Control Engineering
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Module - 06
Lecture - 01
Introduction to Frequency Response

Hi guys, so far what we have done is classified as Time Response Analysis. So, we started by defining the transfer function, we saw how the step response of the station looks like and we characterized things like the overshoot settling time, we also characterized various errors like the position error, the velocity error, acceleration error and so on.

Later on we used some algebraic methods to compute if the system is stable or not, whether or not certain poles or all of the poles lie to the left half of say some point of minus 1 because if I reduce stability, then we had the root locus criterion where we had a graphical technique to see how the roots of the close loop system change as the gain k was increased from 0 to infinity. So, we have not done any design problems so far, but what we learnt through the root locus technique is that I can guess or I can predict accurately the behavior of the closed loop by just looking at the open loop behavior. So, when I do the root locus, I just need to know what is g and h, right and then, I can just tell how my closed loop system would behave based on changes in k from 0 to infinity. I also saw examples where after a while we keep on increasing k, the system becomes unstable and so on.

So, what we will do today is to look at the other part of the analysis which is the frequency domain analysis to see what happens to the system or how the system responds to inputs of different frequencies. So, let us do a little bit of introduction before we actually go to more deeper analysis.

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Fig 6.1.1 – Sinusoidal response of a system

So, I start with as usual systems which are linear time invariant, right. So, the frequency response of a system is the steady state. Now, this is important. The steady state response of the system towards sinusoidal input signal and I will tell you why we are only interested in the steady state response. So, the reason why we study these kind of sinusoidal outputs is if I give pass a signal through LTI system, if this signal is sinusoidal, the output is also sinusoidal of course. What will change is its magnitude and there will also be a bit of phase shift depending on what is sitting inside the system, right. So, we will exactly characterize how the b will change based on what is sitting inside here and similarly, how the phi will change again based on what is sitting inside here, ok.

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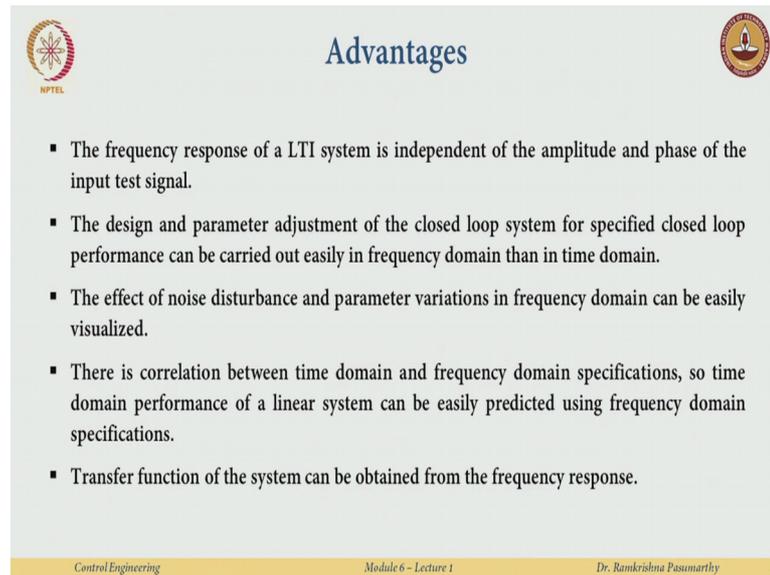
So, why do we need to do all this analysis? So, most of this analysis or what we call also as the frequency response is independent in a way of the amplitude and phase of the input test signal, right. So, we will see why that is true when I go to design a certain system. So, the design and parameter adjustment of the closed loop can be carried out equally or rather easily in frequency domain than in the time domain. So far we have not looked at design problems, but this will be obvious when we start handling design problems in the next, in the coming lectures.

We also saw what the effect of noise and disturbance is in terms of the frequency, right. So, when we were talking about sensitivity and the complimentary sensitivity function, we classified the disturbances as low frequency signals and noise as high frequency signals and now, we will see how we can actually kill the effect of noise in a certain setting, right. So, the effect of noise and disturbance parameters can be easily visualized and then, of course there is always a correlation between the time domain and frequency domain specifications which we will derive shortly.

So, given you know certain performance specifications in the time domain, I can actually even translate it into frequency domain and another thing what I could also do is if I would not know how the transfer function of a system looks like, I can just give it certain signals of varying frequency and based on those frequency response, I can easily estimate what the transfer function of the system is. You may also say well why not just

look at the step response of the system and get the transfer function because the step response included zeta, it included omega n.

I could measure the peak overshoot, I could measure the rise time, the settling time bunch of other things and based on these things, I could get the transfer function. Then, what is the problem? Well, one drawback over there is that we just analyzed second order systems, right and sometimes it might be a little cumbersome to even compute those things. What if the system is over damped, right? What if the system is how do I find out if it is actually critically damped or it is not there could be some difficulties, right in just looking at the step response and looking at and there estimating the transfer function based on that the frequency response analysis or just looking how the frequency, how the system magnitude and phase I am using, these terms without defining them, but it will be clear in the next couple of slides, how the magnitude and phase of a system changes with different frequencies and I just do a plot of that and I can get a very good estimate of how the transfer function looks like. Not only that I could even look at what could be the error constants, right.
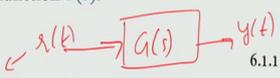
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So, we will see that slowly as we progress through the remaining of this module, as a simple example let us start with LTI system, where I have usual thing is I have a system, some transfer function G of s, I have a input signal which is sinusoid of this form and I will just be interested in how my y of t changes. So, the steady state response of a stable,

well this guy is important if the system is unstable. Then, I do not really need to do any kind of analysis. The steady state response of a stable LTI system to a sinusoid does not depend on initial conditions and therefore, we can just assume that all the initial conditions are 0 and we will also see why this is true. If I just take a signal of this form, I know that r of s would be a omega s square plus omega square. This was just the Laplace transform of this guy and the output now becomes c of s is a omega s square plus omega square times r of s, ok.

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So, what I know I can just do all the partial fraction expansion and all those things which we do to get how the time response of the system or how the solution of c looks like in time domain. I get something like this, right. So, why do I call this as a transient response? Essentially because these guys will consider term will have terms like k1 e power minus t, sorry this will not be t1 e power minus t and depending on where the locations of the poles are I y would come k2 e power minus 2 t and so on. It could also if there are repeated roots, there could also be terms like t times k3 e power minus 2 t, but I know that this term goes to 0 as time goes to infinity and therefore, I am just looking at the steady state response because as t goes to infinity, this entire thing goes to 0 and therefore, I also said earlier that I am interested in stable systems.

If the system is unstable, it would mean there is a pole on the right half plane and that could be something like k4 e power plus 60 and you know that this guy actually blows

up to infinity as time goes to infinity, right. Therefore, there would be nothing left to analyze, right. So, these guys go away. So, the output contains of course the transient terms and the steady state components. Now, at steady state, these guys die down and what is remaining is the effect of only these terms k s plus j omega k s minus j omega, right and these are essentially due to the input, right this guy, ok.

Now, what is this response at steady state when the input is a sinusoid? If you look at the earlier analysis in the step response, even though we had oscillations in the system. Those oscillations were dying down, right say if I just take an undamped system, we had response which was like this for the step response and see there were still oscillations, but they were actually going down right with time. So, there you had terms like e power minus t cos omega t and so on, right. So, even though whatever frequencies we had, the oscillations just died out and whatever remains after these things go away with this guy.

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So, I just look at how the response looks like C ss and at the end I am calling as ss because of steady state k over s plus j omega plus k star s minus j omega and I just use the simple tricks from what I learnt in Laplace transform of how to find these constants k and k star. They turn out to be something like this, right. K would be this guy and the k star would be this guy. I do not really need to teach you how to compute these. These are extremely straightforward things, right and then, I take the inverse Laplace transform.

Well, these are also kind of easy to compute and what I get as the result is I do all the computations and what I get you know this is important, right.

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So, the response is b times sin omega t plus some phi, where the input what was how did our input look like. The input looked like a sin omega t. So, a sin omega t now gets transformed to a signal b sin omega t plus some phi. What does this b, where does this b come from? Well, this b is simply a times the magnitude of the transfer function, right the system which you are passing the sinusoid through. Similarly, phi I could just even compute it as just the angle of this guy, right. Phi is just the angle of g of j omega given this is g. I know g is a complex number and I could easily find out what the angle is and what is also important to notice here is that well I used the word the sinusoidal transfer function, right which is essentially the substituting s with j omega because I am only interested now in the steady state things.

Only the frequency components are remaining, though I can just ignore the real component, right. I am just looking at these guys s equal to j omega and therefore, I just take a transfer function, I substitute s equal to j omega, call it the sinusoidal transfer function and that will help me in the entire frequency domain analysis, right. So, this is little important and this is little simple trick also. Here you could see that right all the other guys just disappear and what I am left is just j omega terms, just these two poles are remaining right and this is why I call them as the sinusoidal transfer function because

after all these things, this is what is left, right. So, that is even obvious through the entire steps which we do, right. So, we just compute k is just g minus j omega that is just when you get the partial fraction substitution k would be ok.

So, I would be just I think let us do some computations. Here substitute s equal to minus j omega and then, you get this thing, right. Similarly, this guy you substitute s equal to plus j omega and then, compute k star and then, you see all these are j omega j omega j omega, right. Nothing to do with the real term, right, this is actually a result of a systematic computation and I know that e power j of something e power minus j would actually give me a sinusoidal signal. This is again some basic complex analysis, right. So, what is the conclusion or the first observation here is I pass a signal a sin omega t through a transfer function g of s. What I see in the output is that I have a different magnitude, I still have a sinusoid with a little phase shift and the magnitude is the magnitude of original signal minus, sorry multiplied by the magnitude of g of j omega the transfer function. Now, I also know why I only take this j omega and then, of course the phi is result of something like here, right. Let us do a very simple example to see what this actually means.

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So, let say I have g of s, I have r of t or r of s whatever we call it and then, I am just interested in how c of s looks like or the c of t, right all in steady state. So, let me assume that g of s is something like some constant over t s plus 1. So, this is what is called as the

transfer function in the time constant form. I am not really writing it in the pole 0 form which is just be that s plus something. I am just dividing that and this is called the time constant form and it is little useful to do this in the frequency domain analysis, right. That will actually be obvious when we do things like the Nyquist or the bode plot.

So, I just first write down the sinusoidal transfer function for obvious reason. So, this will be the k over j t omega plus 1 and let me assume that the input is just say a sinusoidal sort of function a sin omega t. So, what this entire analysis here tells you is that the output or the steady state output which I call as C ss of t is simply computed by the magnitude of g of j omega. What is the magnitude here? Well, that is straightforward to compute that is k over square root 1 plus omega square t square. Similarly, the angle of g of j omega is the negative of tan inverse of t times omega. This is just the complex numbers and we know how to compute those angles.

So, C ss of t would be a multiplied with the magnitude of g of j omega. What is the magnitude of g of j omega? That is k square root 1 plus omega square t square and sin omega t plus phi, this is the phi, this is minus tan inverse t omega. So, this is actually straight forward now, right. Just look at what g is. Compute its magnitude, compute its angle and I can easily see what is the output signal. I do not each time, I really would not need to compute the inverse Laplace transform and then, do all that substitute s equal to minus j omega plus j omega nothing, right, how the magnitude changes? Magnitude changes with the magnitude of g of j omega. How does the phase change? Phase just changes with the angle of g of j omega. So, the reason of doing the entire previous steps was just to get this expression in a reasonably straight forward manner, ok.

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Frequency Response of Closed Loop Systems

- For single loop closed loop systems, as considered in previous modules, the transfer function is given by

$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \qquad 6.1.15$$

- Under the sinusoidal steady state, $s = j\omega$, equation (6.1.15) is expressed as

$$M(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)} \qquad 6.1.16$$

- The sinusoidal steady-state transfer function $M(j\omega)$ may be expressed in terms of its magnitude and phase i.e.

$$|M(j\omega)| = \frac{|G(j\omega)|}{|1 + G(j\omega)H(j\omega)|} \qquad 6.1.17$$

$$\angle M(j\omega) = \angle G(j\omega) - \angle(1 + G(j\omega)H(j\omega)) \qquad 6.1.18$$

For single loop systems, well we just have the transfer function g over 1 plus g and then, I am interested in this sinusoidal transfer function. So, I just substitute s with g of j omega and then, I can just compute the magnitude g of j omega over magnitude of 1 plus g of j omega h of j omega. Similarly, with the angle would be the angle of the numerator guy that is the angle of g of j omega minus the angle of this guy, right. So, things get a little easier to compute, right. This has become lin just as addition or multiplication or additions or subtractions, ok.

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Frequency Domain Specifications

Resonant Peak (M_r)

- The maximum value M_r of magnitude $|M(j\omega)|$ is known as the resonant peak. A system with large resonant peak will exhibit a large overshoot.

- In general, the magnitude of M_r gives indication on the relative stability of a stable closed-loop system.

Resonant Frequency (ω_r)

- The frequency at which the output of the system has maximum magnitude is resonant frequency.

Bandwidth (ω_b)

- The bandwidth BW is the frequency at which $|M(j\omega)|$ drops to 70.7% of, or 3dB down from, its zero-frequency value.

So, the idea is to check now how the magnitude changes? With changes in omega is there, some strange thing happening over there which might give us some information. Similarly, what happens to the angle as the frequency is increased from 0 to infinity, right we have all kinds of frequency signals available, right. So, there are few terms here which we will slowly derive.

One is called the resonant peak. It is in some way you can say well this is not directly related to, but you could when I was doing the time response you know as time was going from 0 to infinity, I defined what was or at what time did I reach the peak overshoot m p as I called it, right and then, I just did some put d of the magnitude by d of t equal to 0 and I get some time where it actually reaches the peak value and I could even exactly compute what was the peak value something depending on zeta e power minus pi zeta some formula we had, right. So, similarly here does the magnitude reach a peak at some frequency and this frequency if it exists is called the resonant peak, right. The system with a large resonant peak we will exhibit a large overshoot. We will see that, right. So, in general this guy gives indication of relative stability of closed loop system. We will just keep this in mind for the future thing. There will also be a corresponding to the resonant peak, the resonant frequency, the frequency at which the output becomes maximum.

So, this guy becomes maximum at certain frequency and that frequency is what we call as the omega r. Earlier we had defined it as a peak overshoot m p and t p and the time where actually I reached the peak overshoot, then something called the bandwidth which I think is the term which we use very often, but formally it means the bandwidth is the frequency at which the magnitude drops to 70 percent of its original value or its 0 frequency value. This scaling we will define little later, ok.

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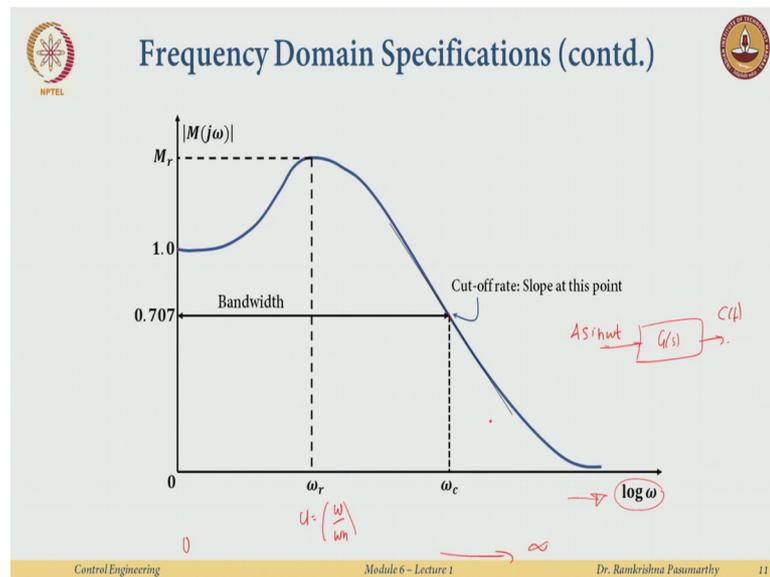
So, this is how a typical response would look like depending on omega r. So, this is on the horizontal axis. I have increasing frequency. This is on I will also tell why we use this n in the log scale. Some books will also give you this in the normalized frequency form, where u would be omega over omega n and this is plotted as u goes from 0 till infinity, so that we will see that a little later, right. So, typically I will also tell you why this is 1 and u equal to 0 or omega equal to 0. I start this way and I could reach a peak and then, I go down this peak is m r as defined earlier. We called this the resonant peak and the frequency at which this occurs is called the resonant frequency.

Now, you see this starts from 1. It goes up and then, falls off to certain value and this is what I call as the cutoff rate, right or the bandwidth, right. This frequency from point from 0 till this is called the bandwidth because for other frequencies, you see my signal gets attenuated. Why I say this is because again I have input signal some a sin omega t. Here you have g of s and you have the output some c of t and you see c of t is what it is a times this magnitude and after this you see that actually the signal is getting attenuated too fast with changing frequencies, right. So, until this I can get see some good output. After this I may not really be interested because my signals just become very small right.

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This is essentially what I call as the cutoff frequency which is where magnitude is 1 over root two times of its peak value. This is not really the maximum value, but the 0 frequency value. I will just change in the slide, right and then, the cutoff rate is the rate of change of slope at the cutoff frequency, right this slope of this line. So, again we start with a typical configuration of the closed loop and I have a certain reference signal, then I have g of s given by this guy, then I have a unity feedback, I have certain outputs c of s plus minus and so on.

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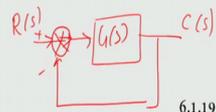
So, the loop, closed loop transfer function which we derived even when we were doing the second order system was this guy and I just substitute s equal to j omega to get the sinusoidal transfer function and this is how it looks like. So, let us see what information does this guy give us. So, m of j omega was omega n square over j omega square plus twice zeta omega n again s substituted with j omega plus omega n square, ok.

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Now, let me define a quantity u which is a normalized frequency as omega over omega n. You may even avoid using this, but I think the computations get a little easier. So, I can just write this guy in terms of u as 1 over 1 minus u square plus j 2 zeta times u. So, how does the magnitude of m look like? This is what I am interested in calculating the magnitude and the phase. The magnitude will decide how my original signal will get transformed, what will be the frequency of the desired, sorry the amplitude of the resultant signal and the angle will decide how much is the phase shift. So, this m of j omega is what the magnitude.

So, in terms of this guy, the real part and the imaginary part with the square root square root of 1 minus u square whole square plus twice zeta u whole square. Similarly, the angle of m or even there is a close loop. So, m of j omega which I call as phi is the inverse tan of twice zeta u over 1 minus u square. Again u is the normalized frequency which is omega over omega n and omega n is the natural frequency of the system. So, let

me take my reference earlier we had a unit step. So, let us keep the unity thing here again one times sin omega t and see how this guy looks like.

So, c of t would be what the magnitude of this guy that is 1 multiplied by the magnitude of the transfer function times the phase sin. This is omega t minus plus phi. What is phi? Phi is this guy tan inverse. This will have a negative sign because all these occur in the denominator minus tan inverse twice u 1 over u square. Now, let see what happens as we change the frequency. Well, I do not really want to do all the values from 0 to infinity, but just look at some typical values first is when u equal to 0, just put u equal to 0 here in that case m becomes 1 and what is phi u equal to 0 phi also becomes 0, right. So, when the input frequency is 0, well the output is just you know it is just the same thing, right. Nothing comes out. What just comes out is 1 sin omega t.

Now, second is when u is 1 which means omega is omega n the natural frequency of the system. What happens to m and u equal to 1? Uequal to 1 m is, so this guy goes away. I have u as 1 over twice zeta and phi. When u equal to 0, this guy becomes very large and tan inverse of infinity I know is pi by 2. So, phi becomes pi by 2 or with a negative sign. Third thing for very large values of frequency when u becomes infinity, the magnitude goes to 0 and what happens to the phase phi as u becomes very large, the phase becomes minus pi, right. So, I know well m starts at certain value of 1 that is for 0 frequency signals, then at u equal to 1, where omega equal to omega n. The natural frequency I have some other value and as u goes to infinity, my magnitude goes to 0 and phi becomes minus pi, then we can ask ourselves a question, ok.

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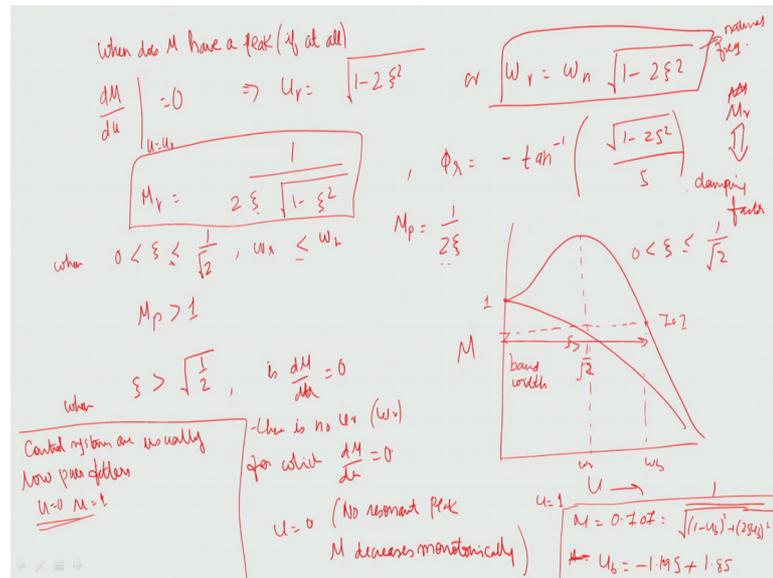
Based on all these three values, u equal to 0, u equal to 1, u equal to infinity, when does m have a peak if at all? So, what do I do? So, what is changing here? The frequency is changing. So, I just look at d m by d u equal to 0 to get its peak. So, this I just do the computation. I just keep here u r is square root 1 minus twice zeta square or equivalently omega r is omega n square root 1 minus twice theta square. So, if I substitute this u r in the expression for m and phi, what do I get that m r is 1 over twice zeta square root of 1 minus zeta square and similarly, phi of r is the negative of tan inverse of square root 1 minus twice zeta square over zeta, right. So, what do we observe from here? Let us see what happens when zeta is between these values, right. You see right from this expression, zeta goes to 1 over square root 2. I see that this goes to 0, right. So, I put an equality here, right.

So, when zeta is between 0 and 1 over square root 2, the resonant frequency w r, it always has a value which is less than omega n right less than equal to and then, the resonant peak m p for these values. Look at these 0 to zeta to 1 over square root 2. So, what was the resonant peak? The resonant peak by the formula was 1 over 2 zeta because this is for zeta greater than 0. Therefore, I am not looking at the value of 1 and I am also not looking at the value of very large frequencies, right. So, this is the thing and therefore, for these guys the peak, the resonant peak will always be greater than 1. So, the second case when zeta is greater than square root of 1 over 2, I will ask myself well

is d m by d u equal to 0 for any of these, right. So, it turns out well when zeta is greater than 1 over square root of 2 which means this number will not exist, right.

So, there is no u r w r for which d m by d u. Sorry this should be u. Here d m by d u equal to 0 or its peak will only be when u equal to 0. So, let us try to just plot these two things, right. I am just plotting over u with m when u equal to 0, m is equals to 1. So, if I look at these things, I get a plot which looks like this. So, for the first case when 0 less than zeta less than equal to 1 over square root of 2 and then, for all other values when zeta is greater than 1 over 2, I just get something like this, right. This is for zeta greater than 1 over 2. There will be no peak, ok. So, now all the bandwidth and all now I can define. So, this is my 1. So, this is say 0.707, this guy will be my bandwidth, this guy will be omega r or where u equal to 1 and so on, right. This is the explanation to why we actually get a picture which looks like this, right.

So, then in this case, well again there is no resonant peak. So, m decreases monotonically as u increases, then you see that well just look at these things over here m r. What is this m r? Well, this m r is again depending on zeta. So, the resonant frequency has some, sorry m r has some direct relation with the damping factor, right and similarly, this guy omega r equal to omega n tells me something about the natural frequency of the system given a particular value of damping, given zeta equal to 0.1, then well omega r has something to do with omega n and if you remember these were essentially the two things which were defining all ten domain specifications zeta and omega n, right and therefore, there is a nice correlation between what was observed in the time domain and what is now observed in the frequency domain.

One more observation is that I will just write down here control systems are usually low pass filters, you see right here. So, when u equal to 0, m equal to 1 all the time, right and you see well if my zeta is less, it actually allows this guy and if zeta is greater than 1 over, sorry this should be square root of 2, then it still allows some low frequency signals to pass through. So, all control systems, most of them are usually low pass filters especially because of this and then, what we observe also from here and then, little more about the bandwidth, right. What is bandwidth? When the magnitude reaches 0.707 or 1 over square root 2, so this entire thing from here till here is the bandwidth. So, if I look at, I know the formula for m right the magnitude and so, m is this just write down here this is 1 over square root of 1 minus u b square plus twice zeta u b square. Then, this

actually gives me that this little approximation gives me that u b is minus 1.19 zeta plus some number 1.85, ok.

So, the bandwidth is also related to the damping factor of the system, right. So, there is also lot of good relation between the system parameters. What were the system parameters when we were dealing in? What were the system parameters when we were analyzing the time response? It was essentially zeta and omega n and these two guys characterize lots of things of the system. Similarly, here give me zeta I can find out what is the bandwidth, give me zeta I can find out the resonant peak and so on. I can determine the bandwidth resonant peak I can even find out what is the omega r, right.

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So, just to summarize we just looked at this is the resonant peak, this is the resonant frequency, the bandwidth. Again equivalently we computed. This way I will skip this computation, but this is again not really difficult to find out just by looking at formula like this, ok.

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So, we just introduced ourselves to the concept of frequency response, we listed out few of those advantages, we just listed out, but we will slowly build upon explaining or justifying those advantages. We define the concept of frequency response, the sinusoidal transfer function, we also defined few frequency domain specifications with respect to second order systems. In the remaining of this lecture, we will or the lecture 2, we will look at what are called as the frequency response plots. Typically we will look at Nyquist plot and define what are the stability margins, right. Relative stability is what we were talking of for a long time and we can see how in the frequency response analysis we can actually characterize this relative stability with the help of what we call as a gain margin and the phase margin.

We will also see some important tool called the bode plot. You may have heard of bode plot, may be while doing circuit analysis or networks and systems, but if you have not, do not really worry. We will also look at the bode plot and we will see how these two actually give us a lot of information about the system, the system stability gain margin and phase margin and how we could formulate our design problems as appropriate bode plots of the close loop system.

Thank you.