Control Engineering

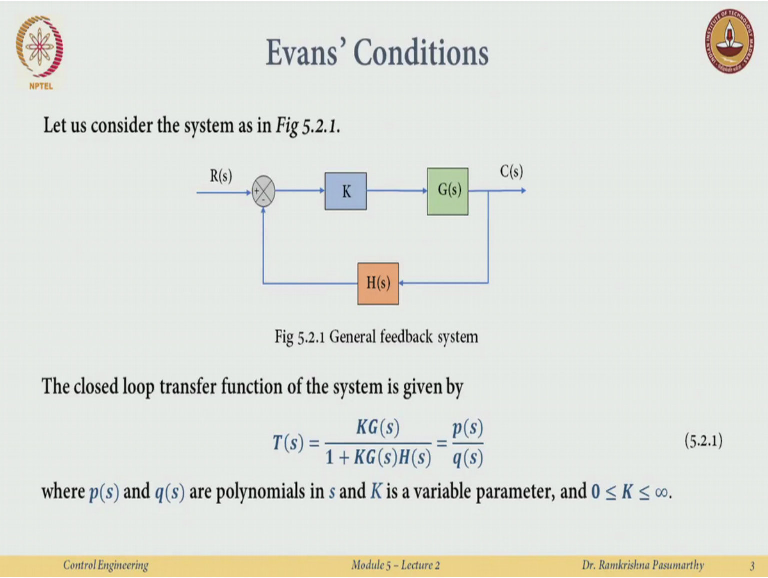
Dr. Ramakrishna Pasumarthy
Department of Electrical Engineering
Indian Institute of Technology, Madras

Module - 05
Lecture - 02
Root Locus Plots

In the previous lecture what we had seen was given a second order system in the closed loop, we saw how the roots moved as the gate increased or the gain was increased from 0 to infinity, and we saw some strange behavior that the roots were moving towards each other they meet and then they go to towards infinity and so on. So, all this was the motivation for this was to try and understand the concepts which we are not which we were not very clear in the Routh stability criterion with respect to relative stability.

So, we will today formalize those ideas into a set of rules and see for given a general system, how would the poles and zeros move; especially the poles move as the system gain is varied from 0 to infinity.

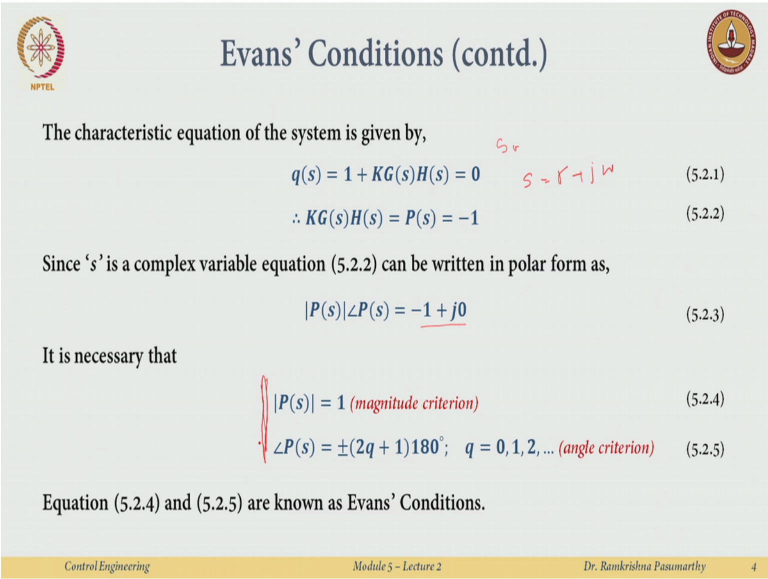
(Refer Slide Time: 01:15)



And this is called the root locus plots. So, how does it go? So, let us consider a typical closed loop system where I have the reference signal a certain output, again K G and a H G in the forward loop H in the backward in the feedback loop plus and minus and so on and I know that for this set of system the overall transfer function is simply K times G 1 over K G H and we now know how to derive this.

So, this is it again just ratio of some polynomial P s in the numerator and q s in the denominator, and will see how the gain varies as sorry how the system behaves as the gain K is varied from 0 to infinity, we could also do for K being negative. So, no minus infinity, but that is a little straightforward extension one; we once we know how it behaves when K goes from 0 to infinity.

(Refer Slide Time: 02:18)



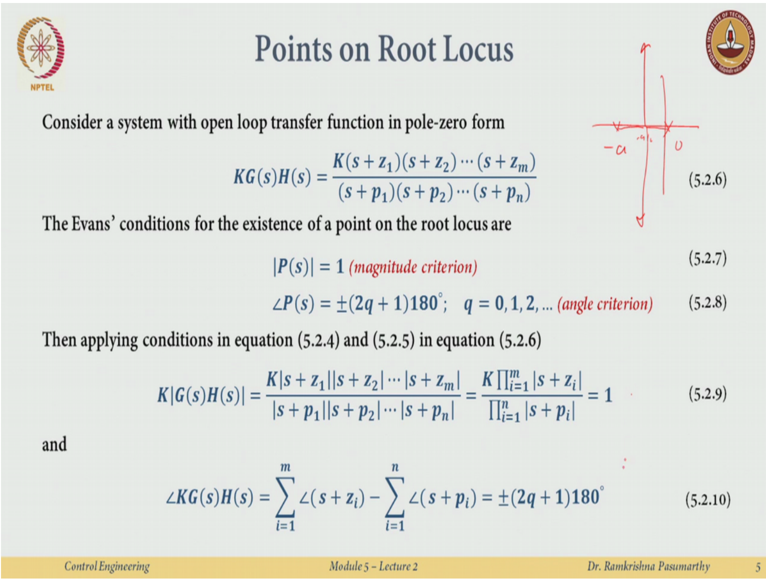
So, where do we begin with well the characteristic equation which we had defined last time is simply the denominator going to see right the q of s is 1 plus K times G is 0.

So, I just do some manipulations K times G the one goes here I call K times G as some, P of s and it is minus 1 and what is s the argument of P is a complex number right sigma plus j j omega right when we do draw things in the in the complex plane. So, s is sigma plus j omega P of s is also then a complex number. Since s is a complex variable P of s. So, this minus 1 can also be then written as a complex number right; so with this guy. So, I can split this complex number in to its magnitude its phase and correspondingly this guy can also be split in to its own magnitude and phase and we see what this exactly looks like right in terms of the magnitude of P s and the angle of P s.

So, what does it mean right. So, this from here to here that the magnitude of this number minus 1 plus j 0 should be the magnitude of P, this magnitude is simply 1 and then the angle should be the angle of this. So, what is this angle? So, this angle is 180 degree or in a way odd multiples of 180 degree plus minus 180 degrees. So, what does this mean right. So, again I am looking for the roots of the characteristic equation as K varies. So, how do I test if a particular point s naught is a solution to this equation I am just finding solutions to this equation.

So, this could be seen just by satisfying the angle criterion and the magnitude criterion right and more will slowly build up a set of rules based on just yes this these 2 things right, and slowly you realize that what we do in the rest of the course or most of the course which is pending from now on is just analyze complex numbers nothing much.

(Refer Slide Time: 04:36)



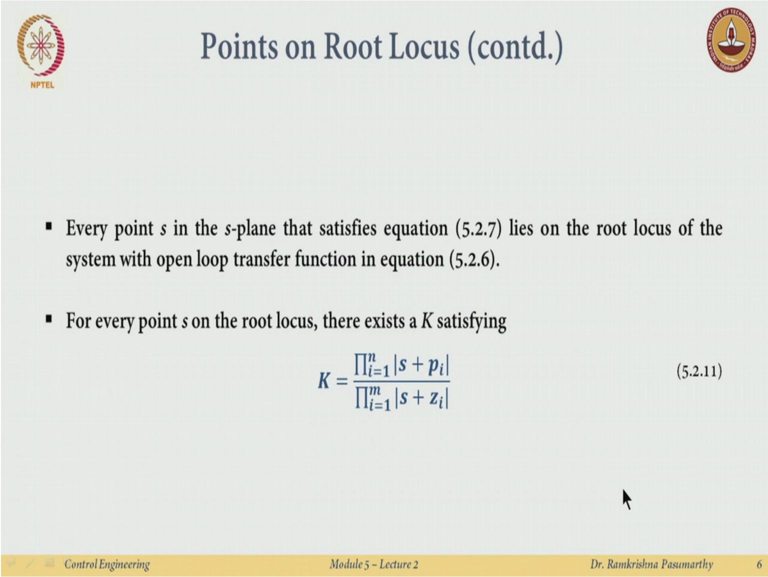
So, a system with the loop transfer function of the form K times G times H is again can be written as K a set of zeros, set of poles, n being greater than or equal to m. So, this gave us what we called as P in the previous slide. So, K times s times H was P and we need to find existence of a point or conditions of when does a point lie on the root locus well the answer is whenever it satisfies these 2 conditions this because we will call this as the magnitude criterion and this the second condition as the angle criterion.

So, how does how does this look like. So, if I just take this guy here and try to compute its magnitude, this will be K times the magnitude of G times H should be K and then K k is a positive numbers are really draw the negative sign here and then the magnitude of all the zeros divided by magnitude of all the poles right and this should be such that the overall magnitude is one. So, K with the product of all magnate magnitude of all zeros divided by the product of the magnitude of all the poles you know s s plus P i this guy should be equal to one right and similarly with the angle. So, this is a complex number or a phasor as you could even call it.

So, the contribution of angle of a constant is 0 there will be some angle contributions. So, these are in the numerator. So, this will be summation of all the angles of the zeros we slowly see how to compute this minus see this guys are in the denominator minus summation of all angles of this poles and what should this satisfy to be on the root locus this should satisfy this angle criterion. So, summation of all angles of zeros minus summation of all angles of poles should be plus minus 2 q plus 1, 180 degrees or like odd multiples of 180 degrees.

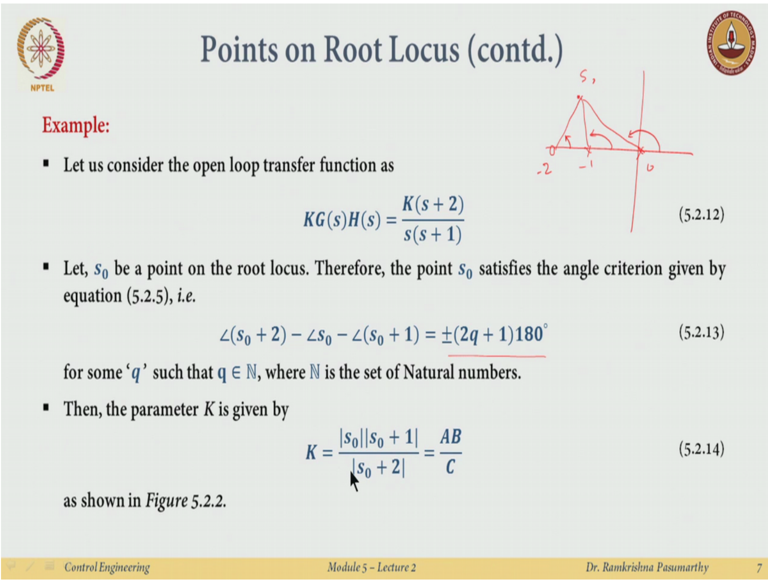
So, these are the 2 conditions which we will verify. So, does the point lie on the root locus the answer is yes if it satisfies these 2 conditions. So, what was the root locus just to just to recall what we have done last time? So, we had started with a with an open loop pole at 0 and at minus a, and what we saw that as the gain increased these 2 guys meet at some point that is minus a over 2 and then this is this go to infinity. So, this movement of poles as the gain K increases from 0 to infinity is the root locus and you can see and we will see slowly of why what this plot means in terms of this and this criterion.

(Refer Slide Time: 07:30)



So, what is the criterion every point s in the complex plane that satisfies this equation, lies on the root locus of the system with the loop transfer function given this right. So, this I am just looking at the open loop open loop transfer function right. So, how does then the magnitude criterion simply becomes? So, from here K is. So, this guy will go to the numerator product of all magnitude of the poles and then the zeroes. So, this should it should satisfy something like this K, should be such that this 2 things should be 2 these 2 things should be satisfied for my any root to lie on the root locus. So, let us see what this means right; so in to in terms of our example.

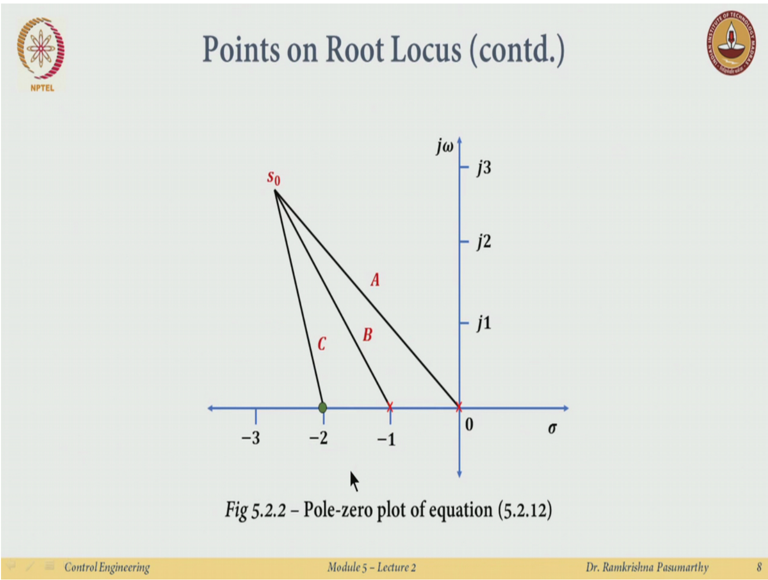
(Refer Slide Time: 08:19)



So, I have K s plus 2 over s s plus 1. So, so give me. So, let me just draw something right. So, I just marked my poles at 0 and minus 1 and there is a 0. Now, I say give me any arbitrary point here right this guy. Now I see if the gain increases will this you know. So, as the gain keeps on increasing will this point lie on the root looks root locus could be any kind of course, on this complex plane right they would also go to the unstable region what will see that a little later. So, for any point s 0 to be on the root locus it should satisfy the angle criterion. So, I will have well this guy, this guy, and this guy and I just measure the angles this way.

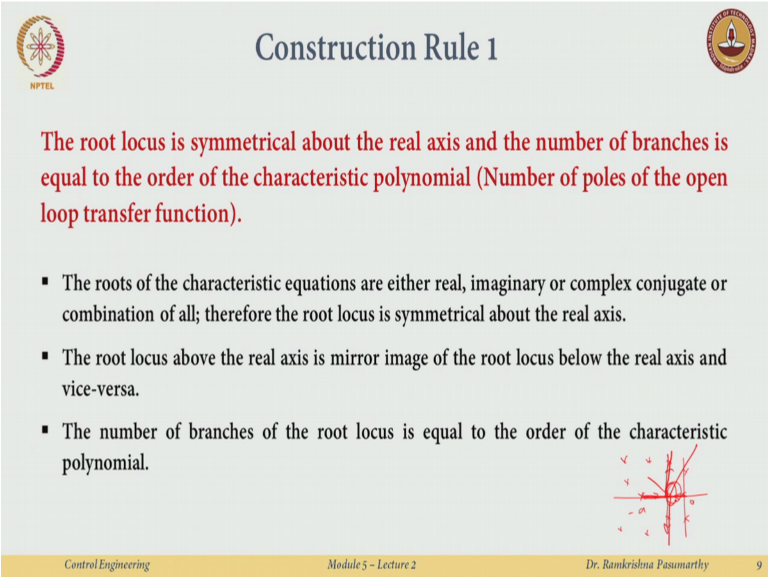
So, this is this how my this is the angle of. So, this is 0 at 2, at minus 2 is a pole at minus 1 pole at 0 0 in the numerator. So, the angle of this guy minus the angle of s of this guy minus the angle of this guy all this should be you know so satisfy this this equality. Similarly K should satisfy this guy right so this one the magnitude criterion. So, summation of this magnitude H 0 plus 1 and H 0 plus 2 right; so this is what this picture says.

(Refer Slide Time: 10:05)



So, if I just go to draw it. So, this a would be the distance of this pole from my desired or the point which I am testing if it is on the root locus this is a, similarly for 1 it is b and c. So, this guy, this if this numbers satisfy this quality then the point is on the root locus together the satisfying this conditions. So, I will. So, then this are these are the corresponding angles.

(Refer Slide Time: 10:38)



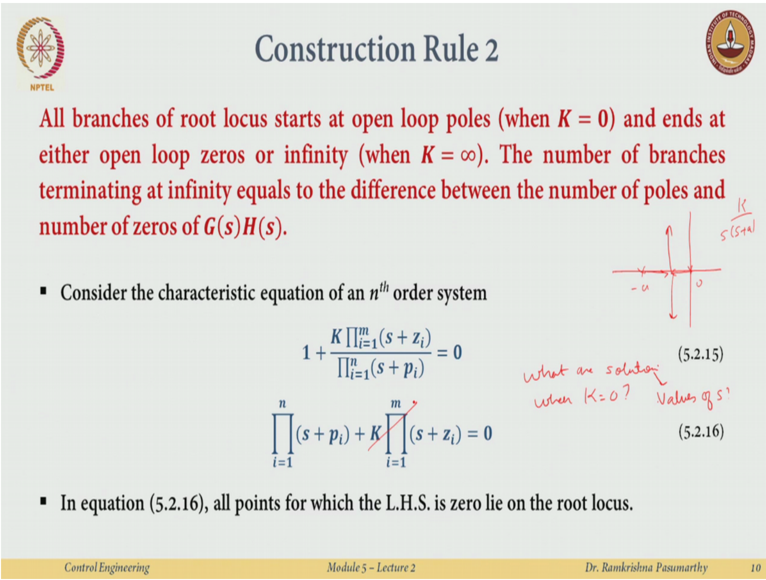
So, the first rule right. So, when whenever we draw a root locus what are we interested in. So, let us just recall the earlier example we did right. So, I had write 0 a pole at minus a, and the root locus yes and also this guy moves this side this side and then they go to infinity. The first thing is the root locus is symmetrical about the real axis and the number of branches is equal to the order of the characteristic polynomial, or the number of poles of the open loop transfer function or the number of poles of G times h. So, what does it mean?

So, first if I: so these are this is this the root locus is such that it will always give me the location of the poles for different values of gain k. So, if there is a pole here, there is always a pole here because all the roots exist in complex conjugate pairs. So, you take any equation or any polynomial s power 6, s power 8 you see that all complex poles and end up or they are in common in they come across as complex conjugate pairs there just be no pole here right and therefore, we can say that if there is a pole here they already be pole here if there is a pole here where does not matter there is a pole here there will be a pole here and so on.

Therefore, the root locus, these roots could be either real there could be imaginary or complex conjugate right, but then if I just look at this thing if it is a pole here there is a pole here, there is a pole here, there is a pole here and so on. And therefore, I can say that well whatever happens it will always be symmetrical on the real axis. So, the first thing is that the root locus is symmetric about the real axis or in other words if I just take the positive half here, and I put a mirror here this guy will just be the mirror image of this guy right and how many branches will the root locus have the root locus will have number of branches that is equal to the number of poles right. So, here I have to branches right one branch comes from here and goes here, another branch comes from here and goes here.

Now, what decides where that why is this guy going here why do not they do here and then come back to their own position or why do not they do this and then go this way by only perpendicular why not this way or why not just keep circling all these things which we will say shortly.

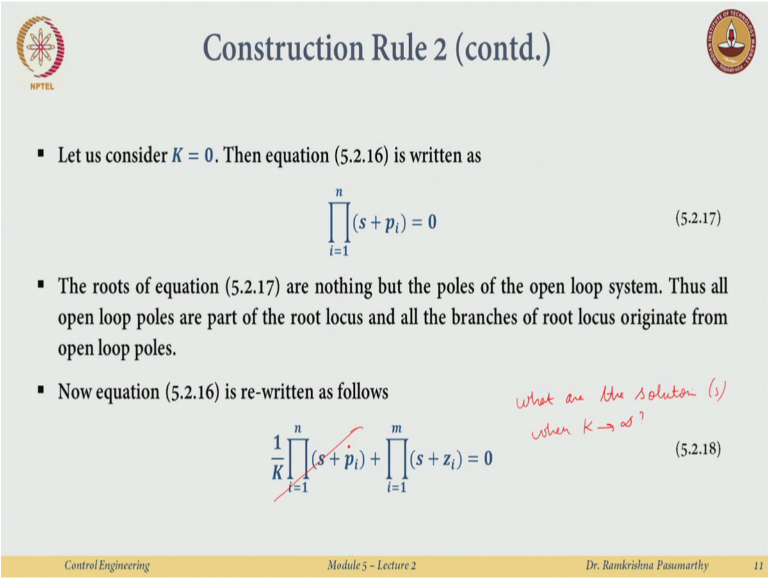
(Refer Slide Time: 13:03)



The second rule says that all branches of the root locus they start at the open loop poles and K is 0, and end at either open loop zeros or infinity as K goes to infinity. So, like what we saw last time right we are a transfer function of the form K s s plus a and we had 2 poles here pole 1 pole at 0 another pole at minus a and these 2 guys will they started moving one guy to the right other to the left they meet and then they diverge; so the number of branches. So, see 2 branches meet and then they go to infinity now why do these 2 go to infinity well the rule says at the number of branches terminating at infinity is equal to the difference between the number of poles and the number of zeroes of G g times H.

So, here there are 2 poles and no zeros. So, both of them will disappear to infinity, now how to generalize this right. So, our characteristic equation is always of this form 1 plus K G times H written as you know the product of zeros and poles it looks like this. So, what I am looking at, how I am looking at, what happens when what are the solutions when K equal to 0 solutions means values of s. So, I take this equation I substitute K equal to 0. So, this term goes away. So, I am left with this guy.

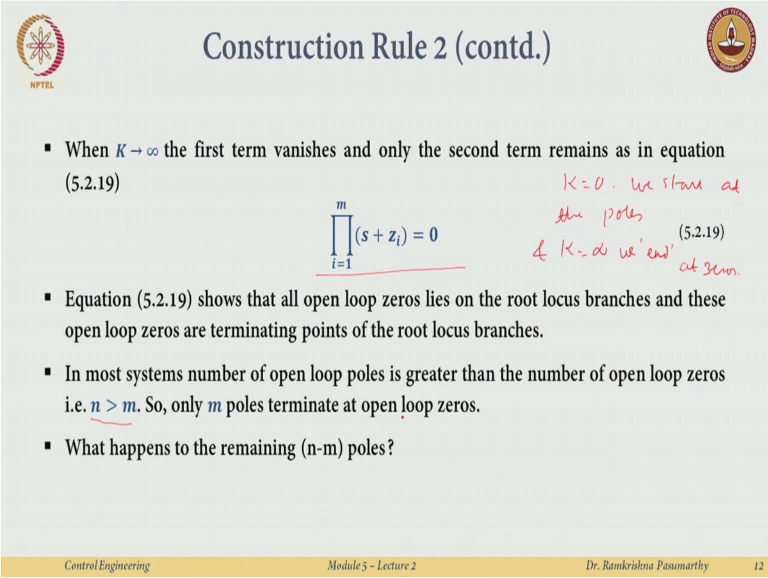
(Refer Slide Time: 14:41)



So, I am. So, what are the solutions what are the values of s here, such that this expression goes to 0 well these are precisely the poles of the open loop system. Thus for all open loop all the open loop poles are a part of the root locus, and the root locus exactly starts from those poles and these are again the open loop poles.

Now, where do they go, what about the zeros. I can rewrite this equation this get divided by K and I get something like this right now the equation can be rewritten in this form. Now what am I looking at I start from K equal to 0 and I ask another question, now what are solutions for again values of s when K goes to infinity. When K goes to infinity I know that this guy disappears if this guy disappears, what I am looking at is an expression which is like this.

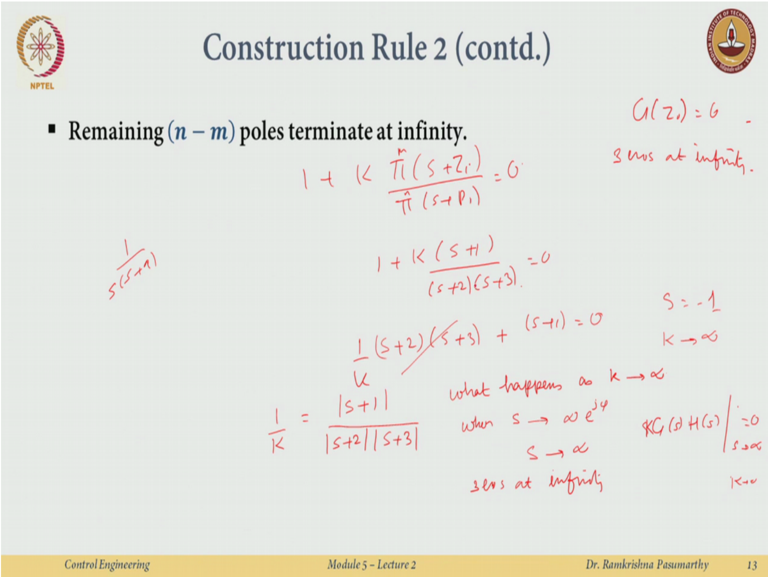
(Refer Slide Time: 15:46)



Now, what are the solutions to these guys right? The solutions of these guys are the open loop zeroes. So, at K equal to 0, we start at the poles and K equal to infinity we end so that is the final point which we look at we end add zeroes.

So, what I can say that also the zeroes lie on the root locus and these are the terminating points. So, I poles at K equal to 0 I start from the pole and I end at the 0, but what we know is that the number of poles is usually greater than the number of zeros right that could also be equal so, but these are m number of poles are n. So, what I could say is that only m poles terminate at zeros right what happens to the remaining n minus m poles.

(Refer Slide Time: 17:06)



So, if I say well this remaining n minus pole n minus m polls they go to infinity. So, let us see what this could mean again I am looking at solutions of the characteristic equation which is 1 plus K the product s plus z i over the product is wrote is over m here it is over n s plus P i equal to 0. So, let us have a little little example and I say well I take an example like. So, say G times H equal to 0 and I want to see what how does it look like is s plus 2 s plus 3, 1 over K s plus one is 0. So, I want to see. So, this is what exactly what we had it 1 over K the product of the poles and zeros this guy should go equal to 0. So, for sure 1 1 solution is s equal to minus 1 as K goes to infinity, but this guy disappears and is looking at this solution.

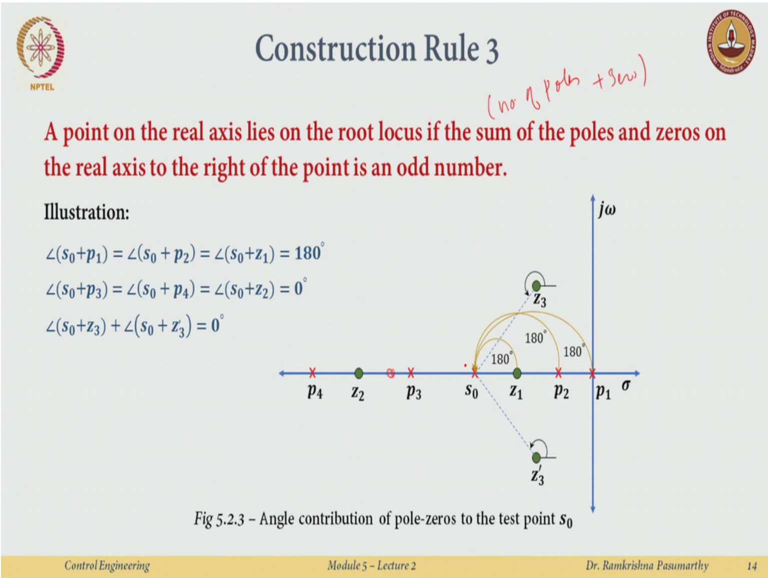
Now are there any other solutions to this equation. So, let us rewrite write it in a slightly different way. So, I have 1 over K is s plus 1 if I set the magnitude part s plus 2 s plus 3; and I am interested again in what happens as K is going to infinity one solution again as K goes to infinity this guy versus zeros s equal to minus 1. So, what we could would could always what we can always also observe is that well something similar happens even when s which is the magnitude goes to infinity and we some whatever angle here.

Another solution could be just you can just substitute t or to take the limit and you see that as s goes to infinity this guy goes to 0 right is series to 1 over divide by a square and the check all the stuff. So, another solution is s going to infinity right what happens then at s equal to infinity? The value of the transfer function or the open loop transfer function

G of s H of s you know then when s is infinity of course, with get K also being infinity this guy goes to 0. Now these are what I call as zeroes at infinity because even at this value at infinity the value of the transfer function goes to 0, because there is a definition at G at some 0 is 0; what it is example.

So, if I just take the other day the other day loading the example of 1 over s, s plus a. Simple thing holds here and both the roots are actually now going to what we call at call as again zeros at infinity. So, the thing is the first. So, here there are 2 poles and one 0, one pole will end up at 0 which is minus 1 and the other pole will go to the 0 at infinity.

(Refer Slide Time: 21:16)



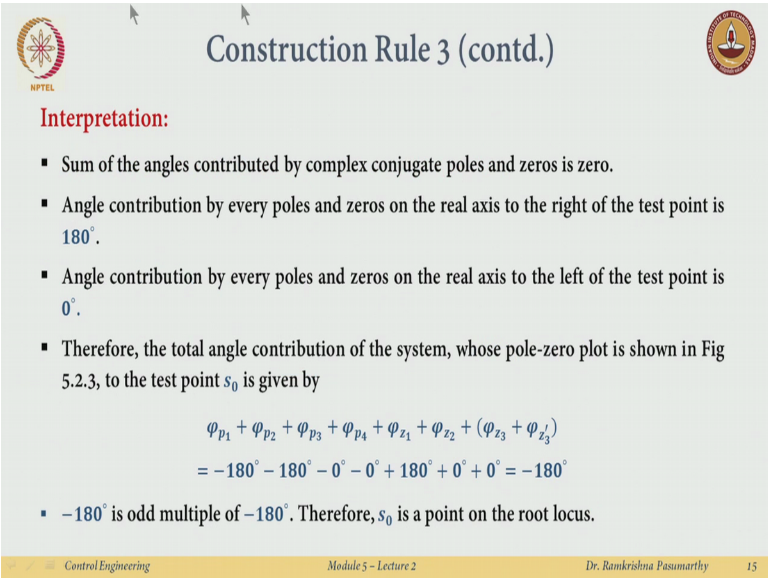
Now, the third rule will say that take a point on the real line say for example, this point right here sorry say this point does this point lie on the root locus well let us see, if we try and find an answer to that a point on the real axis lies on the root locus if the sum of the poles and zeros on the real axis to the right is an odd number.

So, this sum is actually just the number of poles plus number of zeros it is not the values of the poles right it is just say for example, I am looking here from here to the right I just say well there are 2 poles, 2 poles and one 0, 3 poles and one 0, 4 poles and one 0, 4 poles and 2 zeros and so on and I take the total numbers I have here I have 3 poles and zeros, 4 poles and zeros as well right. So, we just we look at the number of poles and zeros not the values of it. So, why is this true? So, let us. So, again the root locus should satisfy the angle criterion. So, let me take a point here and I call this point as s naught.

Now what is the angle sorry there is already s naught is mentioned here is. So, this is s naught right. So, what happens at s naught if I measure the angle well from this pole contributes an angle 180 degrees, this pole contributes an angle 180 degrees, this pole contributes an angle of 180 degrees.

So, this is 0. So, it will contribute an angle of plus 180. So, all these angles will this will be the contribution, what are what about these guys if I just measure the angle from the right. So, they will just be 0 degree they do not do not contribute to the angle; so 180,180, 180. So, we will see if this point satisfies the angle criterion or not.

(Refer Slide Time: 23:19)

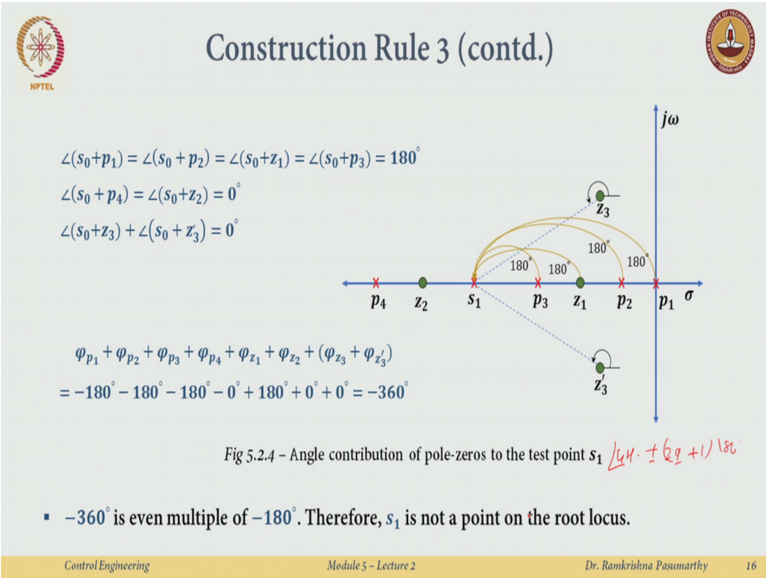


So, why do not we what about conjugate complex conjugate poles, will the sum of the angles is always 0. So, as you can see from here right. So, this angle sorry this angle here plus this angle will always got 0 for 360 degrees order the angle contribution by every poles and zeros on the real axis to the right of the test point is 180 degrees, these are to the right to the right to the right it is 180 degrees, and then to the left of the test point is always 0.

This guys will contribute p 3 z to p 4 they do not contribute similarly with and then you can see the complex conjugate guys here the overall angle goes to 0. So, what is the total angle contribution? By the total angle contribution is angle of all the poles and the angle of all the zeros well. So, here P one contributes 180, P 2 180, z 1 180 P define yourself of your signs 1minus 180 for the poles, but P 1 minus already for the pole at P 2, and all these guys are 0 the first 0 here contributes an angle of 180 with the plus sign and this is right and this guy is an odd multiple of 180 degrees and therefore, s 0 is a point on the root locus right.

So, therefore, this justifies our things saying that the point on the real axis lies on the root locus if the total number of poles and zeros to the right of that point is an odd number.

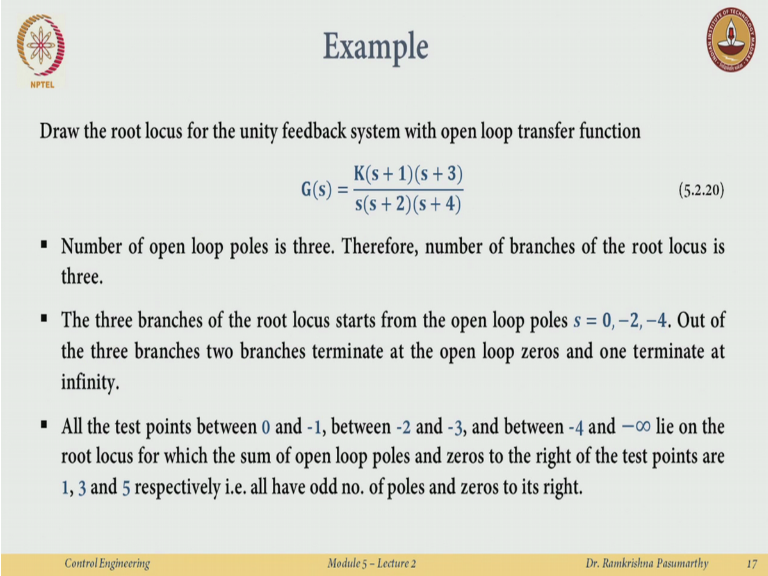
(Refer Slide Time: 24:58)



Similarly, I could even do for some other point say s 1 which is now. So, earlier s 0 was between P 3 and z 1 now I take this point and I call it s 1, and I just check is s 1 on the root locus right and I just look at the angle criterion again. So, again I have P 1 contributing an angle 180 in the negative duration, the pole P 2 again minus 180 I have a 0 that contributes plus 180 P 3 contributes. Minus 180 contribution of P 4 and z 2 will be 0 and the complex conjugate guys will anyways be 0. So, let us do this is minus 180 minus 180, minus 180, plus 180 and this is 360 degree right and 360 degrees it does not satisfy this criteria that it should be an odd multiple.

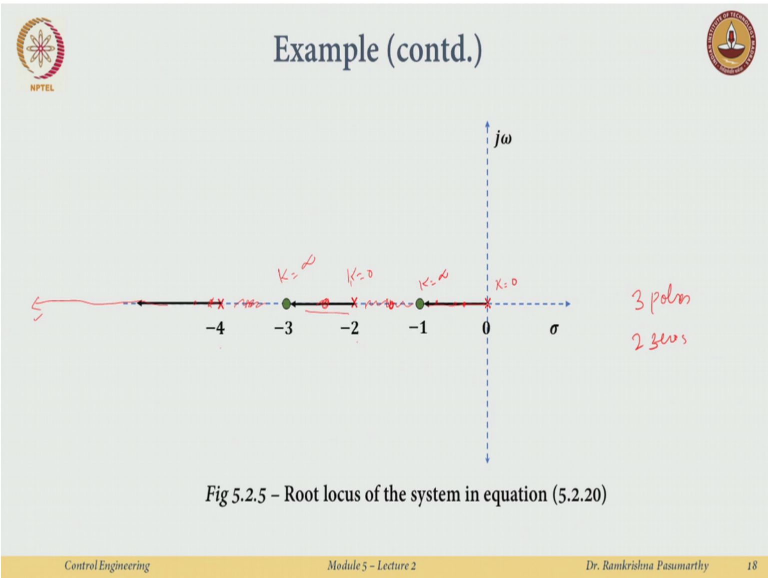
So, 360 is an even multiple. So, if you look at the angle criterion what does the energy angle criterion it should be 2 plus minus 2 q plus 1 times 180 degrees, it should be the angle of G times h. So, this is an even multiple therefore, s 1 is not a point on the root locus ok.

(Refer Slide Time: 26:13)



So, let us end with an example. So, if I take a system with poles at. So, poles at minus 2 minus 4 a pole at the origin and these 2 zeros right. So, what is the first observation is that the number of poles is 3 right, and therefore the root locus will have 3 branches now where will these branches go that depend on what are the locations of the zeros right. So, take these 2 zeros and say well there are 2 zeros sorry and 2 poles sorry 2 zeros and three poles. So, 2 of those three poles will go to the zeros and one remaining one will go to infinity.

(Refer Slide Time: 27:00)



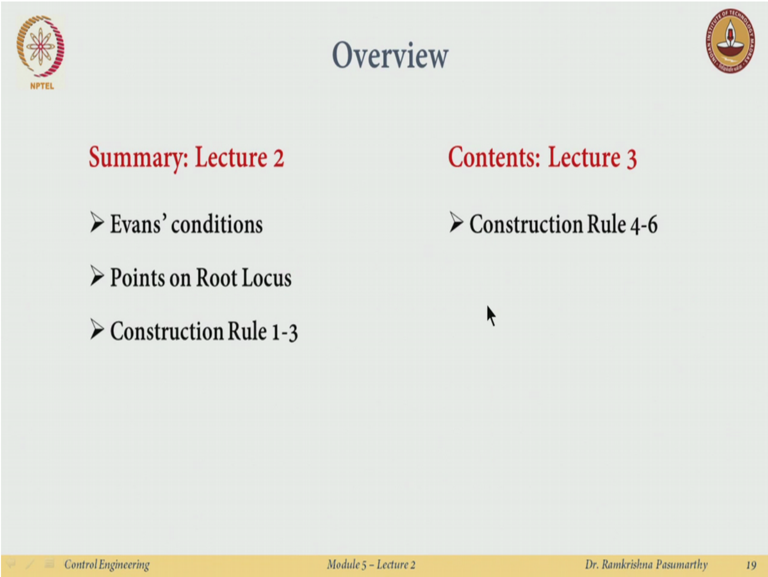
So, let us quickly draw the pole 0 diagram. So, a pole at 0 I have a pole at minus 2, I have a pole at minus 4.

So, let me just take for example, say does this point lie on the root locus I see where there is only one pole to the right. So, of course, this is on the root locus this is also on the root locus this entire thing will be on the root locus right. So, this pole will start here and then move towards this 0, and we know that where the poles terminate at 0. So, this is when K equal to 0 this happens when K equal to infinity. Now look at what happens to this this guy does this guy move towards the right or towards the left we will again see the same the condition number 3. Now is this point on the root locus well the number of poles plus zeros is 2 it is an even number even is not allowed. So, this entire region here is not on the root locus.

Therefore, this guy will you can always check this region is this region on the root locus look to the right there is 1, 2 and 3 total number of poles and zeros is an odd number and therefore, this entire region is on the root locus. So, this guy will slowly move this side and then terminate at 0 again K equal to 0 till K equal to infinity ok now what about this guy can this guy move towards the right. Well, just see whatever is in this region I have 1 2 3 and 4 poles which is 4 poles 1 2 3 and 4 are the number of total number of poles and zeros, which is an even number and that is not allowed. So, this entire region is also all featured.

Now go here right to the left is this does it satisfy the condition number three say 1 2 3 4 5 for the total number of poles and zeros of course, this is on the root locus this is on the root locus and this entire this real axis to the left is on the root locus right. So, what is the second condition say that when K equal to 0, I start at the poles and end up at 0 when K equal to infinity simply this happens here there are in this case there are 3 poles and 2 zeros. So, 2 poles go to 0 and by the by the rule to one pole goes to infinity or the 0 at infinity and you see that entire real line from here from to the left of minus 4 to infinity is on the root locus. So, this is a third guy little region which will go to infinity.

(Refer Slide Time: 29:37)



And this entire this is this will tell me the entire behavior of the system as the gain K is varied from 0 and 0 to infinity. So, these are the first three rules in construction. So, in the next lecture we will see some other rules related to these things.

Thank you.