Control Engineering

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Module - 05
Lecture - 03
Root Locus Plots

In the previous lecture we saw first three steps about constructing a root locus. First step was a little easy to understand that, it is actually symmetric about the real axis, then we also characterize where do the root locus where does the root locus start when K equal to 0 and what happens as K goes to infinity, and then we concluded that they start at the poles and end up at zeros and then what if there are more number of poles than zeroes the remaining poles go to as what we defined as 0 at infinity. We also easily computed to see if what areas of the real line are on the root locus. So, that was just a small test based on how many poles and zeros lie to the right of that particular region of the real line.

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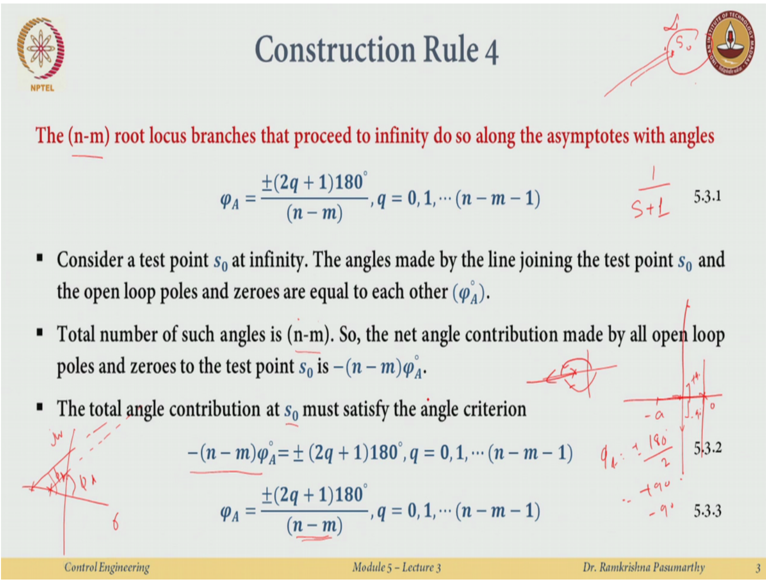

So, continuing on that. The further the next step here is how do the root locus branches go to infinity for example, in the motivating example we saw that we had this guy a pole here 0 and minus a that they meet here and they go to infinity. Now what decides this, why do not they go something like this right or why do not they do something like this or why do not they just instead of going a straight line just you know follow a lazy path and you know do a little more adventure and go this way. So, who decides how they go or in this example it was it easier because it was just like a solution of a second order or a quadratic equation. So, what if there are more poles who meet at this point and then the can they go to infinity in the way they want or they somewhere something that decides that right. So, the statement says that the n minus m roots. So, we are interested now on the roots or on the poles which go to infinity as K goes to infinity.

So, the n minus roots that proceed to infinity they do so, along asymptotes width this this angle right and which I will show you shortly how this is computed right. So, consider a test point at infinity right. So, let us do a little the same just erase this and I say let us say I have my sigma and j omega axis here and say this is my infinity here. So, there is this guy is not at infinity. So, and then there would be you know open loop poles zeroes and whatever. So, there are these 2 poles and I say I just look at how the angle approaches it goes here and then keeps going here and then it reaches this guy.

So, this guy is really far. So, I just do something right go here and then with this guy here. So, this test point which is at infinity. So, what are these angles right those angles made by the line joining the test point and the open loop poles are zeroes, they are almost equal to each other right. So, if I really look at 2 points from very far away they will seem just very close to each other this is this is like the infinity and let say both of them kind of like the same angle phi A and phi a.

So, how many numbers of angles are. So, there would be n minus m number of angles right that would be the difference between the number of poles and the number of zeros. So, the net angle made by all open loop poles and zeros to the test point would just be well this guy right n minus m with a negative sign because I am dealing with poles times phi A right. So, there are 2 things here right. So, this will be phi A and phi a, 2 times of that right as if we are already interested in n minus m guys.

So what should these angles do? So, all these guys they should satisfy the angle criterion. So, what is the contribution at infinity to this guy? So, the total angle contribution at s 0 must satisfy the angle criterion what is the angle criterion it says that the angle should be odd multiples of 180 degrees what is the angle contribution that is this guy minus n minus m phi A is this guy, and this means I can just compute phi A as 2 q plus 1 times 180 degrees over n minus m. So, this angle is now computed just by this m and this is not surprising right. So, let us do that the same example again right I start with 0 I start with minus a; and let us say where I know that to these 2 guys go to infinity from some point. So, what is phi either phi A is say 180 degrees over what was n minus m there it was 2 plus minus. So, what I have is plus 90 and minus 90 that is what exactly happens right.

So, this pole it starts here and goes to infinity at an angle plus 90, and here the other guy this guy starts from here and goes here at an angle minus 90. So, that is what this guy tells us right that the n minus m branches 2. In this case that go that proceed to infinity, do so along asymptotes with angles which are depending on the number n minus n right. If there are more of these guys the angle will change accordingly, if there is only one say for example, if this is say I am just looking at say 1 over S plus L for example, right it is my G times h. So, I can just draw the root locus. So, this is going to infinity. So, the angle would just be minus 180 or plus 180 both are the same right either this or this they just keep on going inside.

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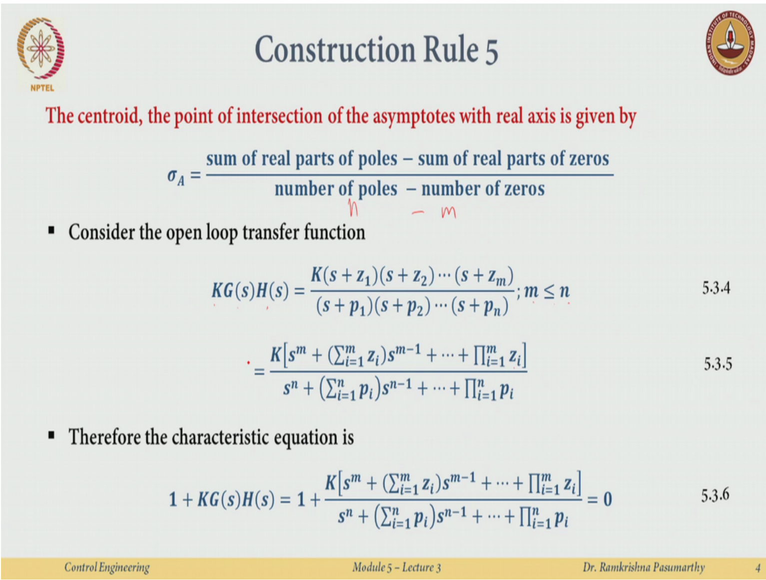

Now, so these asymptotes the will they will somewhere you know they will have some starting point right from on the plane on the complex plane. So, the centroid or the point of intersection of the asymptotes with the real axis is computed in this way, I guess some of all the real part of the poles minus some of all the real parts of the zeros divided by number of poles minus number of zeros this is like an be n minus m n minus m. So, where does this come from? So, what was the open loop transfer function that is K times

G times H. So, what I had was a set of zeros in the numerator set of poles with m being less than or equal to n.

Now I just you know do some so, some high school algebra just to say that this I can write it as decreasing powers of m, s power m s power m minus 1 with the coefficients being the sum of all these guys until I reach the last point s power 0 will just be sorry here it will be the sum of all these guys till I reach here where it will just be the product of all these guys similarly the denominators I start with s power n, the n minus 1 will be the product of all these guys and sorry the sorry the sum of all these guys and the s power 0 guy will just be the product of all this guys. So, I just dividing you know one guy with the other.

So, how does the characteristic equation look like? Characteristic equation I am interested again the solution of the characteristic equation 1 plus K times G times H equal to 0 is what I am aiming to solve.

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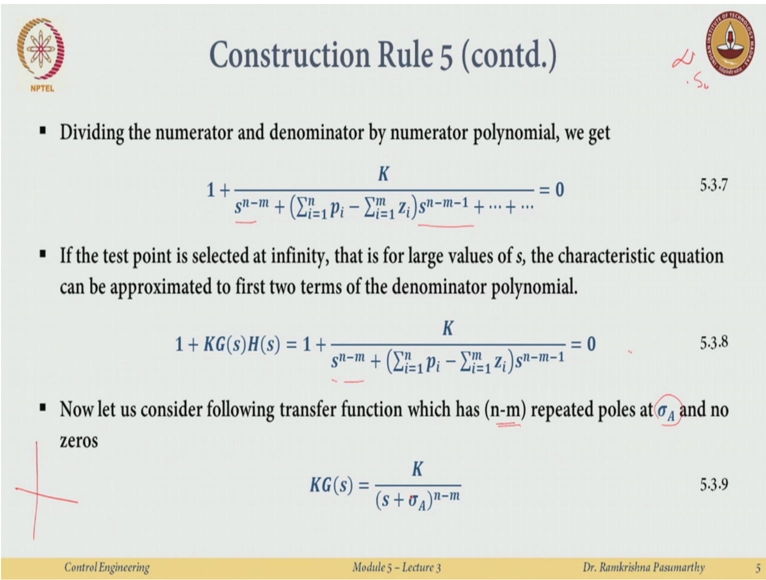

So 1 plus K looks like this I just add a one here and go here right. So, I just divide right I just get this guy down here and then see how it looks like. So, how what will be now the now how the denominator now look like. So, it will have start with powers of n minus m, it will have powers of n minus m minus 1 and so on.

So, what will be the coefficients of s minus n minus 1 that will just be the coefficients of this guys the sum of the poles minus 1, the sum of all the zeros and then you have this series and now very smell I am looking again. So, my axis you see is here and my test point is at infinity here, s c o this is my infinity. So, if the test point is this very far which means I am looking at very large values of x, I can just do my analysis just with these 2 terms and then I can the contribution of the remaining terms would be very small, then really looking at really large very surface.

So, my characteristic equation can equivalently be written as 1 plus K times G times H is just this guys. So, I just have 1 plus K s to the power of n minus m with a coefficient of 1, and s power n minus m minus 1 with the coefficients that is decided by the nature of poles and the nature of zeros. Now let us do something else now right. So, let me start with a transfer function which has n minus m repeated roots at some point sigma A and it has no zeros, how will that look like well the K times G like just say can assume H equal to 1 or this add H does not really matter. So, K times G would be something like this K over s plus sigma A n minus m.

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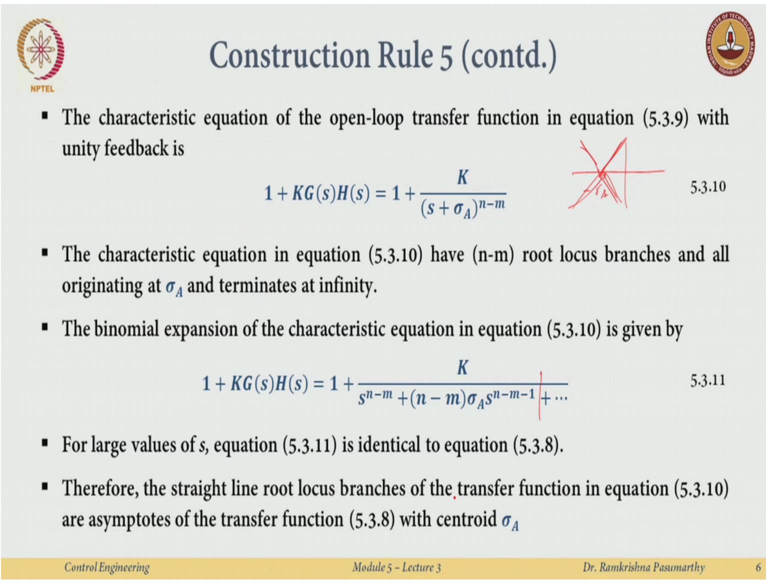

Now, I get the characteristic equation I just add one and I get this one. So, this characteristic equation, how will this root locus look like well this root locus has no zeros and n minus m poles? So, this root locus of this open loop system here with H equal to one will have n minus 1 branches all of them good infinity they start at sigma a.

So, that is what it does it right; so what happens when? So, this is my sigma A and then there are n of these roots all n will go to infinity; so some here some. So, this here where whichever way they wanted to go that is decided again by the nature of these things? So, I am looking at the equation like 1 plus K over this guy, n minus m and all these n minus m branches they go to infinity. Now let me just expand this a little bit right. So, this is like s power a s plus a power n kind of sigma.

So, we do the binomial expansion and in the denominator I have again powers of s n minus m, and then now I am just summing this way right. So, if I just substitute this formula directly summation of all the ones all the poles, how many poles are there n minus m, where are the poles at sigma? Therefore, the second term would simply be sigma A n minus m which is a summation of all the values of all the pools n minus m and so on. Again for large values of s I just stop here and say well this is a good approximation at largest values therefore, the straight line roots the straight line root locus branches of the transfer function in this equation are asymptotes of the transfer function with centroid. So, everybody starts here and then they go to infinity we will say how they go. So, this is just this a very big picture. So, do not really think that I am drawing something very correct here just to give some way idea about sample right.

So, this straight line route branch root locus branches of the transfer function which has sigma A poles sigma n minus m poles at sigma A and then the negative side, they are asymptotes of the transfer function with centroid sigma A all the all the asymptotes start at sigma A. So, what is the relation between this here? So, we started with this one we wanted to find out where what is the centroid point of the asymptotes where how do where do the asymptotes intersect if at all right. So, compare this guy right again. So, what did I assume here is I assumed here that of course, I started with m poles sorry m zeros n poles I approximated my characteristic equation I just did this right, so this was gone. So, this in a way the denominator of 5.3.11 and these are the same.

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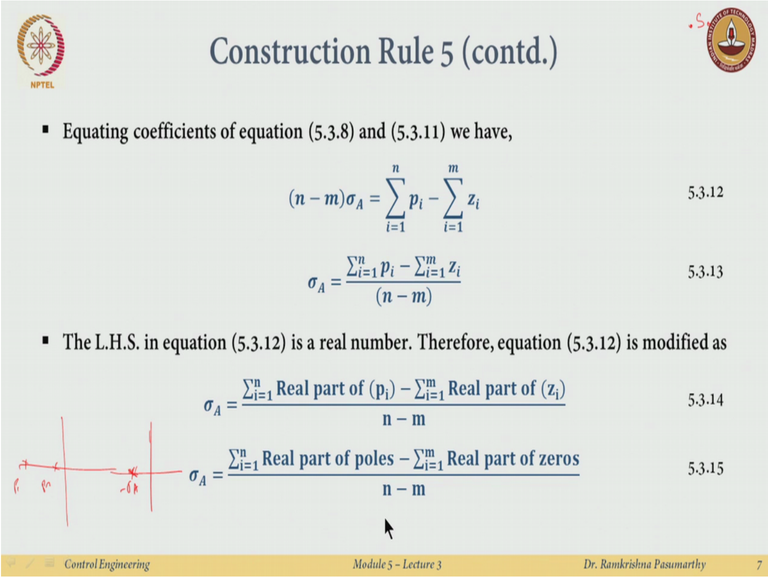

If I just equate this you know what I would get is something like this n minus m sigma A is this guy. Now what is sigma A in the general case because what I showed earlier then we were even looking at the angles is that if I start from here this is my test point s naught and these are 2 poles p 1 and p 2 if I look from very far it might just look at it might just seem as if these 2 poles are in the same location because angle contributions are almost the same. So, I can take this analogy and I can just say well I can equivalently view this as a system where I have 2 poles minus sigma A, I am not changing a location of the poles I am just seeing how they look like from infinity. So, how if there are 2 guys placed here at sigma A and sigma A in the left half plane from infinity it will look. So, these guys p 1 and p 2 will exactly look like this that are actually close to each other and therefore, this centroid point of the asymptotes can be calculated this way, just by equating this expression here to this guy because I know that from infinity they it looks as if all the roots are together.

Because they contribute the same kind of angle correct. So, what is this guy? So, the centroid point. So, I sum the values of all the real parts of the poles real parts of the zeros and n minus right and this is how I calculate the centroid point.

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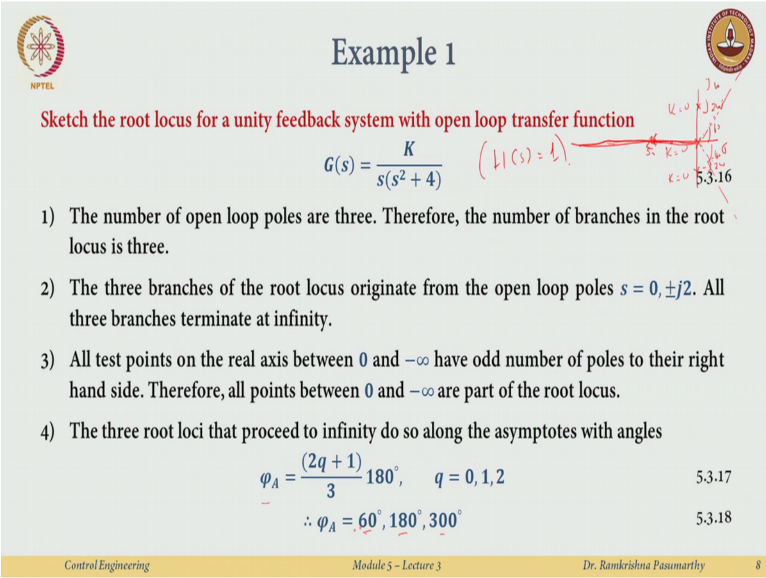

So, let us see how it looks like you know in terms of a small example. So, take G s is K over s s square plus 4, there are 3 poles for the system again H s is assumed to be 1. So, wherever G s is given you can safely assume that H of s is 1 and if we also we all always know how to transfer transform a non unity feedback system to a unity feedback system. So, we can do the same analysis.

So, coming back to this given transfer function of this form I have 3 poles and 3 branches of the root locus. So, these 3 branches let me just approximately draw something here right. So, I have a pole here at origin s square plus 4 equal to 0. So, I will have a complex conjugate pole at sorry at j 2 omega and minus j 2 omega or 2 j omega; however, you want to write this. So, this when K equal to 0 and this here right K equal to 0 just at the poles K equal to 0 at the poles, K equal to 0 my root locus looks like this actually satisfies the thing that I am symmetric about the real axis and all of them go to infinity because there are no zeros.

Now how do they go to infinity? Now first the rule number one told me that it is symmetric rule, number 2 told me that all will go to the to the infinity, rule number 3 tell told me about well is a point like this on the root locus you know is a point is 0 which is on the real line on the root locus, then I compute well see if I am here I just look at here and see how many poles are there well there are 1 2 and 3.

There are no zeros this is the odd number and therefore, this point will be a root locus this point will also be a root locus, any point on this real line if I look towards my right I will have 3 poles which is an odd number. So, this entire line will be on root locus although all the entire negative real axis. So, what about the other 2 guys? So, then we look at how do they go to well this guy goes to infinity in this way right because entire real line is on the root line. Now how do we other guys go? So, I look at the way they look they go to go to infinity what kind of angles they make.

So, the angle here is computed as again 2 q plus 1 over 3. So, I get 60, 180 and 300. 60 is this guy then I have 180, and I have 300 here 300 I can also write as minus 60. So, this is here right because at 180 already this guy is branching out at 180 degrees. So, this is a good a good validation and then these guys they will. So, these are how my (Refer Time: 18:42) go. So, these guys will meet this line at 60 degrees at infinity, and this guy will meet this line at infinity the guy at minus 2 j omega.

So, at infinity not that they just come here and then they go along; so they keep moving and they meet this line asymptotically this guy makes this line asymptotically right. So, this is about the angles.

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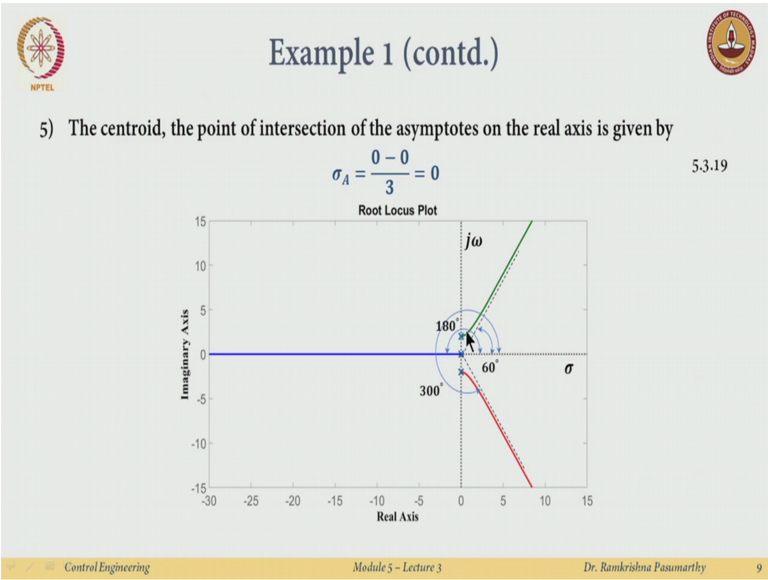

So, next where do these this asymptotes originate and I just look at this this formula here about computing the sum of the real parts of the poles, real parts of the zeros with n minus m.

So, this is 0 minus 0 and a 3 and 0. So, these asymptotes actually start here and just going here. So, they meet exactly at this point. So, now, how the root locus will look like well this guy will start moving little towards this direction, will see what direction exactly it moves in a little in a little later, but I can fairly assume it move something here it does not it never touches this line it just goes here and then it only touches it at infinity. This guy starts moving this way just a mirror image of this, and then goes to infinity this way.

So, we have found out which direction this guys go or at least I know that now this I may not exactly know this look at this direction now, but I know that at infinity it meets this line and therefore, it has to move somewhere wiggly in this direction right and we will define this a little later similarly with this guy. So, we know how in what way or what decides how these roots branch out to infinity.

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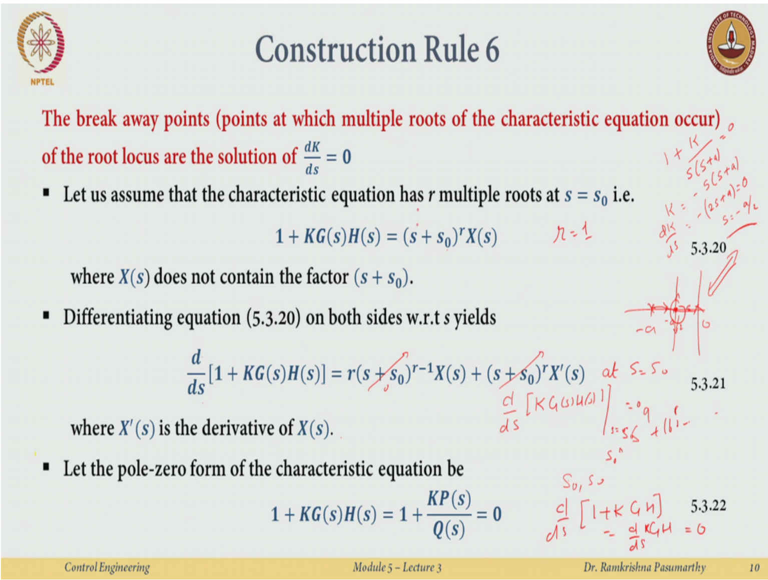

So, again coming back to the first example: so I started 0, I start at minus a and I know here that this guy moves here this guy moves here they both meet at minus a over 2 and then go to infinity. At this point where they meet and then they again go away right they do not really stay together just come here come together just say hello and then you know go in your own path. Now, who decides why where this is y minus a by 2 why not something here, why not something here again this I know because I could just compute from the computations of my quadratic equation, but what in general right I just I may not be able to again compute all the roots and then it might be a little little cumbersome.

So, this was good enough for understanding. So, who decides this point and does that point exist right that also is the thing. So, if I am just giving you a characteristic equation which is a polynomial of order 9 plus 16, s 8 and lots of things. I should first know that does a breakaway point excess or not right. So, who tells me that again everything is from the characteristic equation now let us assume.

So, at this point what happens that there are multiple roots right the root here the root is 0, the root is minus a or the poles and then they come here, here both the poles have the same value minus a by 2 and minus a by 2. So, when this happens my characteristic equation 1 plus K G times H, well I will have here the root at s 0 with some multiplicity r and times X of s. I get this X of s does not contain this factor and we will be interested in situations where r is at least 2 right. Now I differentiate this guy I differentiate this on the left hand side I have d over d s of this guy, on the right hand side I differentiate again with respect s differentiating with the first guy I will have r minus 1 X s, then again same guy here and x prime is where X prime s is again the derivative of X of s with respect to s. Let us leave this here for a while and then come back to this again.

So, the in the pole 0 form I can write 1 plus G 1 plus K times G times H as 1 plus K some polynomial in the numerator and some polynomial in the denominator now let us see what this guy has to do.

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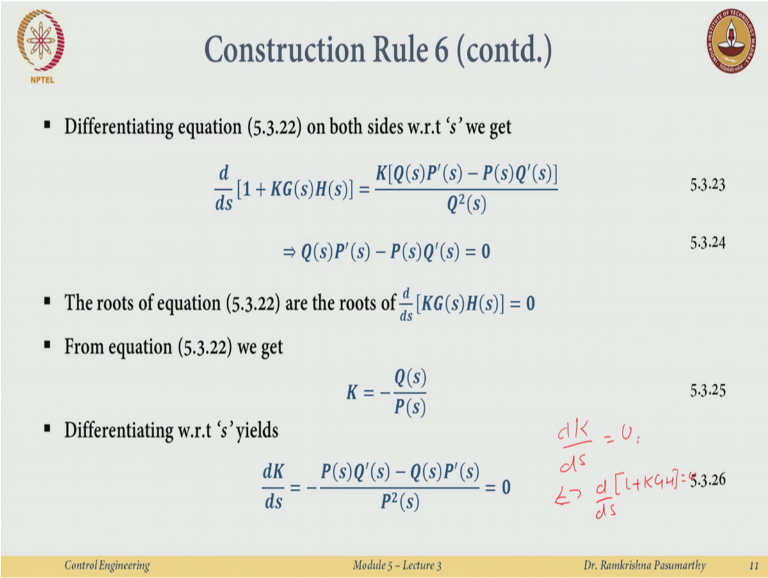

So, I can differentiate this guy and what I have is this thing here is that I was just doing the same thing right I am differentiating this and I am differentiating this again here. So, in terms of p and q, the condition that d H or d H by d s equal to 0 looks like, this just the numerator being equal to 0.

Now, look at look at this guy here. So, I know of course, that at s 0 this is s 0 is a root the characteristic equation is obviously 0. Now look at the derivative right a derivative what is the value of derivative at s equal to s naught, let s equal to s naught or this guy goes to 0 this guy equal to yours goes to 0; so d over d s this. And this in this expression d over d s of K G s, s H at s equal to 0 is 0 sorry s equal to s naught where s naught comes to multiplicity of r. So, that is what is going to the roots of this equation are the roots of this guy. Now I can equivalently you know tried my characteristic equation something like this. So, 1 plus K G times H or 1 plus K P over Q, I can just write it as K equal to Q over P ok there is a reason why you are doing this now. So, first is well I am just looking at how to detect if there are multiple roots and that is when d over d s of 1 plus K times G times H goes to 0 that has a multiple root right at that point right then that point here is.

So, therefore, I can just say that well the solutions are of this form right. Now let me do this another something else. So, K is again now a function of s d K by d s looks like this and when does d K by d s go to 0? D K by d s goes to 0 when the numerator of this guy goes to 0 it is P Q prime, minus Q P prime now look at this. So, these 2 are the same sorry this expression and this expression are the same right; now if the equivalent the condition here. So, when does r have a multiplicity at least 2 sorry if r was just 1, d K by d s would not be 0 it could be something else because this guy would go away.

But if r is 2 this guy would go to 0 this guy would also go to 0. So, if s naught occurs twice it means d over d s of K times G times H equal to 0. So, if s naught occurs twice, that meant that d over d s of 1 plus K G H is also equal to d over d s of G times H equal to 0 and you can have the K or naught. Now this being 0 tells me that there is a multiple root this being 0 also means that this is 0.

Now compare this condition and compare this condition here they both are the same and therefore, I can check I can just equivalently say d K by d s equal to 0, this is equivalent to d by d s of 1 plus K G H being equal to 0 and for obvious reasons this is for me equal to easy to compute because I know Q and P and this might be because I may have to you know G and H might have numerators denominators and may have to do a lot of other stuff, but then this is easier to compute. So, this condition tells me that there is a breakaway point right the breakaway points are the points at which multiple roots of the characteristic equation occur, the solution of this will be the exact size breakaway point you can compute this I thing here it would be very trivially satisfied. So, let us I can.

So, see compute for this example in check right you know in this way now why this is minus a by 2. So, what was the characteristic equation that was 1 plus K over s, s plus a is 0 or K was minus s, s plus a, d K by d s would be minus twice s plus a and if I equate this to 0 I get the solution as s equal to minus a by 2 and this is the breakaway point and of course, in this case it also turns out to be the centroid point and it can happen in many cases. So, this is like you can see why this is a by 2 using this explanation over here.

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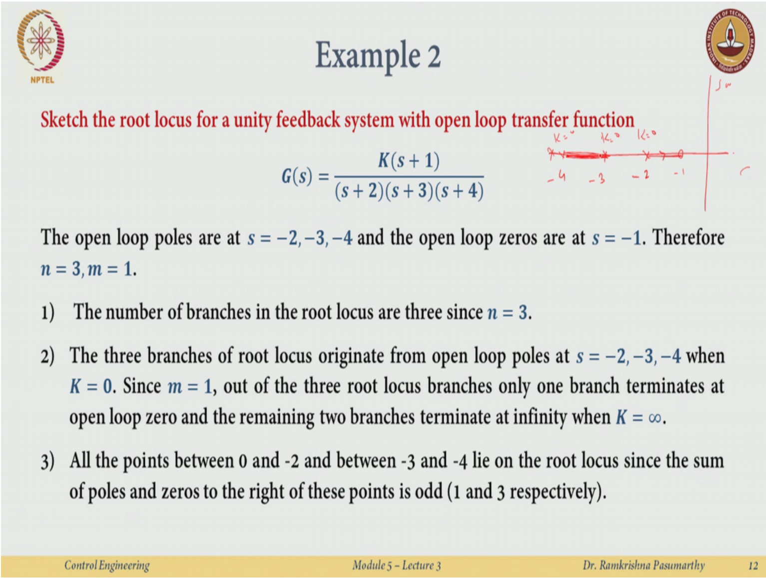

So, let us do a sketch a root locus for this guy. So, I have again a pole at minus 2 minus 3 minus 4, 0 at minus 1, 3 poles 1 0. Again the number of branches in the root locus would be the number of poles that is 3 and these of these 3 branches one will go to the 0 at minus 1 and the other 2 will disappear to infinity. Again starting from K equal to 0 they will start at the poles and then go to infinity. So, let us first draw the basic conditions. So, at K equal to 0 my root locus will be at this poles which is minus 2 minus 3 and minus 4.

So, rule number 3 which part of the root locus on the real line or the real axis will be on the root locus. Let us start from here there is there is nothing here. So, there would not be anything here. So, and then let us look at this area between minus 1 and 2 if I look to the right there is 1 0. So, I am good here right there is an odd number this one. Now look at the area here between minus 2 and minus 3 how many guys are sitting out here one pole one 0 that is an even number I do not like it. So, I just go here between minus 3 and minus 4 just look to the right. So, there is 1 2 and 3 oh it is it is an odd number.

So, I like it right it is here and then what happens that after minus 4? I just look to the right I have 1 2 3 4. So, this area is also not good for me. So, one thing for sure I know that K equal to 0 points are these guys, K equal to 0, K equal to 0, K equal to 0, this guy will go quickly and then sit here right now these guys have they have to find their way to infinity they both try to run away this way right, but it is not allowed because these points are not on the root locus. So, they have to somewhere just be around this one and then decide how to go to infinity.

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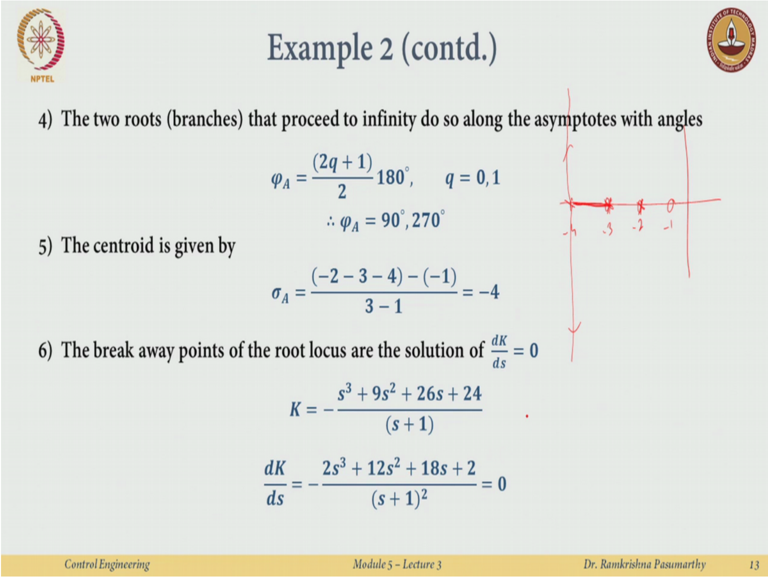

Now, let us see how who tells these guys where to go to infinity. So, how do they go to infinity? There are 2 guys who are going to infinity and they proceed to infinity along asymptotes with angles again given by 2 q plus 1 divided by n minus m; at n minus m was 2. So, you will have 90 and 270.

So, I know at least that. So, these 2 guys let us me just draw it for. So, I have was 0 here this is at minus 1, minus 2 sorry this are all poles, pole a pole, the pole. So, whatever they do in between they will go to infinity at these angles 90 and 270. So, how is the centroid point given? The centroid point where of this is again you compute the sum of the poles of the real parts and zeros and you get this one. Now what I also know is that now this is a centroid point right. So, asymptotes will be like this at minus 4. So, these guys will move and they will meet these guys at infinity and where do these lines at infinity come from? They come from minus 4. So, I know that one now right. So, I just I just remove this. So, I did not know which direction they go to infinity, but now I know they go from minus 4.

Now the sec the next thing is these guys will where do they meet, does this guy wait for this guy to come all the way here and then they just move away or they meet somewhere in the middle or then go to then move apart. So, that is we will find. So, there is a breakaway point right is breakaway point minus 4 or minus 3 or whatever I just try to find out and that is given by solution of d K by d s equal to 0.

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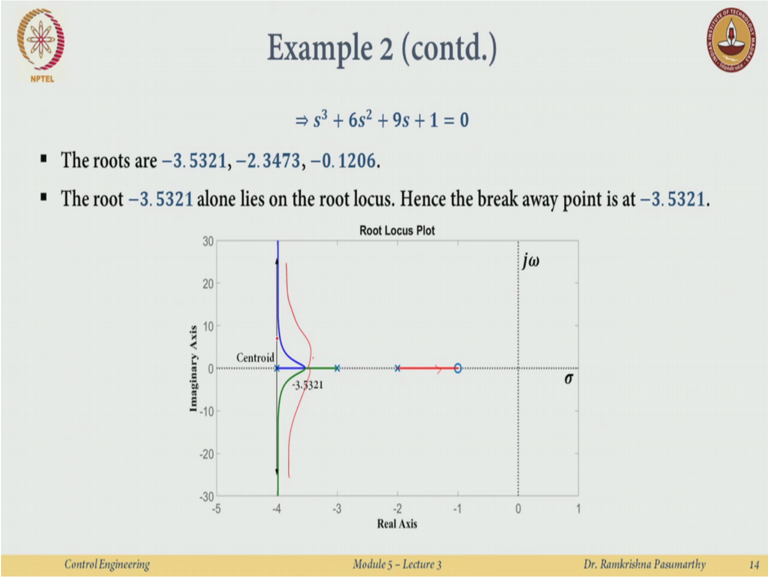

So, I just do all the math here d K by d s equal to 0 would mean well it gives me 3 roots minus 3.5, minus 2, and 0.126. Now look at these points minus 2 that these regions are not on the root locus, this region is not on the root locus, this region is not on the root locus, this region is not on the root locus. So, only point where they are allowed to meet is minus 3.5. So, that is the point here.

So, what happens is well this guy this guy goes here, these 2 guys from the asymptotes centered computation I know that they will meet these 2 lines at infinity. So, they will just start from here this guy goes starts here and goes to infinity this guy starts here and goes to infinity along this line. So, that is a very compressible again which how do they move why not they do something like this or you know something like this you know more.

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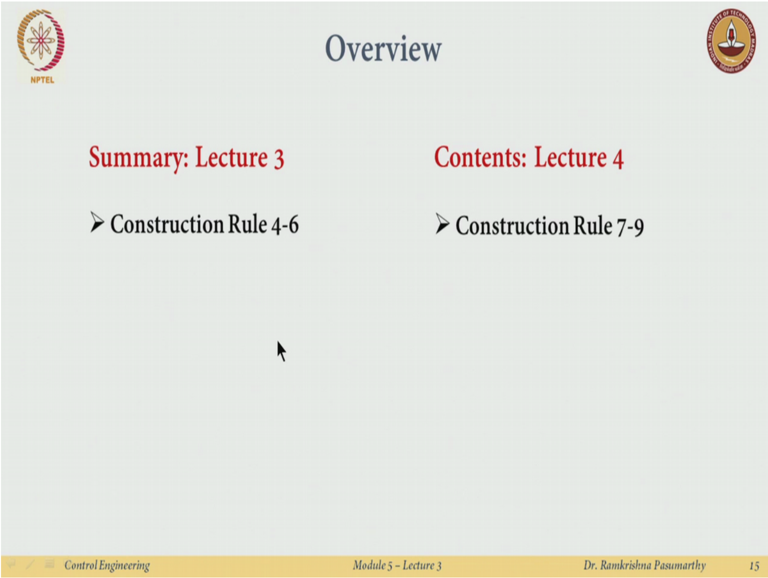

So, that that will see a little later where at the moment that seems interested now little towards the asymptotes behaviour of this. So, we to this we will end this lecture over here and then we will see a little more details and then conclude the root locus by looking at the last 3 rules.

Thank you.