Control Engineering
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Module - 01
Tutorial
Lecture - 05
Solving Problems in Modelling of Systems

Hello everybody. In today's class we will do few problems related to the theory which we have learned so far in terms of several steps of modelling. So, I start with the question which I had asked you last time.

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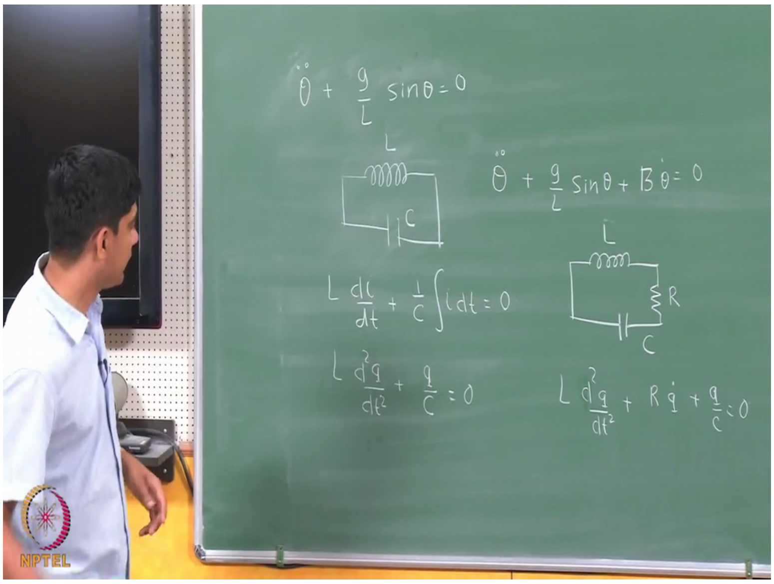

So, starting from a lossless pendulum for which the equations look like this right your theta was the angular displacement and theta double dot plus g over L sign theta was equal to 0, and this represented a simple harmonic motion is there an electrical analog for such a such a behaviour or such dynamics

Well let us take this circuit with L and a C right. So, if I write down the voltage loss I will have L d I by d T plus 1 over C integral i d t is 0right. Now if I write it in terms of the charge I will have L d 2 q by d t square plus q over c is equal to 0. So, this is the solution to this will be periodic right in the same way as it will be to this equations. Only different is difference here is that I am just using both of them are linear elements. So, here the (Refer Time: 01:56) terms comes because of the non-linearity of the element of course, we will discuss about the relation between this model and this model when we do linearization of non-linear systems, but solutions to both of them will just be periodic right and when we had the (Refer Time: 02:13) or the pendulum with some fresher element we had an additional term.

So, my equations looked like theta double dot plus g over L sin theta plus B theta dot was equal to 0, and analog is here would just be a RLC circuit. So, just add a resistance r. So, this is an L R and C and my equations would just be L d 2 q by d t square plus R times q dot plus q over c equal to 0 right and then the behaviour again it is the same this is again the I am just ignoring the non-linear element here for a while, but the linear version of this would look something like this and this, the analogous behaviour of the simple pendulum in the mechanical system case.

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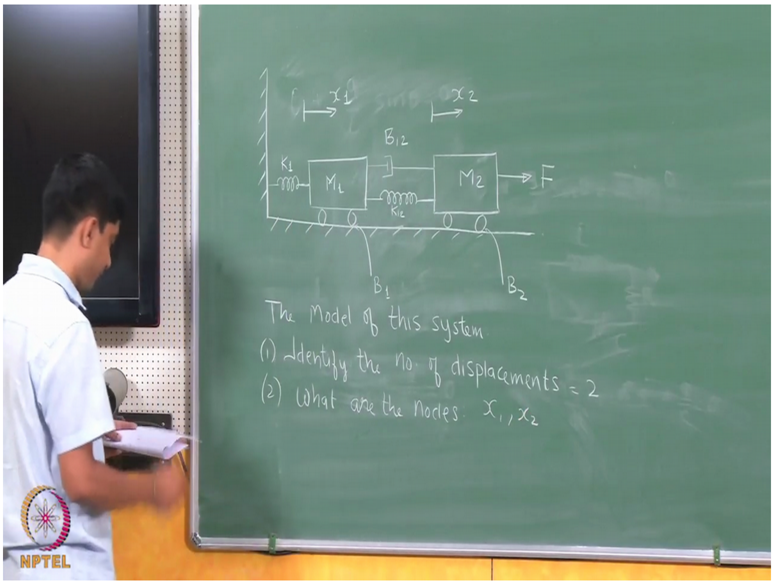

So, we will do some more examples related to this. So, let us take a system which looks like this. So, all this is my reference points or reference node as I would call it, I have mass which I call M 1 and which goes around this surface between the mass and the reference point is a spring with constant K 1, this another mass M 2 right and between these 2 masses is a damper its value B 1 2, a spring connecting these 2 masses with spring constants K 1 2, the friction of the surface is B 1, the friction over this is B 2 right and there is an external force which is being applied here called F, which results in a displacement here I will denote this as x 2 and this displacement I will denote as x 1.

And then my problem here or what I want to write is the dynamical equations governing the system or the model of this system. So, if I go through the steps of the modeling, the first thing to do is to identify the number of displacements right, in this case it is x 1 and x two. So, the number of displacements is 2 then what are the nodes right if I follow the step I would choose x 1 and x 2 as my nodes together with a reference node.

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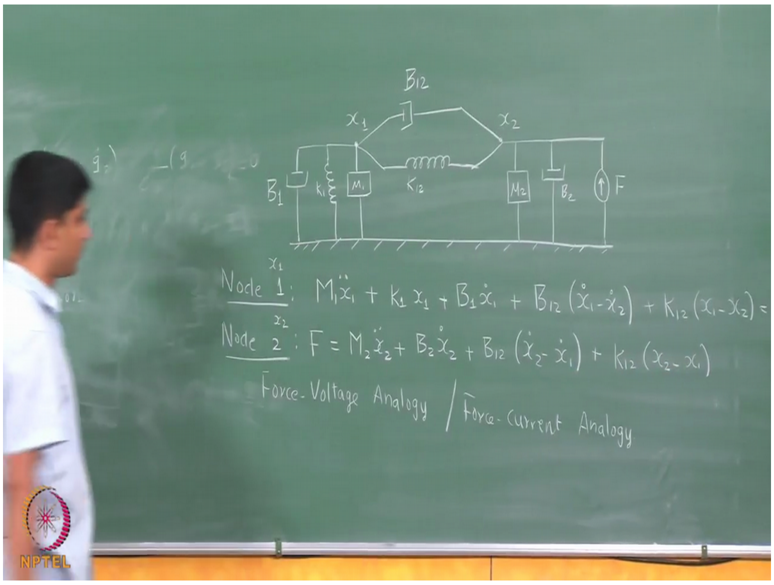

So, if I just mark my nodes here, I call this x 1 and let me call this x 2 and may be a reference node here. Now if I look at each of my element right the first element M 1 is between x 1 and the reference node right. So, I will just draw it this way. So, x 1 and the reference node in between is M 1. Now look at even K 1; K 1 is also between x 1 and the reference node. So, I could draw it and similarly also with B 1. So, B 1 the damping element would show up here call this B 1.

Now, between the nodes x 1 and x 2 is a dashpot or a damping element or and also a spring. So, I would write this as a. So, between x 1 and x 2 a damping element and a spring with coefficients B 1 2 and K 1 2; now I go to node number 2 there is a mass between node number 2 and the reference similarly a dumping element here so that would look something like this. So, between this there is a mass M 2, there is the frictional element denoted B 2 and there is a force which is being applied which looks like this ok.

Now, once we have the system written in form of its nodel components, then it is easy for me to write down the dynamics or I just write down the conservation laws at node 1 right: node 1 where I have the mass, a spring, a damper or actually node x 1. So, how will the equations look like all the forces corresponding to which of this elements this would be M 1 x 1 double dot, plus this is modelled as K times x the damper goes like B 1 x 1 dot that takes care of these three elements. So, node 2 there is also these other 2 elements and these equations would be B 1 2 x 1 dot minus x 2 dot plus K 1 2 x 1 minus x 2 is 0.

So, the sum of all the four sets here is 0; now allocate the second node 2 or denoted by x 2. We have an input force, force of this guy over here, force of the mass and these 2 elements right. So, this would just look like F is M 2 x 2 double dot, plus B 2 x 2 dot correspondent to this element and I have these 2 guys here B 1 2 now with x 2 dot, minus x 1 dot plus K 1 2, x 2 minus x 1. The sum of all the forces here is equal to the external force which is been applied through F. So, these are the entire equations which govern the dynamics of the system.

So, we also learnt how to write this in the force voltage analogy and also the force to current analogy. So, let us just recollect how this analogy looks like. So, here we learnt the analogy between electrical and mechanical systems via the force voltage analogy, and also the force current analogy.

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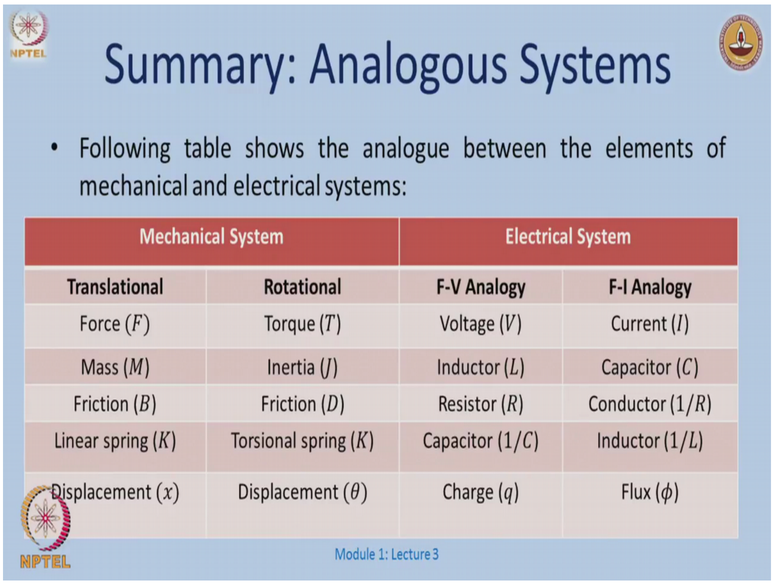

So, let us draw this equivalent mechanical system in terms of its force say starting with voltage analogy.

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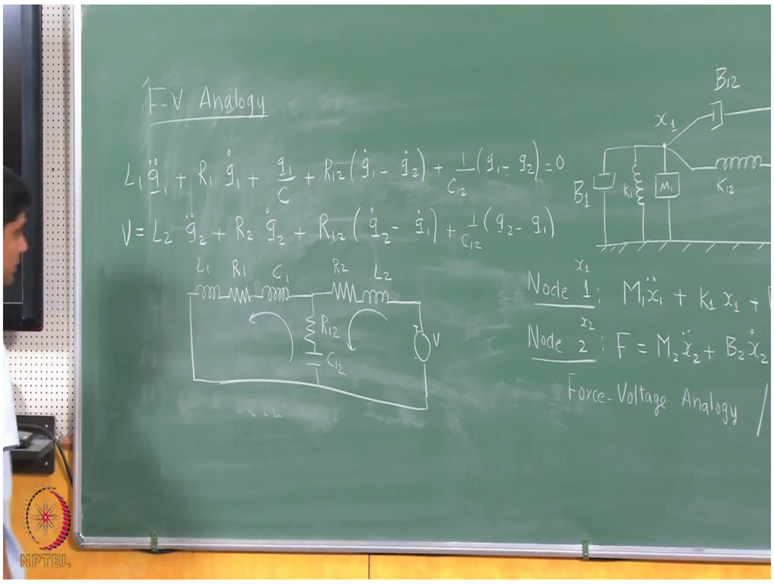

So, the force to voltage analogy tells me that force is related to the voltage, mass with a inductive element, friction with a resistor spring with a capacitor with 1 over C being the value. So, if I use those directly my equations let us write down the equations first that might help us drawing the picture a little better.

So, M becomes L 1, the basic elements here are the displacements the displacement will be mapped to the charge. So, I have L 1 q 1 double dot plus R 1 q 1 dot plus q 1 over C plus R 1 2 q 1 minus q 2, plus 1 over C 1 2 q 1 minus q 2. So, the mass is replaced by this inductor B 1 by R 1, K 1 by 1 over C, you can look at the analogy table why this is 1 over C then R 1 2 replaced is represents B 1 2 and 1 over C 1 2 represents K 1 2 all this equivalent to 0.

The second equation would look like the force would be a analogous to a voltage L 2 q 2 double dot, R 2 q 2 dot, plus R 1 2 q 2 dot, minus q 1 dot I am sorry will be a dot here right because of the x, x 1 dot minus x 2 dot will transform to q 1 dot minus q 2 dot right plus 1 over C 1 2, q 2 minus q 1.

Now, if I were to draw the equivalent circuit well I have the voltage source V which is an analogy of force then I will have R 2, L 2 representing the mass M 2 then these 2 elements B 1 2 and K 1 2 would look something like this I have a R 1 2, C 1 2 and then the mass M 1 with L 1 the element B 1 with R 1 and the spring K 1 is analogous to C 1. So, if I write down treat of this as a circuit right and then write down the equation with possibly maybe this direction of current, I just get these 2 equations right and this is essentially the force to voltage analogy right.

Similarly, we could also write down the force to current analogy; in the force to current analogy my force is analogous to current mass instead of a inductor now is analogous to a capacitor, the frictions element is now a conductor, spring is now equivalent to a inductor, and the displacement is now analogous to a flux element . So, let us see how the first equation goes M 1 x 1 double dot plus K 1 x 1 B 1 x 1 dot, B 1 2 x 1 dot minus x 2 dot plus K 1 2 x 1 minus x 2 go into 0. So, M 1 from the table would represent the capacitor C 1 x is equivalent to the flux phi.

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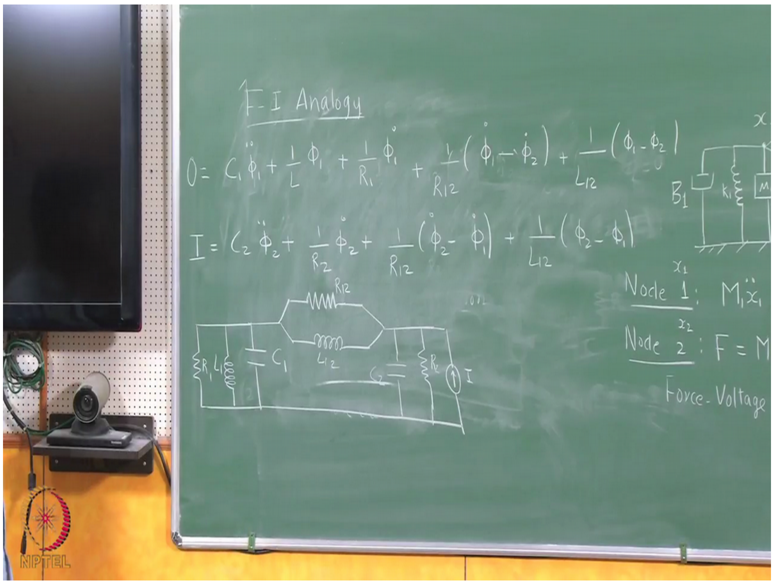

I will call it phi 1 double dot plus the spring element is equivalent now to a inductor I have 1 over L phi 1, now with a resistor I will have 1 over R 1 phi 1 dot plus 1 over R 1 2 phi 1 2 sorry phi 1 dot minus phi 2 dot plus 1 over L 1 2 of phi 1 minus phi 2 all this is equal to 0.

Now, look at the dimensions of all this, all this have dimensions of current. So, I am just writing down a current conservation law at a particular note. Earlier I had written down the voltage conservation law. Similarly second equation my source F is equivalent to a current and again going back to the analogy table I can write down the equations in following way. So, how would this look like in circuit diagram, the mass I will start with the mass will be replaced by C 1 then I have the spring with L 1, B 1 replaced with R 1 then I have these 2 elements here. So, this is R 1 2, L 1 2 then I have the other three elements following right the current source I have M 2 replaced with C 2 and B 2 replaced with R 2, and this is my equivalent circuit in the F I analogy right.

So, we started with a system we had that in a form where we could write the equations with node 1 and node 2 thus the summation of all the forces is 0 then we came to the force voltage analogy using the table and the force current analogy right and we even could come up with the equivalent circuits.

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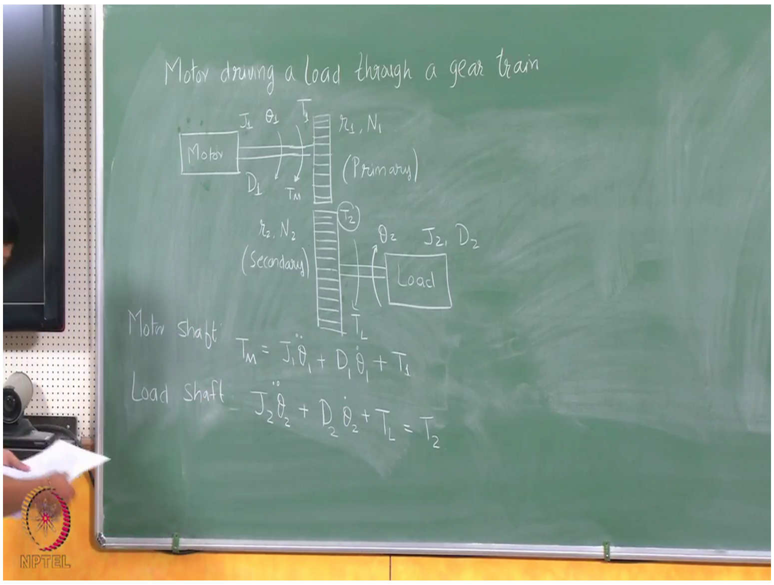

So, the second example is a motor driving a load through a gear train. So, how does it look like let me denote this block as my motor with now this motor is connected to a gear train, and this gear train is an inter connected to a load ok. So, the motor produces a torque T m right I also denote the moment of inertia here as J 1 right the displacement as theta 1 again in this direction right and a friction coefficient D 1 over here.

Similarly, here if this is moving this way theta 2 will go in this direction right imagine this as gear rotating this way then this will force it to rotate the other way, and there will be a certain load torque. Now this gear has a radius of R 1 with the number of T th (Refer Time: 21:35) as N 1, here it is R 2 and N 2. So, I can call this as the primary and secondary right analogous to what I call in the transformer. Now I want to write down the dynamical equations for this or the model for this.

So, start with the motor shafting first what are the elements here well the motor generates or gives me this torque T m, now I have J 1 theta 1 double dot right plus D 1 theta 1 dot. Now this would be the dynamics if there is no load or there is no gear train also right. So, the input power would be consumed by the inertia here the J 1 and the frictional element; now what this load does is well it requires some torque and how is this torque reflected here, it is reflected via this gear train to something here. So, there will be additional torque which is experienced here or additional load because of or additional torque because of the load let me denote this as T l.

So, T m now compensates for J D or J 1 D 1 and also this T L right L not call this T L I will say I will call. This T 1 this T L is reflected here through the gear train as T 1. So, I will have a T 1 here now at the load side at the load side. So, I have a J 2 here and a D 2 similarly I have J 1 and D 1 here. So, here this T 1 is experienced here as some T 2 right I will tell you the relation between T 1 and T 2 shortly. So, here I have the term corresponding to J 2 as J 2 theta 2 double dot, then I have the friction element here theta 2 dot plus I have the load torque T L. So, where does all this come from? So, I have to compensate for the load compensate for J 2 compensate for D 2 and this comes via T 1 through the gear train and let me call this as T 2.

Or let us see the entire process here right. So, I have T m the motor torque it first compensates for J 1 and D 1 and gives me a torque T 1 here, this T 1 is transferred via the gear train to the secondary side called T 2 and this T 2 as to compensate for the load compensate for D 2 and compensate for J 2 right and each of this models which we had seen earlier right why is this J 2 theta 2 double dot why is this D 2 theta dot and so on ok.

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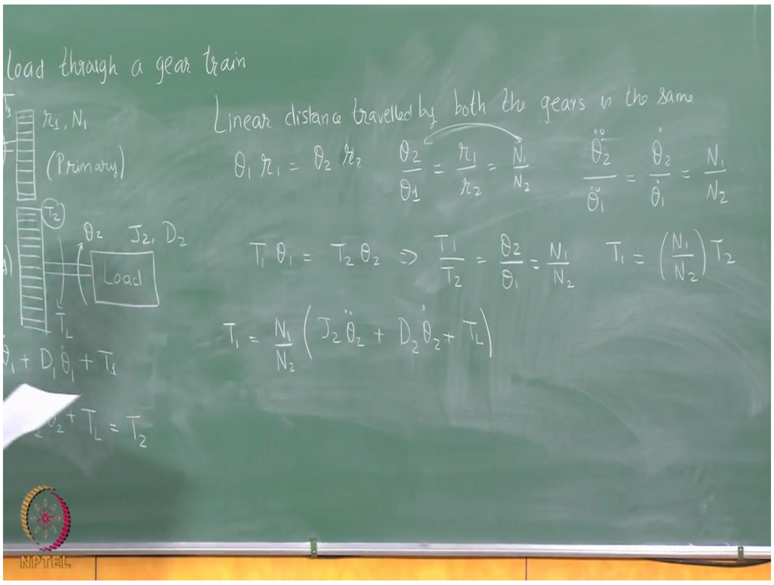

Now, what is the relation between T 1, T 2, R 1 N 1 and R 2 N 2. So, we will we will use the relation that the linear distance travelled by both the gears is the same which means theta 1 times R 1 is theta 2 times R 2. Since the linear distance travelled by both the gears is the same I have that theta 2 by theta 1 is R 1 over R 2 is N 1 over N 2. Further I assume that in the transfer of power between the primary side of the gear to the secondary side there is no power loss, which means T 1 theta 1 is T 2 theta 2 now this means that T 1 over T 2 is theta 2 over theta 1 is N 1 over N 2.

Now, the relationship here can also be written in the following way at theta 2 double dot theta 1 double dot, the angular acceleration, the angular velocity is N 1 over N 2. Now T 1 can be written as N 1 over N 2 times T 2; now what is T 2? T 2 is J 2 theta 2 double dot D 2 theta 2 dot plus T L. So, the relationship between T 1 and T 2 is given via substituting for T 2 over here that is J 2 theta 2 double dot, plus D 2 theta 2 dot plus T L.

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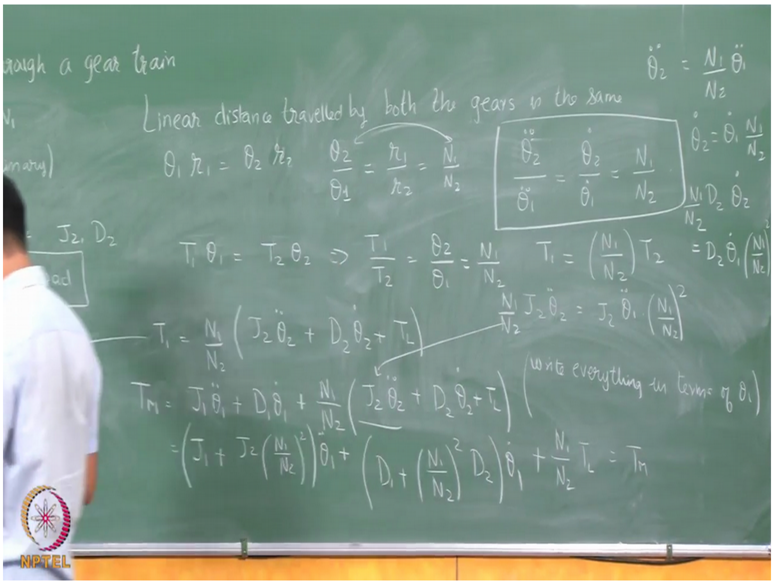

Now, substituting for this T 1 in this equation gives me the following. The T m is J 1 theta 1 double dot this 1 plus D 1 theta 1 dot plus T 1. So, I am substituting for T 1 here that is N 1 over N 2, J 2 theta 2 double H dot plus D 2 theta 2 dot plus T L. Now from this expression I can re write this expression again and then everything in terms of theta 1 right I want to write everything theta 1 or theta 1 double dot and theta 1 dot. So, this will look like the following J 1 theta 1 double dot let us do let us take this term; J 2 theta 2 double dot is in terms of theta 1 would be J 2 again from this expression what is theta 2 double dot? Theta 2 double dot is N 1 over N 2 theta 1 double dot right. So, J 2 theta 2 double dot would be J 2 n theta 1 double dot with N 1 over N 2.

Now, here I have N 1 over N 2, J 2 theta 2 double dot therefore, N 1 over N 2, J 2 theta 2 double dot would simply become this guy. Now there is this term corresponding to theta 1 double dot and also J 2 theta 2 double dot can be written in terms of theta 1 double dot.

So, substituting for this guy over here I get the following. So, I have J 1 plus J 2, N 1 by N 2 whole square times theta 1 double dot. Similarly I could do for the second term right I can write D 2 theta 2 dot in terms of theta 1 right here just using the relation that theta 2 dot is theta 1 dot N 1 over N 2, which means N 1 over N 2 here N 1 over N 2, D 2 theta 2 dot is D 2 what is theta 2 dot theta 2 dot is N 1 over N 2 theta 1 dot. So, I have theta 1 dot N 1 over N 2 square.

So, second term would be D 1 plus N 1 by N 2 square, D 2 theta 1 dot, plus now the remaining term is N 1 over N 2 times T L all this is equal to T M.

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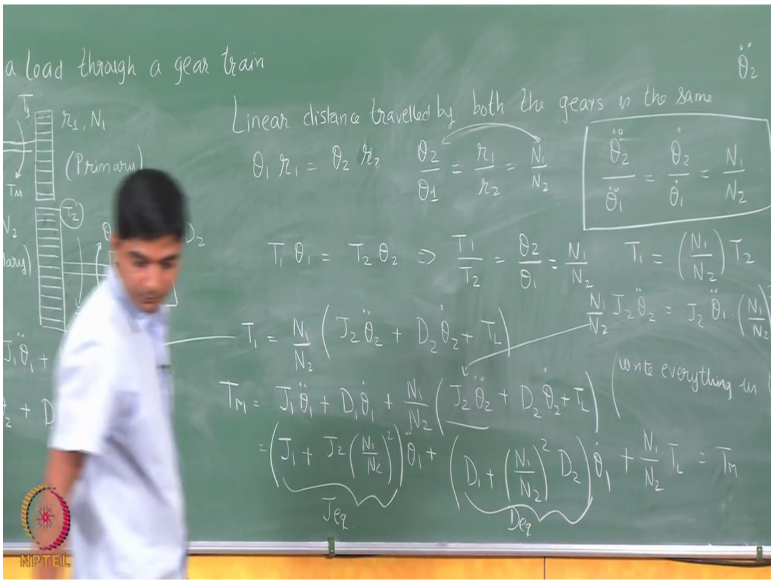

Now, I can call this J equivalent I can call this entire guy as D equivalent and write down my overall equations is that J or let us call J 1 equivalent or D 1 equivalent.

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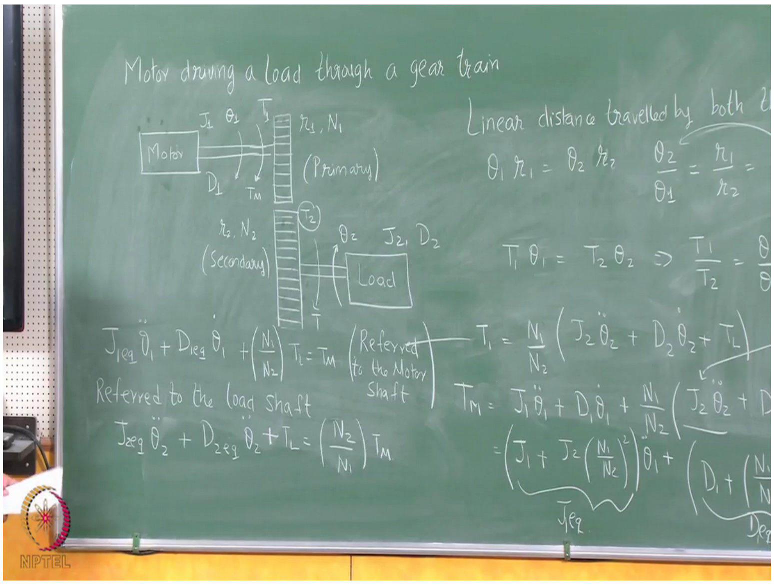

Let J 1 equivalent theta 1 double dot plus D 1 equivalent theta 1 dot, plus N 1 over N 2, T L is T m. So, this is the dynamics referred to the motor shaft and if you remember something from transformers this would also resemble the equivalent circuit dynamics referred to the primary side.

Now, I could also do it referred to the load shaft, where I just write down the equations. So, you will have J 2 equivalent theta 2 double dot, where I just want to write down all equations now in terms of this one this equation in terms of not theta 1 double dot, but now in terms of theta 2 double dot, plus D 2 equivalent theta 2 double dot plus the T L would remain as it is and T m would get transformed via the gear ratio to N 2 over N 1 of T m. So, exactly the same procedure which we did here in terms of writing in theta 1 double dot we just write in theta 2 double dot using this relations, and the first 2 equations which govern the dynamics.

So, these are 2 ways of writing down the entire equations of motion of this one referred to what we in the transformer terminology call as a primary side or the motor shaft in this case or referred to the second risks side or as the load shaft in this case sorry.

Student: (Refer Time: 35:28).

Sorry this will be theta 2 dot thanks for correcting that. So, this will be J 2 theta 2 double dot D 2 will come always to the theta 2 dot plus T L is N 2 by N 1, T m.

Thank you for your attention.