

Control Engineering
Dr. Ramkrishna Pasumarthy
Department of Electrical Engineering
Indian Institute of Technology, Madras

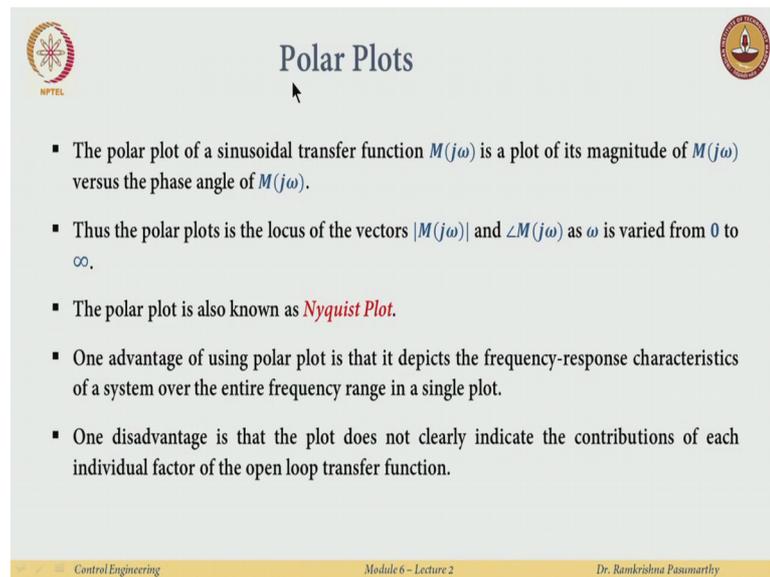
Module - 06
Lecture - 02
Frequency Response Plots

Hello everybody. So, to continue on our discussions with in the frequency domain specification, so what we had seen last time were few very basic things. That is why this analysis is important for us. The first part is that well I can see how my system behaves at different frequency signals, right. This will also help me as we saw in terms of sensitivity of my system, sensitivity to disturbances, sensitivity to noise and so on and these signals occur at different frequencies.

We also saw that if the input to a linear time invariant system is sinusoidal signal, the output is also sinusoidal signal. The only difference would be that the magnitude would change and the change in the magnitude would depend on what is sitting inside my transfer function and also, the phase shift which is observed and that phase shift is again depending upon the transfer function or the angle of the transfer function. Then, we characterized few things which we called the resonant peak. We called you know we had characterized the bandwidth and so on and all this were very beautifully again related to the system specifications like the damping, coefficient, the natural frequency and so on. So, we still did not worry about stability while doing that because we were just dealing with second order systems and the kind of systems which we were dealing with in second order, we always know that it is stable, right.

When we write $\omega_n^2 + s^2 + 2\zeta\omega_n s + \omega_n^2$, we know it is always stable. So, what do we do if I have a system which has more than 2 poles, if 5 poles or 6 poles.

(Refer Slide Time: 02:06)



Polar Plots

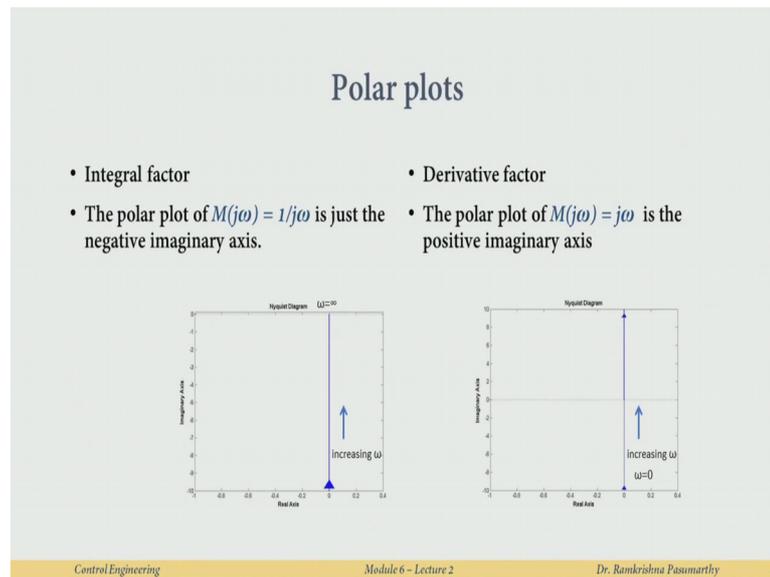
- The polar plot of a sinusoidal transfer function $M(j\omega)$ is a plot of its magnitude of $M(j\omega)$ versus the phase angle of $M(j\omega)$.
- Thus the polar plots is the locus of the vectors $|M(j\omega)|$ and $\angle M(j\omega)$ as ω is varied from 0 to ∞ .
- The polar plot is also known as *Nyquist Plot*.
- One advantage of using polar plot is that it depicts the frequency-response characteristics of a system over the entire frequency range in a single plot.
- One disadvantage is that the plot does not clearly indicate the contributions of each individual factor of the open loop transfer function.

Control Engineering Module 6 - Lecture 2 Dr. Ramkrishna Pasumarthy 3

So, today we will try to answer few of those questions with the help of some frequency response plots. So, these plots are what are called as polar plots. So, what is a polar plot? The polar plot of a sinusoidal transfer function, again this is important and we know why in the steady state analysis in the frequency domain, we just replace s with $j\omega$ because after all the transients die out, only the sinusoidal components remain and we had a little derivation also of that last time.

So, we are actually plotting the magnitude versus the phase angle as the frequency is varied from zero to infinity, right. This polar plot is also known as Nyquist plot. The only thing we have to be careful in Nyquist plot as we will see shortly is that well, we also deal with negative frequencies and I will tell you why that is important. So, why do we need this polar plot? So, advantage is that it depends, it depicts very nicely the frequency response characteristic over entire frequency range just in one single plot. Just one graph will tell me what happens at zero frequency till infinity, but then I really would not know the contributions of each of the individual factors of the open loop transfer function. I just look at everything as one single block and just one single graph, nevertheless there are still lots of nice things which we could do, ok.

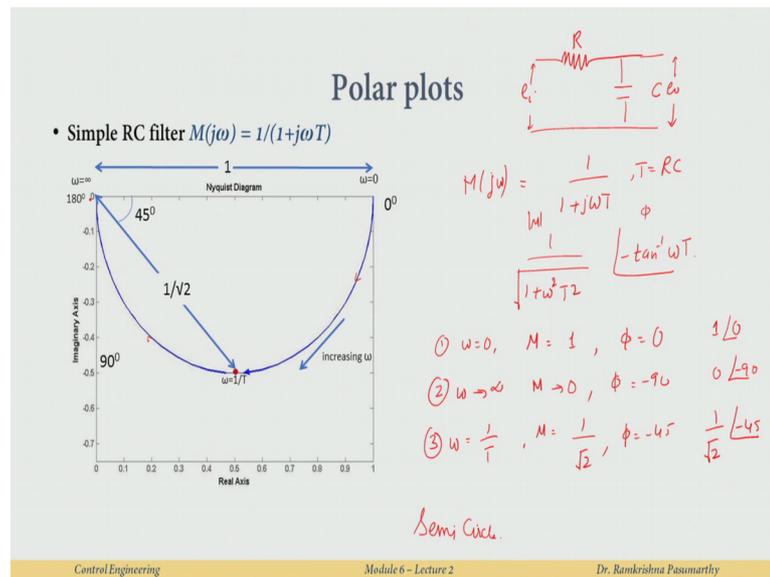
(Refer Slide Time: 03:40)



Starting with some very basic transfer functions, so an integral factor right which has a transfer function over 1 over s, the polar plot I just write down the sinusoidal transfer function which is 1 over j omega. It is just the negative imaginary axis. Why is this true? When I just say omega equal to 0, well I have magnitude which is well at minus infinity and as omega tends to infinity, my magnitude goes to 0 and it is just 1 over j omega. My angle is always over here, right. I am always on the imaginary axis.

Similarly, I just take the derivative factor of just a pole with just s is a transfer function. The sinusoidal transfer function becomes just j of omega and this is just the positive imaginary axis as you would see, right. So, when omega is 0, the magnitude is 0 and it keeps on increasing. As omega goes to infinity, you may ask sir this is not a good transfer function because the number of poles are less or there are actually no poles. There is just one 0, not physically realizable. All that is I am just giving you a little indication of how these things look like in the frequency domain. Nothing about saying that this is a transfer function. Therefore, I do not even call it transfer function. I just say it is a derivative factor, right M of j omega j. When omega is 0, I have a magnitude of 0 and I just keep on increasing the frequency, ok.

(Refer Slide Time: 05:28)



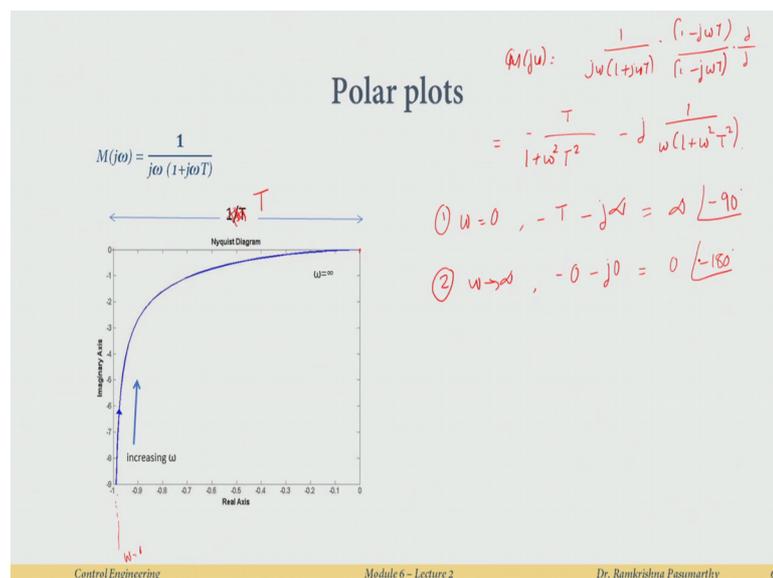
Actually this omega is equal to 0 should be somewhere here. I will just correct. So, something else this looks a little more useful right because I can realize this system and what this system is, it is just a simple RC filter. I can just draw a little circuit diagram for it just for us to make a little sense. So, I have R element. We have done this while we were deriving basic transfer functions. So, I have R I have C. The input voltage here and the output voltage here. So, I will not derive a transfer function, but I will assume that you know this, right. So, I will just directly write down the sinusoidal transfer function, not even the overall transfer function.

So, this M with j omega, it looks like 1 over 1 plus j omega T, where T is the time constant, T equals to RC. Now, again I am interested in how this guy, this is a complex number, right. It will have a magnitude and a phase or a phasor. How will this guy move or change as the frequencies go from 0 to infinity? So, this guy I can write this in terms of the magnitude as 1 over square root 1 plus omega square t square and the angle would be the negative of tan inverse of omega t. This is called the magnitude M and this is the angle phi, right. So, let us observe what happens first is when omega is 0, right this is just the d c signal. When omega is 0, the magnitude M, I just omit the mod symbol. It should be obvious the magnitude is 1 and the angle phi is 0, right. Just put omega equal to 0 and 0, it is obvious, ok.

So, at omega equal to 0, I just have a phasor which is 1, angle 0 magnitude 1 angle 0 angle 0. My horizontal axis is my reference. Therefore, this will be this one, right. So, this will be the entire thing when omega is equal to 0. Now, let us do the other extreme. What happens as omega goes to infinity? As omega goes to infinity, what would happen as omega goes to infinity? What you see is that the magnitude goes to 0 and the angle phi becomes minus 90, right. The tan inverse of, negative tan inverse of infinity is minus pi by 2. So, I have a phasor which looks like 0 with an angle minus 90. So, it will be just some tangential, right. This one, this is 0 angle minus 90, ok.

So, third thing is what happens when omega is 1 over t, in that case you can compute the magnitude M to be just 1 over square root 2 and the angle phi is minus 45. So, we are looking at the phasor which I want to plot 1 over square root 2 angle of minus 45 degrees. That is this one and at omega equal to 1 over t, the magnitude is 1 over root 2 and the angle is negative of 45 degrees. So, if I just keep on you know computing for other values, I just see this plot looks like a very nice semi circle from omega 0 and here I increase all the plot travels, this direction, this direction. Omega is infinity and it stops here. So, I will tell you why are actually doing these plots, but we just do some other interesting plots before we even try to understand why we are doing this, ok.

(Refer Slide Time: 09:51)



So, this guy just had a pole to the previous transfer function. So, what do I get here is, well let me see how I can analyze this. So, I have g, sorry M of j omega is 1 over j omega

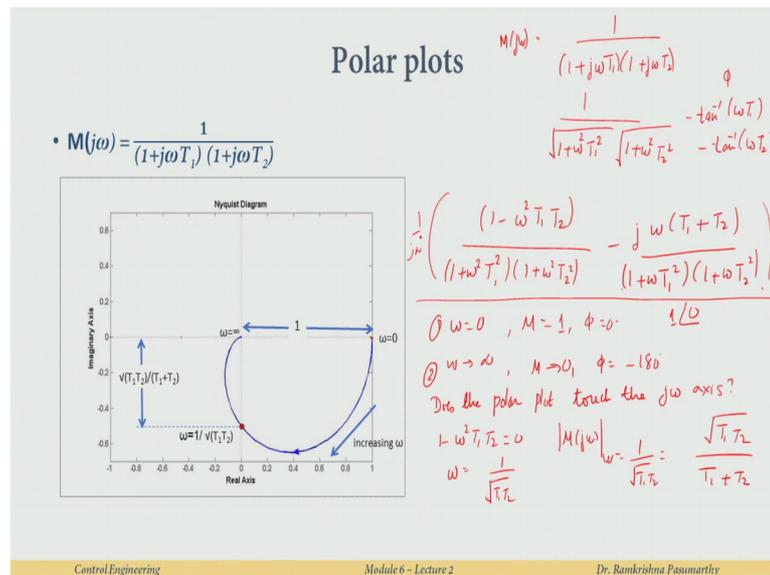
$1 + j\omega T$, right. I just do the rationalization which I learn in school. So, I just multiply and divide this by $1 - j\omega T$. Similarly for this guy, I just multiply the numerator by j and the denominator also by j . That is how I get rid of j , the denominator. So, while doing this, I just get this one, right. If I split up the real and the imaginary parts, I have $\frac{-T}{1 + \omega^2 T^2}$ as my real part and the imaginary part $\frac{j}{1 + \omega^2 T^2}$, ok.

I just do the computations and I just say well what happens first when ω is 0. So, when ω is 0, the magnitude becomes T and well, this guy goes up to infinity, right. So, I have some complex number which looks like this $\frac{-T}{1 + \omega^2 T^2} + j\frac{1}{1 + \omega^2 T^2}$. So, what I am interested in how can I write this as a phasor, right? How can I write this as a phasor in terms of its magnitude and the angle? So, let us look at it carefully, right. So, magnitude if I have a complex number $a + jb$, the magnitude is square root of $a^2 + b^2$. So, look at this closely. So, magnitude is actually infinity and the angle is \tan^{-1} of the complex part over the real part. So, \tan^{-1} of the infinity divided by T . So, that would just be minus 90 degrees, ok.

Next thing when ω approaches infinity, when ω approaches infinity, well this guy becomes 0. Let us keep this $\frac{-T}{1 + \omega^2 T^2} + j\frac{1}{1 + \omega^2 T^2}$. So, this looks like a phasor of 0 and an angle of minus 180 degrees. So, at ω equal to 0, so somewhere here, right. So, I just cannot plot that because I will never reach that at ω equal to 0. What happens is, well I have a magnitude in the real part. So, I can just split into the real part and the imaginary part, right. Just given this, just looking at this complex number over here, so the real part has a value of T , sorry this should be just T instead of $1/T$. This is just T . Sorry about it.

So, the real part goes or it touches this line $-T$ asymptotically. So, this is the phasor $\frac{-T}{1 + \omega^2 T^2} + j\frac{1}{1 + \omega^2 T^2}$. So, at infinity I am at this one. I just asymptotically touch this line right the real part is T and the imaginary part goes to infinity as ω equal to ∞ goes to infinity I then come back to the origin here at somewhat here right, 0 and minus 0 at an angle minus 180, right. This is how the plot would look like for increasing ω . So, these are just two things. So, it is to understand from here, right. We just write down this explicitly as the real part and the imaginary part and compute these guys, ok.

(Refer Slide Time: 14:17)



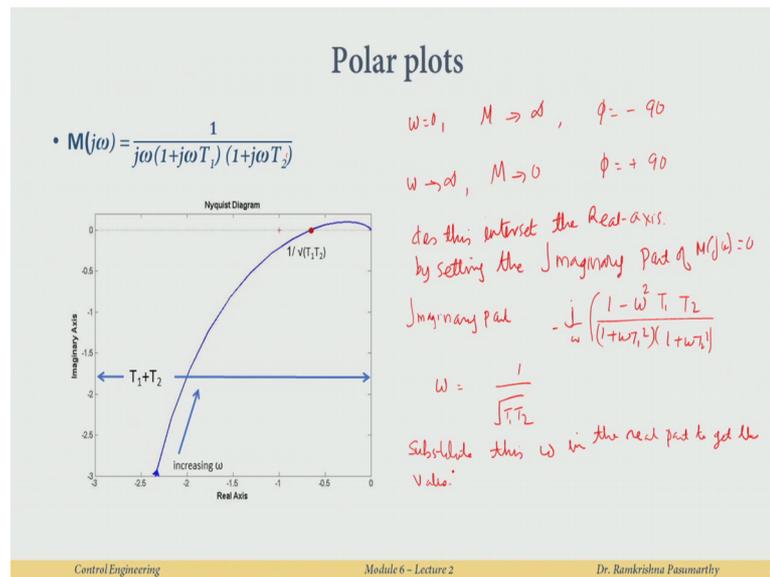
So, the next thing what happens if I have two poles, $1 + j\omega T_1$ and $1 + j\omega T_2$ and just be careful when you do these polar plots is that your transfer function should always be in the time constant form and not like $s + 1$, but something like $j\omega T + 1$. That should be very easy to derive, right. We are not going to the derivations of these things. So, $M(j\omega) = \frac{1}{(1 + j\omega T_1)(1 + j\omega T_2)}$. So, this guy has magnitude. It is the magnitude of this times. The magnitude of this, this is square root $1 + \omega^2 T_1^2$ square root $1 + \omega^2 T_2^2$ square angle would be ϕ is minus \tan^{-1} of ωT_1 minus \tan^{-1} of ωT_2 . Now, again I could do the rationalization and all I can write it down in the real and the imaginary part and it would look something like this. $\frac{1 - \omega^2 T_1 T_2 - j\omega(T_1 + T_2)}{(1 + \omega^2 T_1^2)(1 + \omega^2 T_2^2)}$. We will end with denominator $1 + \omega^2 T_1^2 + \omega^2 T_2^2$.

So, let us do all the steps again. First what happens when ω is 0? When ω is 0, I just get that the magnitude is 1 and the angle ϕ is 0 or looking at the phasor 1 angle 0, right. This one, right at ω is equal to 0 and this here just this point, not the horizontal axis. We are interested in this point. Therefore, this line is not entire blue color. This one second what happens when ω tends to infinity, well ω tends to infinity what I see is that well this naturally goes to 0 and the angle ϕ is well minus 90 for this guy, minus 90 for this guy. That would be minus 180 degrees, ok.

Now, I will ask myself few questions which we could have answered even earlier, but well for some reasons, we did not because they were not important. So, first thing is does the polar plot touch or intersect the $j\omega$ axis, right? This is a valid question to ask. So, just say there is some point here which it touches at this point. The real part goes to 0. So, look at this expression, right. So, does the real part go to 0 for any value of the frequency, you just equate this real part to 0. So, what you get is $1 - \omega^2 T_1 T_2 = 0$ or $\omega = 1 / \sqrt{T_1 T_2}$. So, this is how you can substitute ω . $\omega = 1 / \sqrt{T_1 T_2}$ over here and what you get is that the magnitude or the magnitude of M of $j\omega$ at $\omega = 1 / \sqrt{T_1 T_2}$ is simply $\sqrt{T_1 T_2} / (T_1 + T_2)$. This could just be easily obtained by substituting this in the magnitude expression for magnitude that is marked here, right.

So, this is $\omega = 1 / \sqrt{T_1 T_2}$ and this magnitude here is this much and then, again as ω goes to infinity, you again reach the origin, you arrive at you know if you use the root locus terminology, arrive at an angle of minus 180 degrees to this point. So, again the same steps. Look at the magnitude, look at the phase, just compute simply what at $\omega = 0$, $\omega = \infty$. Just check if the polar plot touches the $j\omega$ axis. You could also check you know why don't we, you may also ask why do not we see the intersection for the real axis somewhere here or here. That you could just check by computing or equating the imaginary part to 0 and you will only get the solution that it only touches this guy at $\omega = \infty$ and $\omega = 0$ that you can just check from this text.

(Refer Slide Time: 19:49)



Now, next thing is, so what if I add a polar j origin to the previous transfer function. So, how does this go? So, this is just the previous guy to which I add j omega. Let us do some simple computations before when omega equal to 0. I can easily see that the magnitude M goes to infinity. When omega is 0, this guy does not contribute to the angle. Only contribution comes from this guy that is as phi equal to minus 90 which is similar to what we had in the second example, right. The magnitude becomes infinity and phi is minus 90, right. So, this one here also see that this number which was earlier T_1 is now T_1 plus T_2 . So, here it was just T . Now, it has just become sum of these two, T_1 plus T_2 , ok.

Then, what happens to this plot as omega goes to infinity? At omega goes to infinity, the magnitude goes to 0. What is the angle contribution as omega goes to infinity? Well, it will just be phi would be minus 90, right because this guy has a minus 90 minus, sorry it will be plus 90 minus 90 minus 90 minus 90. It will be minus of 270 and therefore, you see everything is infinity, right. Tan inverse of infinity, tan inverse of infinity, tan inverse of infinity because all with the negative sign because these are poles and you have minus 270 or even plus 90. So, this is what happens here, right. I can also see now does this intersect the real axis, ok.

So, this can be obtained by setting the imaginary part of $M(j\omega)$ to 0. So, I will just keep those expressions, right. So, we can easily just do this, right. So, to this magnitude,

what you do? You just add 1 over j omega. So, you can just see, right. So, if I add 1 over j omega, this guy will become my imaginary part now because of this j, let us you know quickly do this how does the imaginary part look like. So, this imaginary part is just the real part of the previous guy 1 minus omega square T 1 T 2 divided by again 1 plus omega T 1 square 1 plus omega T 2 square and I just do one over j omega. This will also be similar to writing. So, I just get rid of j from the denominator. I just have j in the numerator with a minus sign. Again when does this go to 0, this goes to 0 when omega is 1 over square root of T 1 T 2, ok.

(Refer Slide Time: 23:59)

Stability in the frequency domain

- Stability depends on the location of the roots of the characteristic equation
- The Nyquist stability criterion determines the stability of a closed-loop system from its open-loop frequency response and open-loop poles.
- It relates the open-loop frequency response $G(j\omega)H(j\omega)$ to the number of poles and zeros of $1+G(s)H(s)$ that lie in the right half s plane.
- Consider a function $q(s)$ that can be expressed as a quotient of two polynomials.

$$q(s) = \frac{(s - \alpha_1) \dots (s - \alpha_m)}{(s - \beta_1) \dots (s - \beta_n)}$$

- As s is a complex variable $(\sigma + j\omega)$, the function $q(s)$ is also complex $(u + jv)$

Control Engineering Module 6 - Lecture 2 Dr. Ramkrishna Pasumarthy 9

Now, substitute this omega in the real part to get the value. I just leave that simple computation to you, ok. So, based on these things, we can just keep on adding one more pole here j omega square. I possibly add 0 here and then, I just keep on doing bunch of things, but we will just stop here, otherwise we will have to spend few years doing all those plots. So, let us see what do these plots mean, right. What do these plots mean and then, can I infer something more just than saying what happens when omega equal to 0, what happens when omega equal to infinity, what is the angle, what is the magnitude and so on, ok.

So, Nyquist gives us a nice answer. So, before we do that, let us just do some stuff. What do we know? Now, we know this for a long time now that stability depends on the roots of the characteristic equation that the roots should be or the poles should be on the left

half s-plane is what the transfer function stability criterion tells me. Now, this Nyquist stability criterion, it determines the stability of a closed loop system from its open loop frequency characteristic and also its open loop poles, we will all today, what we will do is just to see what this statement means, ok.

So, again what does this do? It relates the open loop frequency response. So, again all the analysis, this is important because I just look at the open loop transfer function G times H and I can tell you what the closed loop behaviour is, right. So, that is what Nyquist will tell us very shortly, right. The Nyquist plot relates the open loop transfer function, open loop frequency response to the number of poles and zeroes that lie in the right half s-plane of the characteristic equation. Now, we will see how we arrive at this. Just consider a function q of s which is again in the usual terms you can say it is a transfer function which has M number of zeroes, n number of poles and usually these n is greater than or equal to M , ok.

So, what is s ? Here s is a complex variable real part σ imaginary part ω , right. If this is complex variable, the function q of s is also complex variable. Let me call that u plus jv in the new axis. So, I start with the σ $j\omega$ axis and I go to u jv axis via this map q , right. So, q takes s , it computes all these things and gives me q of s . No, not complicated. Don't get worried, right. So, what does this mean is that for every point s in the s-plane.

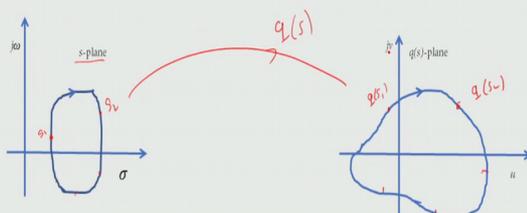
(Refer Slide Time: 26:44)



Stability in the frequency domain



- This means that for every point s in the s plane at which $q(s)$ is analytic, there exists a corresponding point in the $q(s)$ plane.
- Function $q(s)$ maps s plane into the $q(s)$ plane.



Control Engineering
Module 6 - Lecture 2
Dr. Ramkrishna Pasumarthy 10

So, if I take let me just see here, right. So, this is my s-plane. So, for every point here s, let me call this s 1, there might be a corresponding point q of s 1 here, not there might be there will be, right.

For all the points at which q s is analytic, where I can actually compute this, right. So, we will not go into the definitions of analytic functions, but we just say that well I can compute it almost at all points except when q is analytic, right. So, for s1, there will exist a q of s1. Similarly I take s 2 here, there will exist q of s 2 and so on. Whatever points here, here, here and at some corresponding points here, here and here, right. So, the function q s, this maps the s-plane. So, this is my s-plane and this is via q of s. This function q maps the s-plane which equals sigma and j omega to the q s-plane with the coordinates, I can call u and j v, right. This is what this particular expression tells us, right. This is nothing surprising and nothing new, ok.

So, we can just say well you know we avoid points like s equal to beta 1, right. That time q of s may not exist. So, we avoid all those points or what we call as singular points, ok.

(Refer Slide Time: 28:21)



Stability in the frequency domain



- The contour in the s-plane which does not pass through any singular point, there corresponds a contour in the q(s) plane.
- We will not necessarily be interested in the shape of the contour.
- Our aim is to detect (graphically) if there are any poles of the closed-loop system in the right half place (stability!)
- Lets now ask a few questions here
 - What does it mean when the q(s) plane contour encircles the origin?
 - Consider the case when the s-plane contour encloses *only* one of the zeros of q(s)
 - All the other poles and zeros are *outside* the contour.
- $|q(s)| = \frac{|s-\alpha_1||s-\alpha_2|\dots}{|s-\beta_1||s-\beta_2|\dots}$

Control Engineering Module 6 - Lecture 2 Dr. Ramkrishna Pasumarthy 11

That is what the contour in the s-plane which does not pass through any singular point, there corresponds, there exists a corresponding contour in the q s-plane. The contour is like this. It is entire map which I take here, right. I just take this little closed loop here and I say something else happens here. Well, I just may have to make sure that I do not

hit a point, where s equal to beta 1 or s equal to beta 2 or I do not hit s at the poles and because at that point q of s is not defined, but for all other points, it is defined, ok.

We will not necessarily be interested in the shape of the contour, right. We will just say there is a contour here which maps to a contour. On the other side, our aim here is to detect if there are any poles of the closed loop system in the right half plane and when I detect this, I have you know there is direct implication on stability. So, I do a trick where say well, I ask myself a question. Does there exists a pole on the right half plane? I do a graphical technique. The answer is yes. System is unstable. Answer is no. Well, system is stable. So, let us just ask few questions what does it mean when the q s -plane encircles the origin, right. So, this is little important here, right. So, why I am not really interested in the shape? It could be this one or it could just be something I am just interested in this origin. Does the contour on the q s -plane contain the origin? Not necessarily here. Here it could be whatever I will tell you in some other stuff related to this, ok.

Now, let start by considering the case when the s -plane contour encloses only one of the zeroes of q s . So, what does that mean? So, q of s has n zeroes and let say that this guy encloses 1 0. So, let me draw 0 here and I call it as alpha 1. So, all the other poles and zeroes are outside the contour. Now, given this complex number I can compute the magnitude of q s as blah blah blah and I can also compute the angle, right the angles of the zeros minus the angles of the poles.

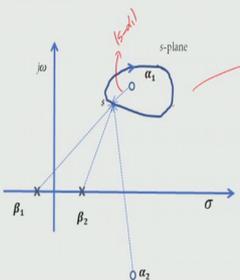
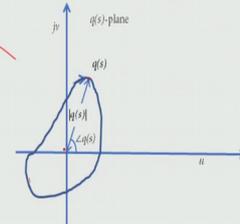
(Refer Slide Time: 30:48)



Stability in the frequency domain



- As the point s follows the path in the clockwise direction on the s plane contour, the phasor $s - \alpha_1$ generates a net angle of -2π , while all other phasors generate zero net angles.
- This means that the $q(s)$ phasor also undergoes a net phase change of -2π .
- And the tip of the $q(s)$ phasor describes a closed contour about the origin of the $q(s)$ plane in the clockwise direction.

Control Engineering
Module 6 - Lecture 2
Dr. Ramkrishna Pasumarthy 12

So, let us see here. So, this is the contour in the s -plane which encircles a 0, ok.

So, let us see as the point s follows the path in the clockwise direction. Now, let see I am here and I just take a clockwise direction. I come all the way and I just end up here now with respect to this, I just say looking at this particular phasor, right. What does this do? This actually has to rotate all the way and come back which is it does a net angle or it generates a net angle of minus 2π because the positive angle we measured as the anticlockwise. Now, what does this do? This angle from the pole and beta 1, well it goes here, this phasor goes here, here, here, here and again comes back here, right. So, it just goes this and just comes back this, right. It just goes this and comes back this, whereas the other way it just starts here and does the complete revolution like this. So, the pole just as this and this comes back over. So, how much angle does this generate? Well, the angle is 0 and similarly, this guy again this goes again and again comes back to same position. It does not really take a revolution or a rotation similarly with alpha 2, si alpha 1 generates or the phasor s minus alpha 1. So, if I just was to mark this, this guy would be the phasor s minus alpha 1. The phasor s minus alpha 1, first you have to write it here. It generates an angle of minus 2π while all other phasors generate 0 net angles, ok.

Carefully we will go to the next statement. What does this mean that q s phasor also should undergo a net phase change of 2π . So, it is something like this, right. If I take a q s here, I know that q maps the s -plane to the q s -plane. So, if s undergoes the phase shift of 2π , so this also should undergo the phase shift of 2π just from the expression of q s and the tip of the l q s phasor which is this guy, it describes a contour about the origin. What happens when s equal to alpha 1? This is the origin. So, this guy maps to the origin here, ok.

So, again I have here two zeroes, alpha 1 alpha 2, two poles beta 1 beta 2. I just know like this, pick this guy alpha 1 and I just generate a contour along this in the clockwise direction. So, I go here I see that this guy also generates a contour, right a close curve again in the clockwise direction, right and I can just measure this phasor and also the angle that is just from the alpha ones and beta ones and beta twos. Not only that if I am encircling 0 here, 0 I will encircle the origin here, that is because when s equal to alpha 1, my map what does q of s do when s equal to alpha 1, it maps with the origin. It is a very simple way of understanding this. There is actually much complicated proof, some complicated, I am sorry some complex analysis which we will not go into, but we will

just learn what complex analysis is telling us, we just believe what people from complex analysis told us, but we will just use it in our own way, right. We will just be a little smart and translate in our own language and say well, this information which they proved is correct and will actually help me in my stability analysis.

(Refer Slide Time: 34:53)

Stability in the frequency domain

- Further, if the s -plane contour encloses two zeros, the $q(s)$ plane contour encircles the origin twice, again in the clockwise direction. (This generalizes to encirclement of n -zeros).

$j\omega$ s -plane $q(s)$ -plane $j\omega$ u

- If the contour does not encircle or enclose any pole or zero, the corresponding contour in the $q(s)$ plane will *not* encircle the origin.

Control Engineering Module 6 - Lecture 2 Dr. Ramkrishna Pasumarthy 13

So, what happens when my contour encir or encloses two zeroes. So, let me just draw this, say this is 0 here, there is a 0 here and I just got on this. So, this is my s -plane with sigma and j omega. So, what will happen here is that I will encircle the origin twice, once like this and once like this, something like this again. The shape is not really important. What we are interested is in the encirclements of the origin, right. So, you have a u and j b , ok.

Similarly, if there are n zeroes, I will encircle the origin n times. Again n this is clockwise direction, this is also the clockwise direction. Lastly if the contour does not enclose any pole or 0 as I said earlier, so if I just here I will just be here, right. I will not trouble everything if I am just here, I will just be here. Nothing happens. It will never encircle the origin. That is what the q map that tells me, ok.

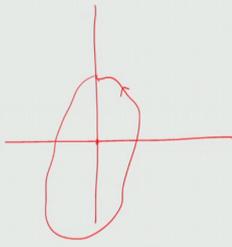
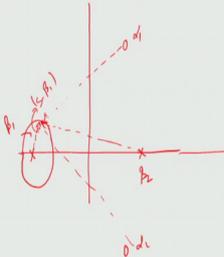
(Refer Slide Time: 36:01)



Stability in the frequency domain



- What about encirclement of a pole in the s-plane?
- Consider encirclement of a pole $s = \beta_1$ by the s plane contour. The phasor $(s - \beta_1)$ generates an angle of -2π as s traverses along a closed path.
- Correspondingly $q(s)$ experiences an angle change of $+2\pi$. ($s - \beta_1$ appears in the denominator of $q(s)$)
- This means one counter clockwise encirclement of the origin.



Control EngineeringModule 6 - Lecture 2Dr. Ramkrishna Pasumarthy 14

Now, what is encirclement of a pole? So, let us say again I and do the same thing just draw it for here. So, let us say I have a pole here and this was my beta 1 here. When I had a pole here, earlier this was beta 2, I had a 0 somewhere here and 0 somewhere here, alpha 1, alpha 2, ok.

Now, again let us say I am just having a contour, where I do this, right this guy and I start from here. So, this phasor is my s minus beta 1. So, what happens again let us see all the angles, right. If I am starting from here, this is my point s and say there is a phasor from here, there is a phasor from here and a phasor from here. Now, how does the phasor alpha 1 change? If I just pro this pen, the phasor alpha 1 and the clockwise, it goes here, traverses all the way and just is here, right again which means it generates an angle of 0, right. So, again just look at here, it goes here here and again comes back here.

Similarly, beta 2 contributes an angle 0, alpha 2 also contributes to an angle 0. What is beta 1 doing? Beta 1 is in this position. It just goes again all the way, this way, right. Again the phasor s minus beta 1, I will just show this here, it generates or it traverses an angle of minus 2π as s traverses along close path. Therefore, what will happen here is that q s will now have an angle change of 2π say somewhere here and this is plus 2π because s minus beta 1 occurs in the denominator angle always becomes minus when I am talking of the angle of a pole and this means, again I just encounter or encircle 1, I encircle the origin once in the counter clockwise direction.

Now, again we are not really see why exactly we are encircling the origin. It is the same arguments as before, right. So, what we will just try to understand is the following that if in the s-plane, I encircle 0 in the clockwise direction, the corresponding contour in the q s-plane encircles the origin once in the clockwise direction. If there are more than one zeroes, say 5 zeroes, I will encircle 5 times. If I am encircling a pole in the clockwise direction correspondingly in the q s-plane, I will encircle the origin once now in the counter-clockwise direction because I am looking at the pole which contributes an angle of negative. It goes to the denominator, ok.

(Refer Slide Time: 39:20)



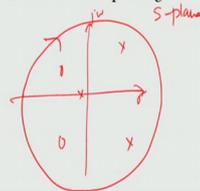
Stability in the frequency domain



- Therefore, if there are P poles and Z zeroes of $q(s)$ encircled by the s plane contour, then the corresponding $q(s)$ plane contour must encircle the origin
 - Z times in the clockwise direction and
 - P times in the counter clockwise direction.
 - Net encirclement of the origin is $(P-Z)$ times in the counter-clockwise direction.
- The standard pole zero form of open-loop transfer function is

$$G(s)H(s) = K \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$q(s) = 1 + K \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} = \frac{(s+p_1)\dots(s+p_n) + K(s+z_1)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)}$$



S-plane

Control Engineering
Module 6 - Lecture 2
Dr. Ramkrishna Pasumarthy 15

So, therefore, I can just say something more now if there are P poles and Z zeroes of q s encircled by the s -plane, so let us say you know I just draw this and I say I am not really interested in where these poles are. There they could be stable, they could be unstable on the imaginary axis at the origin, whatever. I am not talking about stability at all. I am just saying well say there is a pole here, a pole here, a 0 here, a 0 here and say one more pole here, right and I say I just want to have some fun and I say I just enclose this all the time. Now, I just you know say in the s -plane, I encircle all of them. So, this is $\sigma + j\omega$ in the s -plane, ok.

Then, the corresponding q s -plane contour must encircle the origin z times in the clockwise direction. In this case, it should be two times in the clockwise direction, p times in the counterclockwise direction of poles two 0 two times in the counter

in the clockwise direction, three poles three times in the counterclockwise direction and the net encirclement is P minus Z in the counterclockwise directions. That way if this number is say well, I will always say this number will always be greater than 1 possibly or even if it is less than 1, it does not really matter because I am just its minus one times counterclockwise means one time in the clockwise direction. So, the sign will take care of themselves, ok.

So, we now know about encirclements, the mapping from the s -plane to the q s -plane. What happens if I encircle poles and zeroes in the s -plane? How does it reflect on the q s -plane? That I know now what is the relation to stability, the standard pole zero form, I know that my open loop transfer function is $g h$ has again a set of zeroes, a set of poles, right and my characteristic equation q of s is 1 plus k , again this side of zeroes and this side of poles and I can write this as an equation of the form s plus z prime till z n prime. This number will no longer be m , but will be n because this will go here say this is s plus p 1 till s plus p n plus k times s plus z 1 till s plus z m over again these open loop poles s plus p 1 s plus p 2 s plus p n . So, this will also be s power and this will also be s power. So, this will have n zeroes and n poles, ok.

(Refer Slide Time: 42:14)



The Nyquist stability criterion



- The zeros of $q(s)$ are the roots of the characteristic equation.
- The poles are the same as the open loop poles.
- For the system to be stable, the roots of the characteristic equation (*the zeros of $q(s)$*) must lie in the left half of the s -plane.
- What if the open-loop system has poles in the right half plane? Meaning the open loop system is unstable.
- The expression for $q(s)$ suggests that even if the open-loop system is unstable the closed-loop system may be stable.
- We need to investigate if there are any roots of $q(s)$ in the right half s -plane.
- This leads us to the Nyquist stability criterion.

$$1 + K \frac{(s+3)(s+2)}{(s+1)(s+4)} = 0$$

Control Engineering
Module 6 - Lecture 2
Dr. Ramkrishna Pasumarthy 16

The zeroes of q s are the roots of characteristic equation. So, this is the transfer function, it is the set of zeroes and a set of poles, q s going to 0 would give me that all these s plus z 1 prime multiplied by s plus z 2 prime all the way till s plus z . M prime should go to 0

and therefore, the zeroes of $q(s)$ of $q(s)$ nothing to do with $g(s)h(s)$. Now, we are just doing with $1 + k$, this one and this one or $1 + g(s)h(s)$. Only the characteristic equation and the poles of $q(s)$ are the same as the system poles, ok.

Now, when is this system stable? For the system to be stable, the roots of the characteristic equation $1 + k$ or $k(s)$ plus s minus z , I can use also what s plus, this does not matter. $1 + k(s)$ plus z $1 + s$ plus z m over s plus p $1 + s$ plus p m equal to 0 or this guy is 0, right. So, this guy is equal to 0. So, the roots of this equation must lie in the left half of the s -plane. That is what we know from the earlier lectures, ok.

Now, these roots of the characteristic equation turns out to be the zeroes of this $q(s)$. So, if zeroes is confusing, you just leave it. You can just say the roots of the characteristic equation, nothing else even if I write root in italics. You can also just not read this if the term 0, why am I interested in 0 is confusing. You do not read that. Just say well the roots of the characteristic equation, that is all whether it happens that these guys just are in the numerator, right and then, the numerator terms you always call them zeroes, ok.

Then, the question arises what if the open loop system has pole in the right half plane, right which means that the open loop system is unstable say this p_1 is at plus 1 does not matter, right. This p_1 even if it is in plus 1, I can or I may have all these z ones till z m to be on the left half. The expression for $q(s)$ is therefore such as that even if the open loop system is unstable. The close loop system may be stable because these are the open loop poles and these are the closed loop poles. We will see an example also. So, meaning to say even if the open loop system is unstable, I can make it stable. This is not what Nyquist will teach us, how to do that Nyquist will just tell us though is the close loop system stable or not.

Now, what we need to investigate are there any roots of $q(s)$ in the right half s -plane, right or any of these numerator guys z_1 prime. Till z_n prime is there, any one sitting in the right half plane is z_1 prime z_2 prime z_n prime is any one sitting on the right half plane. That is what we will try to investigate and that is what leads us to define the Nyquist stability criterion.

(Refer Slide Time: 45:51)



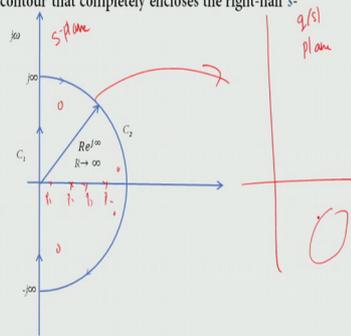
The Nyquist stability criterion



- To investigate the presence of any right half zero, choose a contour that completely encloses the right-half s -plane.
- Such a contour is called a Nyquist Contour.
- It is directed clockwise and comprises of an infinite line segment C_1 along the imaginary axis and an arc C_2 of infinite radius.

along C_1
 $s = j\omega$, ω varying from $-\infty$ to $+\infty$

along C_2
 $s = R e^{j\theta}$, θ varying from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$



Control Engineering
Module 6 - Lecture 2
Dr. Ramkrishna Pasumarthy 17

So, now the encirclements will come into picture. Now, the polar plots will come into picture and so on, ok.

So, what do I need to investigate? I need to investigate the presence of any right half 0. This is again the right half 0 of q of s , right. The zeroes of q of s are my close loop poles. Just you write down that and I will just try to understand what it means and I am not really talking of zeroes here. I am still talking of the closed loop poles which happen to be the zeroes of the denominator transfer function $1 + g h$ or $1 + k g h$ which is q . Now, choose a contour that completely encircles or encloses the right half s -plane, right and this is called as the Nyquist contour again. It is directed clockwise and comprises of an infinite line segment $j \omega$ c_1 which is infinity and then, an arc of again infinite radius right, ok.

What happens along c_1 ? Along c_1 what is happening is s is just $j \omega$. This is c_1 , this line s is all $j \omega$ with ω varying from minus infinity to plus infinity. So, I am starting from minus infinity all the way till plus infinity, the entire imaginary axis and along c_2 . What is c_2 s is $r e^{j \theta}$. Just the phasor here again θ . What does θ do? θ goes from here till here or here, till here at whatever whichever if you measure from here to here, this is a θ varies from $\pi/2$ to minus of $\pi/2$, right. So, this means actually I am taking care of the entire right half plane including the imaginary axis, ok.

(Refer Slide Time: 48:20)



The Nyquist stability criterion



- Why do we need to define such a contour?
- It encloses all the right half s-plane poles and zeros of $q(s) = 1+G(s)H(s)$.
- Suppose there are Z zeroes and P poles of $q(s)$ in the right half s-plane.
- As s moves along the Nyquist Contour in the s-plane, a closed contour is traversed in the $q(s)$ plane which encloses the origin

$$q(s) = \frac{(s+z_1)(s+z_2) \dots (s+z_n)}{(s+p_1)(s+p_2) \dots (s+p_m)}$$

$$N = P - Z$$

- times in the counter clockwise direction.
- Now, how do we test stability with these encirclements of the origin?
- For a system to be stable, there should be no zeros of $q(s)$ in the right half s plane.

$$Z = 0$$

- This is met if $Z = P$. $N = 0$.

Control Engineering
Module 6 - Lecture 2
Dr. Ramkrishna Pasumarthy 18

Now, we will ask some questions. First is why do we need to define a contour, such a contour I just have to start from 0, go till infinity in you know j plus j omega infinity minus j omega infinity. Not only that, I also have to travel $r e^{j\theta}$ with r going to infinity. Why do I need to do this? Well, such a contour encloses all the right half plane, all the right half poles and zeroes of my characteristic equation. Now, suppose this starts everything will now start to make sense of what we did so far. Suppose there are z zeroes and p poles of $q(s)$ in the right half plane, again we just write out z zeroes. So, zeroes of $q(s)$, where $s + z_1$ prime $s + z_2$ prime and so on until $s + z_n$ prime and then, you have poles $s + p_1$ prime. These are all open loop poles. In other words, all these z_1 prime till z_n prime are the closed loop poles, ok.

So, as s moves along the Nyquist contour in the s-plane, this is a Nyquist contour which we define as the entire right half plane. Now, as s moves along the contour in the s-plane, a close contour is traversed in the $q(s)$ -plane which encircles the origin. So, let us say, sorry this is my Nyquist contour. So, it has zeroes here, poles here, few more zeroes, few more poles. Whatever now if I map this contour, what I wanted to go from the s-plane, so this is in the s-plane, the Nyquist contour and I want to see what happens in the $q(s)$ -plane. This $q(s)$ -plane will encircle the origin, right. That is what I learnt from these previous slides here, right these things encirclement of the origin, ok.

So, as s moves along the Nyquist contour in the s -plane, the closed contour is traversed in the q s -plane which encloses the origin and equal to p minus z times in the counter clockwise direction. That is what we also observed somewhere here, encirclement of the origin p minus z times in the counterclockwise direction. Now, what has this to do with stability? I am still talking of so many things. I talked of polar plots, I talked of omega from 0 to infinity, I talked of encirclement of poles and zeroes and then, correspondingly I said something in the clockwise, something happening in the clockwise direction. Now, we will see actually make sense now how do we test stability with these encirclements of origin.

Now, I will again ask myself a question when is this system stable. Well, for the system to be stable, there should be no zeroes of q s , this guy in the right half s -plane because this zeroes of q of s are the roots of the characteristic equation and these are also the poles of the closed loop system. Now, this system for this system to be stable, z should be equal to 0. There should be no zeroes on the right half plane. Now, this is met if, sorry this is met if the number of encirclements is p . It is not z ; it is n . Z is anyways 0. So, if the number of encirclements in the counterclockwise direction and this is n the number of encirclements in the counterclockwise direction is equal to the number of poles of q s in the right half plane because I am just my contour is just right half plane.

So, there might be p_1 sitting here, p_1 p_2 p_3 p_4 and so on. See there are four of these guys sitting here. Then, this n should be equal to 4 if the system or for the system should be stable, there should be no zeroes. All these z_1 till z_n should be on the left half plane. So, none of them should sit on the right hand side. So, z should be equal to 0. Now, if z is equal to 0 and I measure the number of counterclockwise encirclements, what should that match? That should match the number of holes from p_1 till p_n on the right half plane, ok.

(Refer Slide Time: 53:24)



The Nyquist stability criterion



- This means for the closed-loop system to be stable, the number of counter-clockwise encirclements of the origin of the $q(s)$ plane should equal the number of right half s-plane poles of $q(s)$.
- These are the open-loop (*unstable*) poles of the transfer function $G(s)H(s)$.
- If the open-loop system is stable, then the closed loop system is stable if

$$N = P = 0$$
- Which means, no encirclements of the origin.

$$G(s)H(s) = [1 + G(s)H(s)] - 1$$
- The contour of $G(s)H(s)$ corresponding to the Nyquist contour in the s-plane is the same as the contour of $1 + G(s)H(s)$ drawn from the point $(-1 + j0)$.
- Thus the encirclement of the origin (of $1 + GH$) is equivalent to the encirclement of the point $(-1 + j0)$ by the contour of GH .

$(\sigma - j\omega) - 1$
 $1 + j2$
 $0 + j0 - 1$
 $(-1 + j0)$
 $(0 + j2)$
 $(\sigma - 1, +j\omega)$
 $q(s) - 1$
 encirclement of origin of $(1 + GH)$
 encirclement of $(-1 + j0)$ of GH

Control Engineering
Module 6 - Lecture 2
Dr. Ramkrishna Pasumarthy 19

This means that for the closed loop system to be stable, the number of counterclockwise encirclements of the origin of the q s-plane should equal the number of right half s-plane. That is what I just explained right through these things, ok.

These are open loop unstable poles. So, I do not really worry now is the open loop is some unstable, open loop is unstable. I cannot do anything. So, let us close the books and go home. No, there is still hope. If n equal to p , then my closed loop system is stable, right and if open loop system is a nice guy, if it is stable, then close loop system is stable if and only if n equal to p equal to 0. So, if all these guys are well behaved, then there is no p which is on the right half plane which means this number p here, this number p here will go to 0. So, n equal to p equal to 0.

So, my stability now translates to the number of times I encircle the origin in the counterclockwise direction, right. For example, if all these poles are stable and say n is equal to 1 or minus 1 to say n equal to minus 1, then my system is unstable because well there are no poles of all. P_s are equal to 0 and whenever z is greater than 1 or greater than or equal to 1, then I am unstable, right. I always want z to be 0. This is a necessary condition. Now, n equal to p equal to 0 which means no encirclements of the origin. So, in this entire region here, I do not encircle zeroes which I should never encircle. I should never encounter zeroes and I also never encounter poles here. So, may be my contour here when there are no poles and zeroes might just be here and therefore, I am happy, ok.

So, well you might say you know sir this is actually too complicated because I start with an expression which already looks complicated. There are m poles, sorry m zeroes n poles. I have to compute all this, count these numbers that might be cumbersome. However, this is actually given to me, right. The plant is given to me, the feedback loop h is also given to me and what I promised you or what I told you is that just by looking at the open loop transfer function g and h , I can tell you the closed loop behaviour, but no, well I actually made you compute all these z^{-1} primes till z^{-n} primes to see where they are and then, I said something to do with you know z^{-p} , p minus z encirclements and so on, but we will just simplify things a little bit. This is just to again get to the point which I promised you that just by the open loop behaviour, I can tell the close loop system stability, ok. So, this is my close loop characteristic equation $1 + g$ times h and I can just rewrite this as the q of s minus 1 .

So, this is just my q of s I am just rewriting this as q of s minus 1 . Now, this is a complex number just say I am looking at $0 + j 0$ and I am just doing minus 1 . So, what happens is that the real part shifts with minus 1 minus $1 + j 0$. Similarly, I have $1 + j 2$. So, I am just doing a minus 1 . So, what happens I will just you know have $1 - 1 + j 2$. Similarly, if I have just any complex number $x + j \omega$, I do a minus 1 . I will just get a $x - 1 + j \omega$, right. The imaginary thing always remains the same and the real part is shifted by 1 , right. If this is the σ , it just becomes $\sigma - 1$, ok.

If the contour of $g h$, this is the same as the contour of one times g times h just that I am just shifting the real part to minus $1 + j 0$. I will show you a plot of this and therefore, if I say well have I encircled the origin of $1 + g h$, this is actually equivalent to saying have I encircled or asking a similar question have I encircled the point minus $1 + j 0$ of $g h$. So, I am looking, I am comparing encirclements of region of $1 + g h$ to this is minus $1 + j 0$ of g times h . So, I will just show you a plot, right. So, this is $1 + g h$, ok.

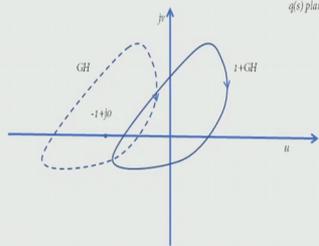
(Refer Slide Time: 58:48)



The Nyquist stability criterion



- If the contour of GH of the open-loop transfer function $G(s)H(s)$ corresponding to the Nyquist contour in the s -plane encircles the point $(-1+j0)$ in the counter clockwise direction as many times as the number of right-half s -plane poles of $G(s)H(s)$, the closed-loop system is stable.



- If the open-loop system is stable, then the closed-loop system is stable if the contour of GH does not encircle the point $(-1+j0)$.

Control Engineering
Module 6 - Lecture 2
Dr. Ramkrishna Pasumarthy 20

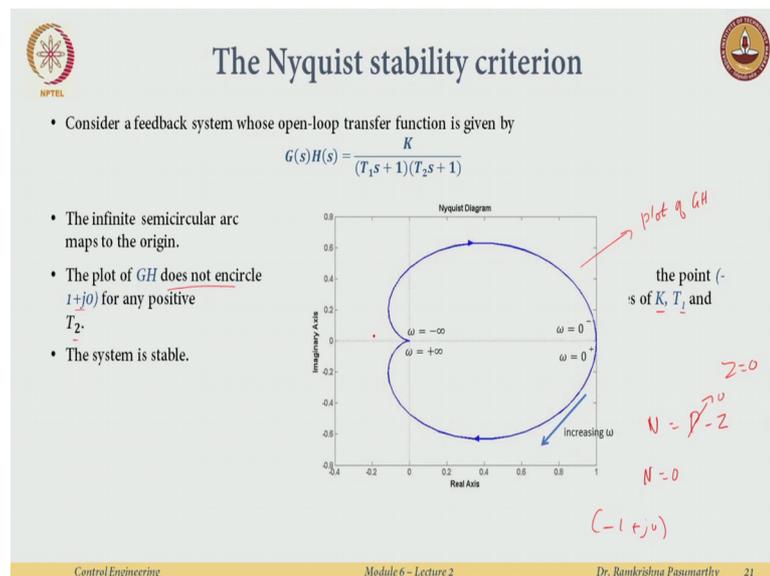
Now, it shifted by minus 1 minus 1 plus $j0$. This is 1 plus $g h$. I just do minus 1. So, this just shifts here which means the origin which I am interested in the encirclement of the origin, now I just do a minus 1 all of a sudden. So, therefore, this origin I was interested in encirclement of the origin of 1 plus $g h$ or the contour of 1 plus $g h$, now I am just doing 1 plus $g h$ minus 1. So, the origin also shifts here, sorry I am really sorry. So, the plot shifts by minus 1 in magnitude. So, you will see all this distance would be minus 1 and then, from here I shift here, ok.

So, encirclement of origin of 1 plus $g h$ is equal to encirclement of the point minus 1 plus $j0$ of $g h$. Now, this is easier to check because I know g , I know h , I do not need to compute z_1 prime all the way till z_n prime. So, now, all the results of n equal to p minus z of the origin will transfer to again n equal to p minus z not of the origin, but now of the point minus 1 plus $j0$. This will be my point of interest. So, let us read the statement. If the contour $g h$ of the open loop transfer function corresponding to the Nyquist contour, if it encircles the point minus 1 plus $j0$ in the counterclockwise directions as many number of times as the right half poles of g times h , the close loop remains stable, everything remains same. It is just that the encirclement of origin now just translates to encirclement of minus 1 plus $j0$, everything you just go to the previous slides, replace the origin with minus 1 plus $j0$ and you will have this thing, ok.

So, if the open loop system is stable, again I am just rewriting all those things which I said. Then, the closed loop system is stable if the contour does not encircle the point minus 1 plus j 0. Earlier I said if the open loop was stable, then the closed loop system or the closed loop contour of 1 plus j g h should not encounter, they should not encircle the origin. Now, you say well if the open loop system is stable, the contour of g h should not encircle the point minus 1 plus j 0. Just origin control c and then, this control v, right or control f and control v, find and replace. So, now, this is our point of interest minus 1 plus j 0.

To summarize all these things, we will just, so this is the statement of the Nyquist stability criteria, right and I am just interested in the point minus 1 plus j 0, ok.

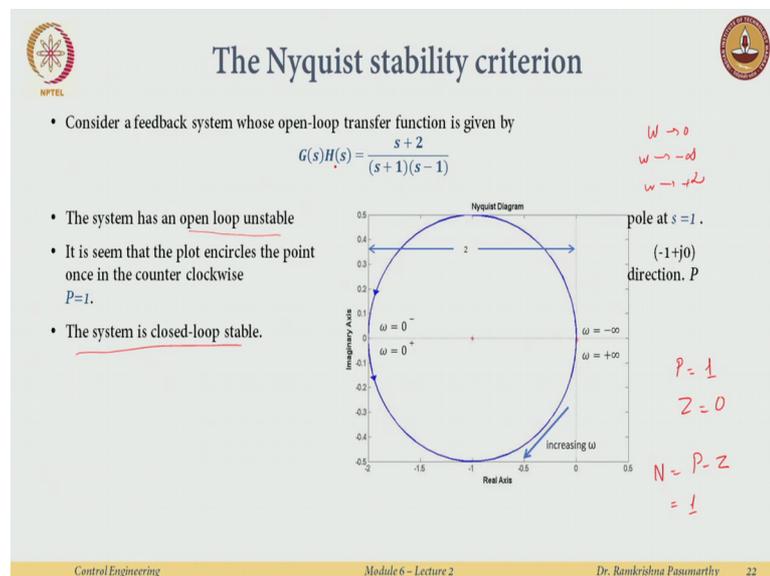
(Refer Slide Time: 61:54)



Let us summarize with the help of example, right. Consider a feedback system whose open loop transfer function is this guy. Now, this is similar of a polar plot which we had in one of the earlier slides. So, let me slowly go there and this one, right. So, just what I am interested little more in doing is there I plotted as omega was from 0 to infinity. Now, I plot a little more, right. I am also interested in minus infinity because if I just plot omega from 0 to infinity, I am just looking at this one, this contour, but I am interested in the entire right half plane. So, I just the same thing, right. I just look at what happens when omega is minus infinity, nothing changes. I just have to compute again the angle and the phase, the angle and the magnitude or the magnitude and the phase.

So, for this example, well the infinite semi-circular arc, it gets mapped to the origin the plot of $g(s)$. So, this is my origin, somewhere here is the point minus 1. It is not even in the graph. So, I am nowhere near to minus 1 plus $j0$. So, what do I conclude is that the plot of $g(s)$ again there is some distortion here. I will just take care of this. So, this is the plot of $g(s)$, right and then, it does not encircle minus 1 plus $j0$ for any positive values of k t_1 and t_2 , sorry for the distortion I will take care of the slide and therefore, n must be p minus z open loop system is stable. Therefore, p is 0, z should be 0 for stability. Therefore, n should be 0 that of the point minus 1 plus $j0$, but if you look here, I am really far away from minus 1 plus $j0$. Therefore, this system is stable.

(Refer Slide Time: 63:49)



Now, look at this example s plus 2 s plus 1 s minus 1. This open loop system is unstable. So, this is some interesting case. Now, I just draw the Nyquist plot, I just look at what happens at ω is 0, ω is minus infinity ω at plus infinity and I just plot this, right. So, I start from ω is 0 and from here and then, I reach here. So, I start from minus infinity, I go all the way here and I go all the way here and I just plot this and I say where am I. So, this is my origin here. I am at a distance of 2 and this 1. So, here my p is 1, z should as always be 0. Therefore, n which is p minus z should be 1 in the counterclockwise direction, ok.

Let us just check this. So, if I just do move in the counterclockwise direction, am I encircling the point minus 1 plus $j0$. So, this is the point minus 1 plus $j0$. I am encircling

it one time and therefore, the close loop system is stable. I start with an open loop unstable system and Nyquist tells me that it is actually closed loop stable, right with $1 + g \times h$ as the feedback and I am closed loop stable, right. It does not tell me still how can I make a close loop system stable, can I do it always. None of those questions are answered. Just that well I can make or I can start from open loop unstable system. Nyquist will tell me well is it close loop, stable or not, right just via these plots. So, the polar plots started from 0 to infinity, here I am even going from 0 minus to minus infinity. That is the only little change, but nothing changes in the analysis, ok.

(Refer Slide Time: 65:46)

The slide is titled "Overview" and is divided into two columns. The left column is titled "Summary: Lecture 2" and lists four bullet points: "Polar plots", "Stability in frequency domain", "Nyquist stability criterion", and "Example". The right column is titled "Contents: Lecture 3" and lists three bullet points: "Special cases of Nyquist criterion", "Relative stability in Frequency domain.", and "Phase margin". The slide also features logos for NPTEL and IIT Madras at the top, and footer text at the bottom: "Control Engineering", "Module 4 - Lecture 1", "Dr. Ramkrishna Pasumarthy", and "23".

So, what we learnt so far is polar plots and we just started to define for ourselves the notion of stability in the frequency domain leading to the Nyquist stability criterion, did a couple of examples and next, we will do some special cases of the Nyquist criterion. What if I hit a singular point, what if there is a pole on the imaginary axis things like that and then, we will also define. So, now, everything was to do with $-1 + j0$, how far I am from $-1 + j0$. So, we will define relative stability in the frequency domain and introduce the concepts of gain margin and phase margin, ok.

Thank you.