, if you start your dynamics initial condition at the equilibrium point the dynamics will always stay there. You start anywhere else around in a neighborhood of the equilibrium point or

anywhere else in your phase space, that is a space of your state variables anywhere else it will be a closed orbit about that equilibrium point. And we will see very quickly that these closed orbits basically mean that the dynamics are basically oscillating with respect to each other. So, this orbit basically means that when the population of the deers is increasing, say assume I am at this particular point, and the population of the tigers is actually reducing at the same time you see the population of the deers is increasing, and it keeps increasing until the population of tigers have completely come down, at this particular point the population the tiger starts going up and the deer population starts decreasing.

So, if you actually plot this response you will actually see that if the tiger population was going like this, the deer population would follow this, but with a slight delay. So, every time the tiger population goes up the deer population would come down, and when the tiger populations coming down the deer population would actually go up. So, you will see this kind of a response; that is basically an outcome of how these closed orbits are there.

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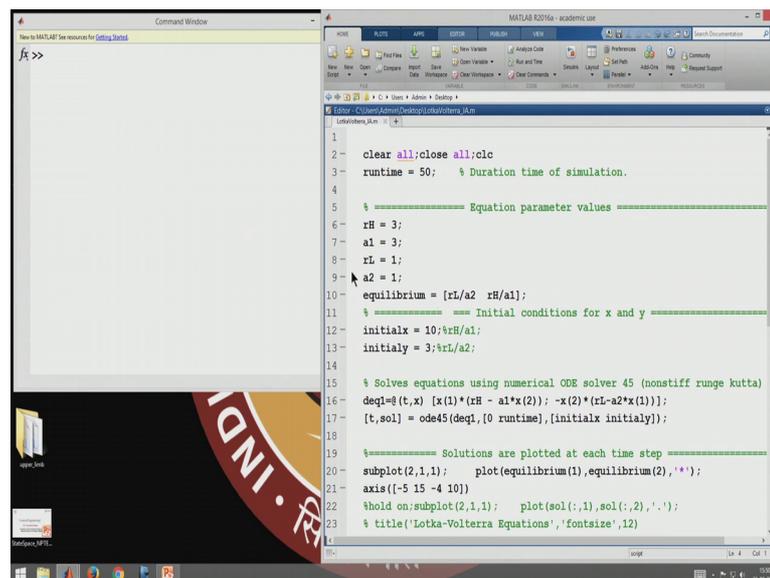


So now, we talked of the center, it turns out the other equilibrium point which we did not consider which was a trivial one where did that go yes. So, where L is equal to 0 and H is equal to 0, it has a specific mathematical property that is called the saddle we will not

really go into those things, we are more interested in this equilibrium point which actually has some physical meaning ok.

So, now that we have our linear system of this form, let us actually and we know that the Eigen values are unlike this and we expect to see a stable center, let us actually go ahead and simulate this in MATLAB and we will see what kind of responses we going to get.

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```
1
2 clear all;close all;clc
3 runtime = 50; % Duration time of simulation.
4
5 % ===== Equation parameter values =====
6 rH = 3;
7 a1 = 3;
8 rL = 1;
9 a2 = 1;
10 equilibrium = [rL/a2 rH/a1];
11 % ===== Initial conditions for x and y =====
12 initialx = 10;%rH/a1;
13 initialy = 3;%rL/a2;
14
15 % Solves equations using numerical ODE solver 45 (nonstiff runge kutta)
16 deg1=@(t,x) [x(1)*(rH - a1*x(2)); -x(2)*(rL-a2*x(1))];
17 [t,sol] = ode45(deg1,[0 runtime],[initialx initialy]);
18
19 % ===== Solutions are plotted at each time step =====
20 subplot(2,1,1); plot(equilibrium(1),equilibrium(2),'*');
21 axis([-5 15 -4 10]);
22 hold on;subplot(2,1,1); plot(sol(:,1),sol(:,2),'-.');
23 % title('Lotka-Volterra Equations','fontSize',12)
```

So, this is a code which basically simulates your predator prey model, it is called the Lotka Volterra because that is the formal name of this one. So, the basically the code runs like this we are going to simulate it for 50 seconds it is actually a little bit too long we will just do for 30 seconds and the parameters which have taken. So, the growth rate of the deers is three the natural death rate of the tigers is 1. So, that is a minus r_l times L of t and then the interaction parameters for the deers is three and for the tigers is 1 ok.

Now, these have been chosen not with any specific ecological model in mind it is more a mathematical simulation, but if you have actual data from say any forest system you can actually substitute these appropriate values and the equilibrium point is defined over here it was r_l divided by a_2 and r_h divided by a_1 . So, that is over here, now let us say the initial condition. So, this is the number of deers and this is the number of tigers and the initial condition we usually assume that the number of deers is much larger than the

number of tigers right otherwise all the deers will die eventually tigers will starve they will also die, but that really does not happen in most ecological conditions.

So, number of deers is more than number of tigers as an initial condition, and then we use this function from MATLAB called the ode 45. So, ode 45 calls a function called deq one which I have defined in the previous line line number sixteen over here and it says that we need to simulate it for about 30 seconds with these initial conditions. For those of you for whom it is not clear what is this ode 45 what is this deq 1, the best thing to do is to just type help ode 45 in MATLAB and it gives an excellent introduction to what you need to do and it. In fact, gives a very specific example as well ok.

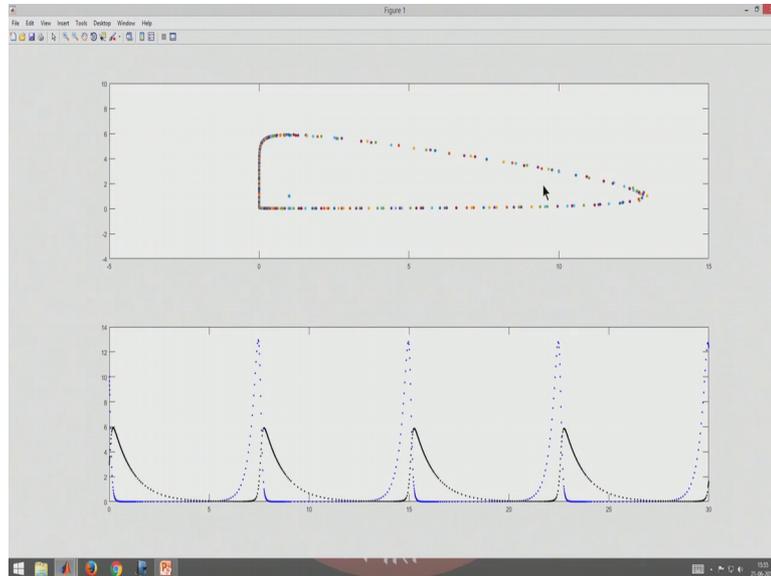
So, you either in this. So, I can go to Google and check any of the examples for ode 45. Let us see how the simulations look like. So, what I am doing here is to first in line number 20 I am plotting the equilibrium point with as a star, and then when I start my simulation. So, you can see that in this particular line over here I have computed the solution of my dynamical system right. So, the solution basically is h of t and l of t and for h in MATLAB I have called it as x of 1, for l I have called it as x of 2. So, x of 2 and the number of tigers and x of 1 are the number of deers.

So, the solution basically will have 2 columns, the first column will be the number of tigers the number of deers as a function of time second column will be the number of tigers as a function of time there will be the solution. T is of course, the time vector which is returned by this ode 45, it is an internal MATLAB function. So, now, this small piece of code it takes each value of my solution each population of tiger each population of deer plots it on the graph, then it holds the graph moves to the next value of the of the tiger population deer population again plots it and so on. It is basically an iterative process this final thing over here it lots the time series of x and y, like what I told you before in the slide. So, this will basically part plot the population of tigers as an up as a function of time on the x axis, and population of deers as a function of time ok.

So, we will see how all of this looks like; now the interesting thing about this particular piece of code is that it allows me to actually see how this plot actually rotates or evolves. So, let us go and click on run and please remember this code will be available for all of you; you can see that this is how the dynamics is actually evolving from the particular

initial condition which we have chosen of 10 and 3, which is somewhere over here right we started at this point and then it evolves like this.

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So, this is the closed orbit which I was talking about initial condition was somewhere at this point and then it just goes around in this.

Now, we can see in black I have plotted the number of deer, in blue I have plotted the number of tigers and always noted there is a slight fish difference between the 2, and that is basically we can explain that as follows. So, let us start at this particular point on the time axis here on 6, it could be months or years around 6 and we can see that the number of deers keeps increasing very fast because there are very few tigers left in the jungle right. So, the number of deers keeps on increasing and at some particular point the tigers have now access to a lot of deers, and it is very easy for them to hunt.

So, then the tiger population also starts to increase all right and when the tiger population starts to increase and reaches a certain critical number, now the tigers are able to hunt the deers very well, and the deer population starts to fall over here right. So, this cyclical process keeps on repeating. For those of you who may actually be a bit skeptical about why these mathematical models will actually represent the true condition, it has been proven in ecology that if you sample data for they have done an experiment with hares and foxes, and they have sampled the data for almost 100 years or even 150 years, the

behavior over a large timescale actually looks like this oscillatory behavior with a slight phase shift all right. Of course, when you actually look at shorter time intervals the behavior will be fairly different, but in the larger time scale they will actually look very similar to this ok.

Now, we started at one particular initial condition 10 and 3, let us see what happens at different initial conditions. So, let me just say I have got 6 7 deers and 5 tigers and you simulate this you are going to get exactly the same kind of behavior all right. So, this goes around and round the equilibrium point, this is a stable center. And once the plot is complete we will get this graph as well I can see the numbers have changed because the initial condition of the tigers and properly and the deers were different.

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The Jacobian comes back...

The Jacobian matrix is
$$J = \begin{bmatrix} r_1 - a_1 L & -a_1 H \\ a_1 L & -r_1 + a_1 H \end{bmatrix}$$

At the equilibrium point $\begin{pmatrix} r_1/a_1 & r_1/a_1 \\ a_1 & a_1 \end{pmatrix}$ we have
$$J = \begin{bmatrix} 0 & -a_1 r_1/a_1 \\ a_1 r_1/a_1 & 0 \end{bmatrix}$$

with eigenvalues $\pm \sqrt{r_1/a_1} i$ stable center

$\dot{x} = Ax$

Compute the Jacobian at the other equilibrium point: saddle point!

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So, this behavior which is a numerical simulation of a non-linear equation actually agrees with what we expected to get from our linearized model, where we predicted that we are going to get a stable center and the simulation of the non-linear dynamics has shown exactly the same behavior.

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Model of the Van der Pol Oscillator

The Van der Pol oscillator equation is an example of an oscillator with nonlinear damping

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

Physics, engineering and biology;
Action potential of neurons, model geological faults in seismology, tunnel diodes and so on.

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So, in the previous example we have seen a couple non-linear differential equation at the predator prey model, and we saw an example of what is called a stable center which is a particular type of equilibrium point. We also showed that with respect to the equilibrium point the predictions which we make true or also we have shown that in the simulations we get the same prediction of the stable center all right.

So, in that particular case we were it was nice that the predictions did with the linearized model did agree with the non-linear case or on the equilibrium point. So, let us move on to another example of the van der pol oscillator, and this is basically an example of an oscillator with non-linear damping which you can see in this particular term over here right. So, if you think of this as a mass spring damper or a RLC circuit. So, you would see that roughly speaking this would look something like this $m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0$ right. So, we talk on the differential equation let us say plus $b \frac{dx}{dt} + kx = 0$ right. So, this would be the example of a mass spring damper, in this particular case we have just assume m equal to 1 k equal to 1 and this damping term is of a specific kind it is of this all right it is a non-linear damping term and there have been various applications of the van der pol oscillator.

So, people in biology when they are studying the behavior of neurons in brain under certain task related conditions, they have actually shown that you can actually do the modeling with the van der pol oscillator it reasonably agrees with the experimental data

as well. And the classical engineering example for the van der pol oscillator is a tunnel diode, this is basically a diode which allows us to do very high speed switching in electrical circuits. So, now, let us take this particular non-linear differential equation, we see that this is where we get the non-linear term we like will again repeat the same process and see what kind of behavior we can extract out of this one ok.

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Model of the Van der Pol Oscillator

Given the Van der Pol oscillator equation

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

introduce two variables

$$\begin{aligned} x_1 &= x \\ x_2 &= \frac{dx}{dt} \end{aligned} \quad \left| \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{d^2x}{dt^2} = -x_1 \end{aligned} \right.$$

Let us assume that $\mu = 0$

we get the harmonic oscillator!

energy conservation
skew symplectic
mass spring LC
eigenvalues

λ	x
λ	x

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So, before we do that let us simplify the problem one step before and what we will do is to assume that this parameter mu is equal to 0 right. So, when we assume that mu equal to 0 this entire non-linear damping term actually goes away, and you basically get this expression $\frac{dx}{dt} = x_2$ and $\frac{d^2x}{dt^2} + x = 0$, and we will now show very quickly the this is basically the dynamics of a harmonic oscillator. So, let us model this as a state space system. So, what we are going to do is to introduce 2 variables as what we did before as well it is a second order system. So, we will introduce 2 variables, we will call the first one as x_1 and I will just you know that to be x itself we will call the second as x_2 and we will call it as $\frac{dx}{dt}$ all right.

So, from this again as we did before we see that \dot{x}_1 is nothing, but x_2 and x_2 dot is nothing, but $\frac{d^2x}{dt^2}$. And $\frac{d^2x}{dt^2}$ now we can write by looking at this expression over here as basically nothing, but minus x and what is x we use this expression over here and x is nothing, but x_1 . So, if I put this now in a state space form we actually get $\frac{dx_1}{dt}$, $\frac{dx_2}{dt}$ is of course,

the states on the right hand side and the interconnection matrix will be nothing, but 0 1 minus 1 and 0.

So, if you actually do the modeling of a mass spring system or if you do the modeling of LC circuit; no resistance in the LC circuit no damping in the mass spring system you will actually get exactly the state space equations, and it is very easy to show that the behavior of the state space system is an oscillatory behavior. So, if you plot any one of the states as a function of time, you will actually get a perfectly oscillatory behavior all right. In fact, a small exercise for all of you please compute the Eigenvalues of this system, and you will see that Eigen values will lie on the a on the imaginary axis. And when the Eigenvalue is lie on an imaginary axis we have seen before it produces the behavior of a centre right a closed orbit.

So in fact, if you plot the phase space and remember the phase space basically is a plot of the states, it is very nice to plot this because it actually gives very interesting insight into the problem you plot the phase space of this and it is going to look exactly like this assuming we start at some initial condition over here. So, I had a different initial condition and you are going to get this kind of response so on and so forth right. So, what I just did in this slide was to consider the van der pol oscillator, we remove the non-linear damping term over here by assuming that μ equal to 0, and we get this expression over here this expression is basically the dynamics either for a simple mass spring system or LC circuit. In the state space form it looks like this not really very relevant in this particular discussion, but this the structure of this matrix this is actually called a skew symmetric matrix.

The structure of this matrix is very interesting and it leads to really beautiful insight into the analysis of such systems, we would not be needing it here, but if anyone of you is interested just look at skew symmetric and property of energy conservation all right. And you see that indeed this structure the 0 1 minus 1 0 actually leads to really interesting properties. So, we are done with a harmonic oscillator which is a special case of the van der pol oscillator.

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The Van der Pol oscillator equation is an example of an oscillator with nonlinear damping

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

Let us suppose that $\mu = 1$

Physics, engineering and biology:
Action potential of neurons, model geological faults in seismology, tunnel diodes and so on.

S.S. model →

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -x_1 + (1-x_1^2)x_2 \end{aligned}$$

x₁, x₂

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Now let us not assume mu equal to 0 let us assume that mu equal to 1 greater than 0. Specifically mu is equal to 1; and when we assume mu equal to 1 and we compute from the differential equation we compute the state space model same way as we did before. So, we will since it is a second order system we will introduce 2 new variables x 1 and x 2, and we will relate these to each other through the interconnection matrix and when you do that you are going to get this state space model.

Now, let. So, dx 1 by dt is x 2, dx 2 by dt is is this one let us see what we can do with this guy.

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Model of the Van der Pol Oscillator

$$\begin{cases} \frac{dx_1}{dt} = x_2 = f_1(x_1, x_2) \\ \frac{dx_2}{dt} = -x_1 + (1 - x_1^2)x_2 = f_2(x_1, x_2) \end{cases}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1x_2 & 1 - x_1^2 \end{bmatrix} = J$$

The Jacobian is $J = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1x_2 & 1 - x_1^2 \end{bmatrix}$

The origin (0,0) is an equilibrium point and around the equilibrium, the state space equation is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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So, we have the model here which I am recalling in these 2 expressions and for this we can compute the Jacobean. Now because it is a second order state space system we will have a 2 cross 2 Jacobean, just to help you calculate this it will be a 2 cross 2 Jacobean and you will essentially just compute the partial derivatives as follows right. For f one we take the partial derivative with respect to x 1 and x 2, for f 2 we take the partial derivatives with respect again 2 x 1 and x 2; and when you do that you are actually going to get this expression.

So, this is my f 1 of x 1 and x 2 in this case x 1 term doesnt exist this is f 2 of x 1 and x 2. So, when you take the partial derivative of this expression with respect to x 1 since x 1 term does not exist this is nothing, but 0 and I take it with respect to x 2 you simply get one. Similarly when you take the partial derivative of f 2 this expression with respect to x 1 you would get minus 1 minus 2, x 1 times x 2; because this gets you take a derivative of this which is basically 2 x 1 then you have the negative sign and finally, when you take it with respect to x 2, you just have this term over here this comes out and you get 1 minus x 1 squared. So, this is my Jacobean which we also have over here.

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Model of the Van der Pol Oscillator

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -x_1 + (1 - x_1^2)x_2$$

equilibrium points
 $\frac{dx_1}{dt} = 0, \frac{dx_2}{dt} = 0$

The Jacobian is $J = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1x_2 & 1 - x_1^2 \end{bmatrix}$ (x_1, x_2) = (0, 0)

The origin (0,0) is an equilibrium point and around the equilibrium, the linearized state space equation is \rightarrow eigenvalues

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\lambda = \pm x$

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Now, of course, we have the Jacobean and in order to go ahead and linearize the system we would also need the equilibrium points. In order to get the equilibrium points we do the same trick as we did before it is not really a trick it is a procedure we just set dx one by dt is equal to 0, and we set dx 2 by dt equal to 0 like I said each one of these equal to 0 and then when you compute the Jacobean the equilibrium point you will see that this is the solution of the above 2 equations. There is no other solution in this particular case x 1 has to be equal to 0 x 2 has to be equal to 0 then you take this equilibrium condition substituted back in the Jacobean you are going to get your wonderful linear system over here in this particular case I have replaced the variables x 1 with x dot and x 2 with y ok.

So, now let us analyze the property of this system, and the way we are going to do that is again and the way we are going to do that is to again as we always do we compute the Eigenvalues remember in transfer function how is to compute the poles that is that is really the first step you would always do similarly compute the Eigenvalues of the system and you will see that the Eigenvalues are on here.

S 45.22

Model of the Van der Pol Oscillator

The original non-linear equation was

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

Around the equilibrium, the linearized state space equation is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The eigenvalues are $\begin{bmatrix} 0.5000 + 0.8660i \\ 0.5000 - 0.8660i \end{bmatrix}$ we have an unstable equilibrium point !!

For $\mu > 0$ the equilibrium is always unstable ! In fact, close to the origin the system is always unstable, but away from the origin, it is always stable – and it goes to a limit cycle

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So, if you look at the and the s plane or the pole 0 map. You will find that the Eigenvalues are over here and what do we learn what have we learned from basic controls that if the poles are in the right half plane you actually have an unstable system.

So, what we have basically done is to consider a non-linear system, linearize it around an equilibrium point get this linear system and we have shown that the linear system here is unstable. Now it is not unstable in any random way it is unstable specifically around the equilibrium point that we have considered, and this is an interesting distinction because what I mean by unstable around the equilibrium point is not the same as unstable at the equilibrium point. So, you will see very quickly that let us say that the equilibrium point was 0 comma 0 right. So, x comma y equilibrium point. So, I will call it x naught, y naught if I start at 0 comma 0 as the initial condition as per my definition I should be stuck there forever and you will be stuck there forever, there is absolutely no way that you are going to escape from the equilibrium point.

So, it is not unstable at the equilibrium point, it is merely unstable around the equilibrium point. So, if I start anywhere else around that equilibrium point, we are going to see what actually happens and actually you will see some very interesting behavior that although your linear system claims it is unstable, and you will see very weird responses when you look at the transfer function, which I will be doing in a couple of minutes. It is not unstable in the sense that this is going to blow up right it is not like your oscillator is going to heat and blow up or cause some catastrophe it is actually going to do a very interesting it has a very interesting behavior. Start anywhere around the equilibrium point

and you will see that it goes into what it goes into what is called as a limit cycle. Start anywhere around the equilibrium point and it will always reach the limit cycle and. In fact, in this specific example we will see that you start anywhere else in your face space any initial condition, it will always converge to this limit cycle and we will see this in a simulation.

So, the limit cycle is a very interesting behavior it is actually telling us that, wherever you start in the system anywhere you will always converge on to this limit cycle except if you start at the equilibrium point, because if you start at the equilibrium point you will be stuck there forever.

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Model of the Van der Pol Oscillator



The original non-linear equation was

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

Around the equilibrium, the linearized state space equation is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

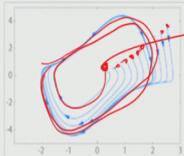
The eigenvalues are

$$\begin{matrix} 0.5000 + 0.8660i \\ 0.5000 - 0.8660i \end{matrix}$$

Thus we have an unstable equilibrium point !!

nonlinear
linearization of
predator prey
van der pol.

BUT: we have a limit cycle !!
A limit cycle is a closed trajectory in phase space such that at least one other trajectory merges into this as $t \rightarrow \infty$



In fact we have a stable limit cycle !

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So we will see indeed that this behavior which plotted over here, this is actually what is called as a limit cycle and you start at any initial condition wherever you want except at the equilibrium, you will always tend to the limit cycle ok.

So, let us see how this actually looks like in MATLAB. What we saw with the van der pol oscillator. So, we saw that the Eigenvalues are in the in the right half plane and which means the equilibrium point is unstable. Now I have got a small MATLAB code again like the predator prey problem we will be sharing the code with all of you, let me show one specific thing that let us actually start the simulation add the initial condition. So, that I will show you that if you started at 0 comma 0 the initial condition, you will

always stay there only because this is the equilibrium point; like we did with the predator prey model we again use ode 45 the same type as we did before what will change of course, is the model of the system good.

So, if I start the simulation by assuming the initial condition is exactly 0 comma 0 the equilibrium point, you will see that there is really no dynamics right everything is exactly at 0 comma 0 . Perfectly there it never moves anywhere else now, we have said that this is an unstable equilibrium point. So, let me just perturb a little bit the initial condition along x direction, I can do it along y it has the same behavior. So, when I perturb it along the x along the x direction by this factor of 10 power minus 7 or 6 or 7 whatever this is and then I do the simulation because it is an unstable equilibrium point, it is not supposed to stay there right and that is exactly what happens. So, let me. So, we started off slightly away from the equilibrium point at this particular place, where my mouse is pointing and because it is an unstable equilibrium point it actually spirals out of that.

Now, the reason it is spiraling out of that is because it needs to spiral and go towards this larger object over here, this larger object is a closed orbit this is the limit cycle, and it will always go to the limit cycle. So, this is this basically proves that your equilibrium point is not stable. What happens if he started different initial conditions right. So, for example, let me start at 3 point whatever that is right, maybe I can make this as 2 and we will see what happens. Will it go to the equilibrium point? It will not be not, it is unstable where will it go well it again tends to the or to the limit cycle all right. So, I started over here and then it tends to the limit cycle. In the previous case we started almost close to the equilibrium point and even that way into the limit cycle. Let us try this for a lot of initial conditions and we will actually see a family of solutions for each of the initial condition ok.

So, what I have done here is to take 20 possible choices of initial conditions and from each of these values 0 , 1 , minus 1 , 3 , minus 4 and 4 I am actually choosing a value for the initial condition. So, in my case an initial condition could for example, be 0 comma 4 it could be for example, 0 comma 4 right it could also be minus 1 comma minus 2 it could be minus 1 comma 4 it could be 0 comma 3 so on and so forth. For each of these initial conditions when you actually compute the solution and plot it you are actually going to get this.

So, this beautiful looking graph basically tells me the following wherever you start from whether this is the initial condition, this is the initial condition or this is the initial condition and so on, wherever you start from the final simulation will always tend to the limit cycle every single time. This is true as long as μ is greater than 0 if μ is less than 0, then the if μ is equal to 0 we see that it will just be the a perfect circle which is a perfect oscillation because it is a harmonic oscillator. So, we also see that as long as the values of x and y are larger than 0, either x or y are larger than 0 it will always tend to the limit cycle, if both x and y are equal to 0 it will get stuck at the equilibrium point ok.

So, there is actually one specific case where we got x was 0 y was 0 initial condition and it is stuck at this equilibrium point this brown color plot. Otherwise you see this beautiful looking graph ok. So, we have seen the example simulation in MATLAB that we obtain what are called limit cycles, as we see in this plot as well we have seen the notion of a stable limit cycle, we have also seen the notion of a unstable equilibrium point. So, that concludes lecture part 2 of lecture 2 which was focused on non-linear systems and the linearization of non-linear systems ok.

So, the linearization of non-linear systems we have taken 2 specific examples the predator prey problem, and this van der pol oscillator both are non-linear systems and we have shown how we can linearize them and what interesting behavior they actually get. So, thank you very much and if you want additional references they are over here all right.

Thank you and good luck.