Control Engineering
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Module – 04
Lecture – 02
Routh–Hurwitz Criterion

Hi guys. In the last lecture, what we had seen was to define properly the notion of stability and related to the transfer function or how does transfer function have or does transfer function have any information on the stability. And we characterize by saying that if all my poles are strictly on the left half of the S plane then my systems are stable. And then we also had some special characterizations when there are repeated complex conjugate roots, how the system goes to an unstable behavior. And even when you just have a single pair of complex conjugate poles where the system was refined to be marginally stable, and all these in fault computing roots of some higher order polynomial could be 6, 7, 8, depending on how many poles there exist in the system.

So, today we will try to find an algebraic way to see if there are better or may be more efficient tests of finding out stability of the system. We may not be directly interested in the exact location of the poles, but an answer to whether the system is stable or not would be enough for the initial analysis purpose. So, what we will do today is to just look at a polynomial of a fairly higher order for which we cannot just compute roots per hand. What are the methods which are which evolved you know historically over time. So, this method is typically referred to as the Routh-Hurwitz criterion.

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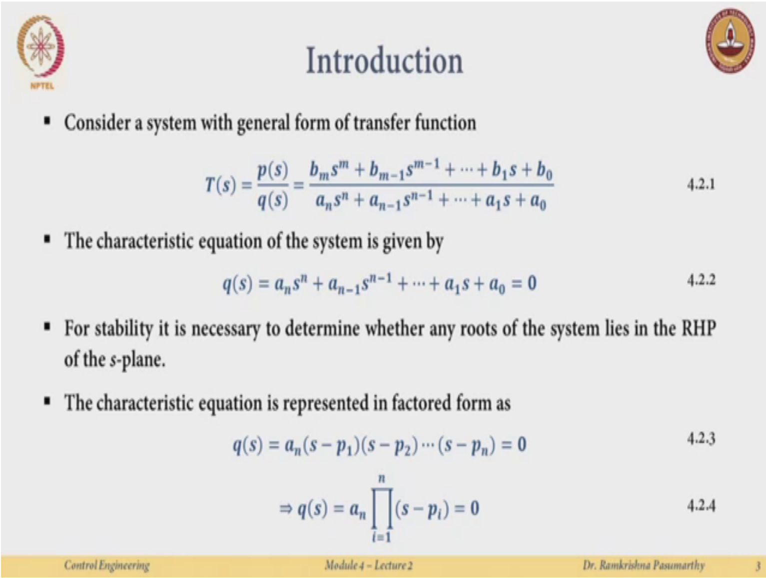

So, we are again interested in the transfer function analysis where I have on the numerator polynomial of the m and n, and then n is usually typically greater than or equal to m. And then the characteristic equation which essentially what happens in the denominator when that is equated to 0, this is referred to as the characteristic equation of the system, no matter if your system is open loop, if you are doing an open loop analysis or even a close loop analysis. We will formalize this characteristic equation shortly, but at the moment you are just interested in the denominator of a given transfer function for which we are interested to perform this stability analysis.

So, the definition of stability leads us to a test which says or which ask the question or are there any roots of the system on the right half plane if the answer is yes then my system is unstable. So, let us do some steps here. So, q of s is a polynomial here which I can I can write it in this way as the product of. So, this will have some roots let me call those roots are p 1 to p n and these roots are the poles of my system. So, if I expand this expression in terms of the poles, the expression would look something like this that again in decreasing powers of n until decreasing powers of s until you reach n equal to 0, this is simplify this.

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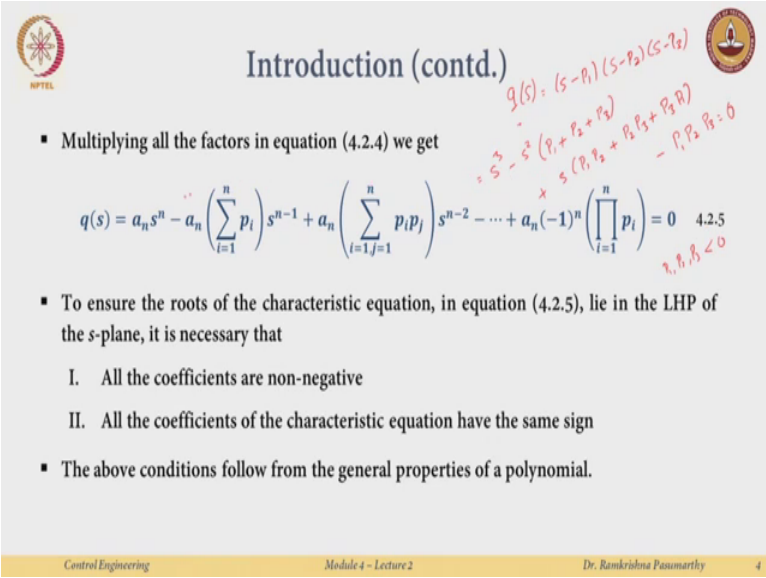

So, what happens when there are three poles right when this expression q of s, q of s let us even assume that a three would be equal to 1, then I have s minus p 1 s minus p 2 s minus p 3. And this would expand to the following you have s cube and s square, so you will have sum of all the roots. So, what is says is p 1 plus p 2 plus p 3 plus the term in s which is p 1 p 2 p 2 p 3 p 3 p 1 minus the product of all these n equal to 3, so I will have a minus p 1 p 2 p 3 is 0. So, stability says that all the poles should be on the left half plane. So, let us assume just see that you know solve p 1, p 2, p 3 are less than 0. Let us just for simplicity assume they are real even though if they are complex nothing much would change also.

So, if p 1, p 2, p 3 are less than 0 then this entire number would also be negative and the coefficient of s square they needs to be positive. So, similarly if p 1 and p 2 both are negative p 2 p 3 are negative p 3 p 1 is negative all this would be the products would be positive numbers. Similarly if p 1 is negative, p 2 is negative, p 3 is negative, their product would be a negative number and this minus would make it positive. So, what does the first observations say- if I have a polynomial like this and if I want to observe or if I want to say something about the location of poles in the stable region, the coefficients of this should be non-negative, thus one of the necessary conditions. It may or may not be sufficient that we will see a little later. And all the coefficients they should well has be they should have the same sign possibly it could be a minus, minus, minus, minus all the time that could thus also allowed, because you just equate to 0 and then all the minuses will become pluses.

So, the first observation just by an expansion like this and keeping in mind that stability would mean the poles are on the left half plane would lead us to say well, the necessary condition is that all the coefficients are non-negative. It could be a 0. What if in this thing, if this is true all the coefficients would be positive? So, this follows just from the general conditions of this of this polynomial this is general expansion when you have it for degree n.

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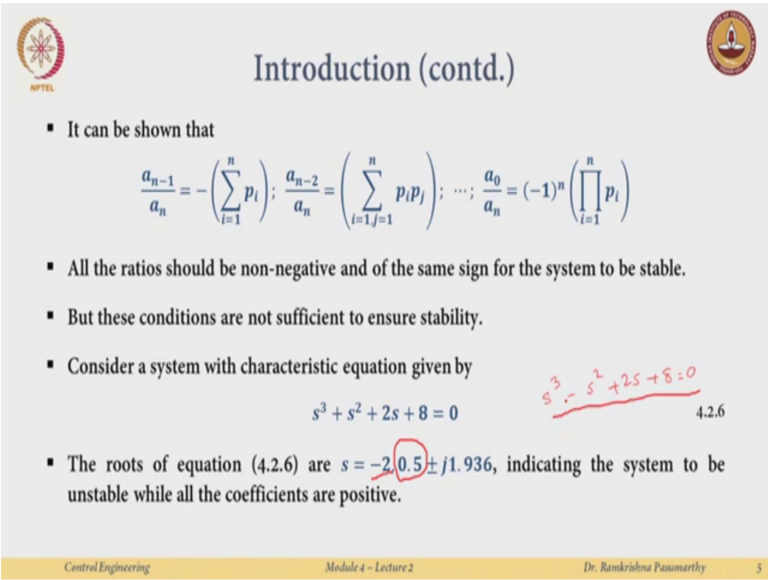

So, some straightforward computation here would show that this ratio or like this some of poles from p 1 to p n could be expressed as a ratio of a n minus 1 and a n, and similarly the products and all. So, based on again these observations here we could also say that all the ratios should be non-negative and of the same sign for the system to be stable and I am just rewriting those conditions in terms of polynomials and use dividing throughout by a n. And since our a n we assumed to be 1 for simplicity I think this would be very straight forward to find out, but these are not sufficient conditions these are necessary.

For example, I take a system which follows this rule over here right the s cube plus s square plus 2 s plus 8 equal to 0. Well this satisfies all this conditions that the coefficients are all positive. Now, does it guarantee stability we will just you know compute the roots and the roots turn out to be minus 2, this is stable guy, and you have 0.5 plus minus j now this guy this guy sits in the minus half plane. So, therefore, the system is unstable even though all the coefficients are positive. So, the test that these things all the coefficients are non-negative is just a necessary condition, it will not guarantee the things. However, if instead of a plus here I have a condition like say s square s cube minus s square plus 2 s plus 8 equal to 0; it violates the condition that all this guy should be non-negative this is definitely unstable, this is because I am violating the necessary condition.

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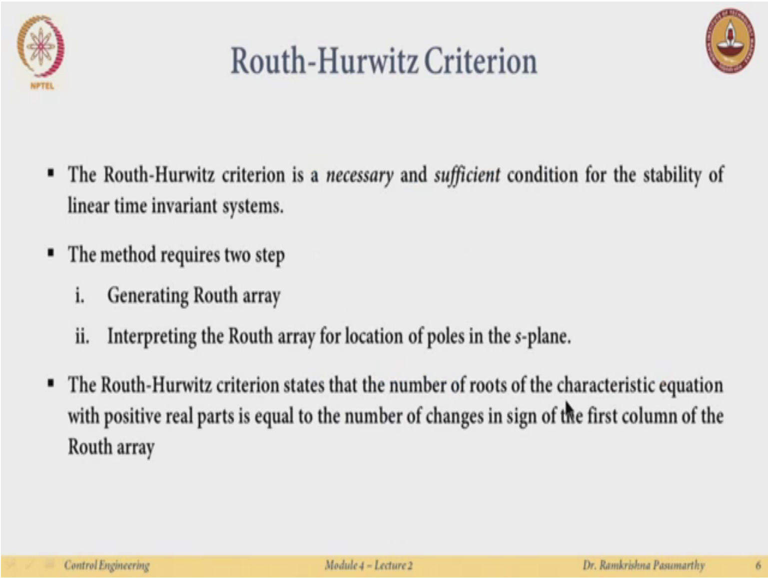

And then to have the roots computed or at least to check where the roots lie in the complex plane Routh and Hurwitz, I will tell you a story of this a little later. They came up with a condition, which was both necessary and sufficient for the stability of linear time in variant systems. So, how does this go we go to the proof of it? But we will just learn the technique. So, the method has two steps, first I just generate something called a Routh array and interpret the array in terms of locations of poles in the s-plane. So, we will read this statement once we see how the array goes.

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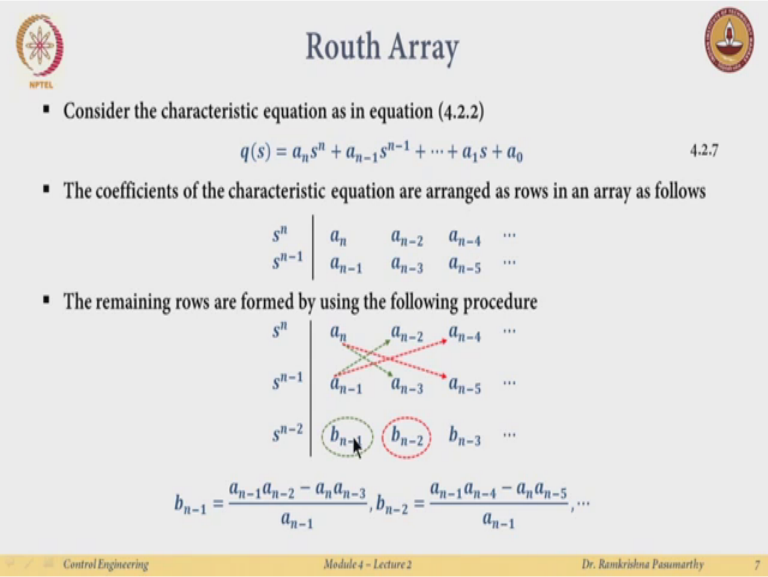

So, I have this characteristic equation. And I just write it into an array, I start with s power n, I take the coefficient here, I put a power n and then I take s minus 1, I took this coefficient put it here and then I do all the coefficients of the of n, n minus 2, n minus 4 and so on until here, and then n minus 1, n minus 3 until I reach the end of the polynomial here. So, these two are given to me. How about the remaining rows? So, given s n, s n minus 1, s n minus 2, so this number is b n minus 1 computed in the following way, but I take a n minus 1 times a n minus 2 minus of this guy divided by a n minus 1 and just a simple formula.

Similarly, I compute b n minus 2 as again I start with a n minus 1, I go to the next row here a n minus 4, again a n, a n minus 5 divided by this guy. So, this is I think if you just look at this as a matrix and see the numerator is just the negative of the determinant of that matrix and I just divide it by this coefficient here.

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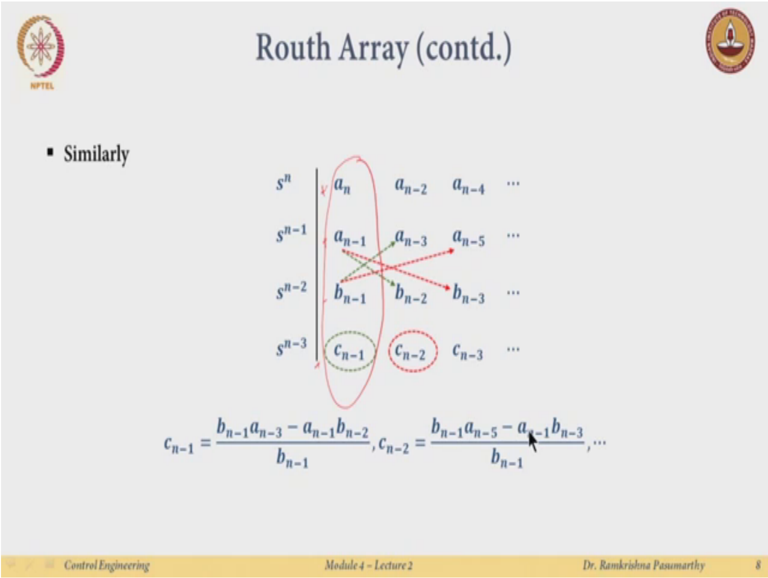

And then keep on doing this going to in this recursively. Then I can have how do I compute the third thing c n minus 1, I start from here I take b n minus 1, I go to this guy a n minus three subtract the product of with the product of a n minus 1 b n minus 2 and divided by this guy. Similarly, I do for c n minus 2, and I go from here till here subtract this guy and divide again by b n minus 1 and so on. So, I do all these steps and what does this last statement tell me. The last statement tells me that the Routh-Hurwitz criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of sign changes in the first column of the Routh array. So, I have this thing here is the first column of the Routh array.

So, a n, a n minus 1 to go to b n minus 1 and c n minus 1 if there is a sign change in any of these numbers. So, by construction a and a n minus 1 would be of the same sign or they would right with they typically both could be positive. So, if the b n minus 1 becomes negative right then there is a sign change and we will see that with the help of an example and then and if this is a plus say- for example, and this is plus, this is a minus and this again becomes a plus. So, you see where there is no sign change here, there is one sign change and two sign changes. So, this sign changes will tell you that there are or they will tell you how many poles are there on the right half plane and that is what again will read the statement again.

That the number of roots of the characteristic equation with positive real parts is equal to the number of sign changes in the first column. It will not tell you the exact locations, but it will tell you a first test is my system stable or not or all the roots of characteristic equation are they on the left half plane or not or if they are on the right half plane how many are on the right half plane and so on.

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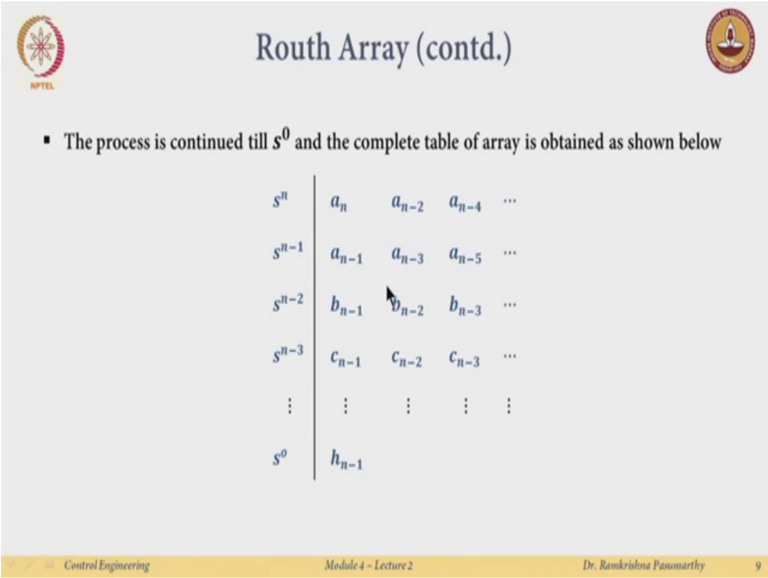

So, you can just keep on computing this, this array until you reach s power 0 and this is you just look at this column. Hence, check if there are any sign changes. If there are no sign changes, then my system is stable. If there are sign changes then there are roots on the characteristic root roots of the characteristic equation on the right half plane. How many are there that will depend again on the number of sign changes.

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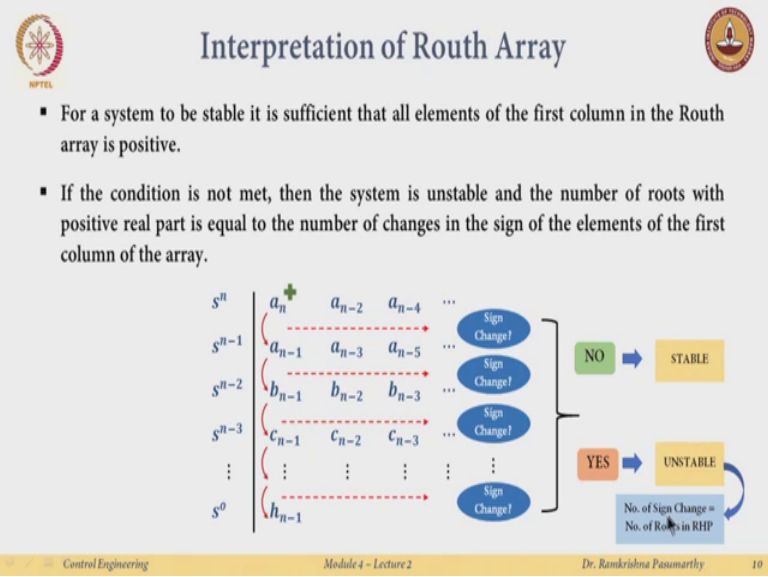

So, just to repeat, so for the system to be stable it is sufficient that all elements of the first column are positive. Again this is just see why it is just a sufficient condition. If this condition is not met, when the way if there is a sign change then the number of roots with positive real part is equal to the number of sign changes. So, if you just look at this way I will be go from a n to a n minus 1, I ask my question is there is sign change I go from a n minus 1 b n minus 1, and I keep on asking this question is there are sign change is there are sign change. And when the answer is no then I can conclude that the system is unstable. And if there are sign changes, if the answer is yes even once then the system is unstable. And the number of times it changes the sign that equals the number of roots on the right half plane.

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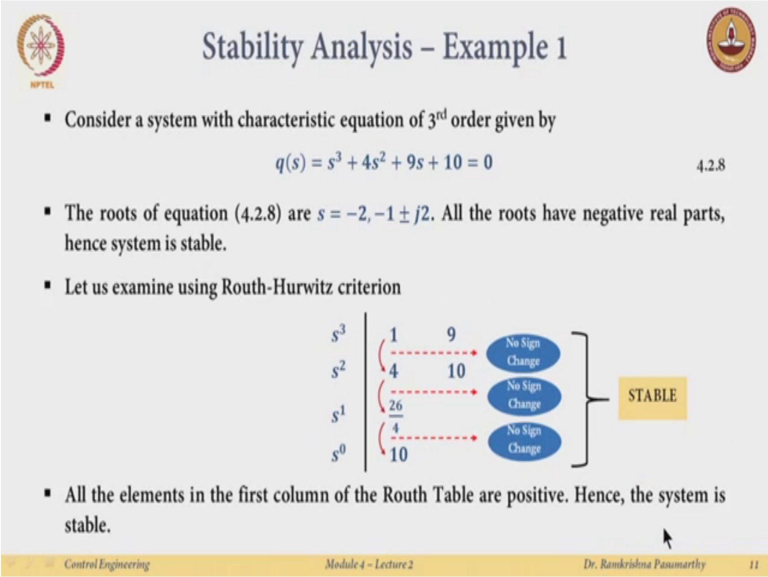

So, just as an example, so I start with the very simple looking polynomial s cube 4 s square 9 s and 10, everybody is of the same sign. So, the necessary condition is met. So, let me go about drawing the Routh table. So, I start with the s cube which is a highest order of the polynomials is have 1 and 1, then I go to s square and this is a 4 here, then s n minus 1 would be I am looking into s power 1, it is 9 here, and then I apply in it 10 here.

Then I just compute how do I do this 4 times 9 minus 1 times 10 divided by 4. So, I get a number here 26 times 4 and then a 10 here just by looking at the formulas for b n minus 1, c n minus 1 in the previous slide. So, what question do I ask I just look at this column, and I ask is there a sign change from 1 to 4, answer is no; from 4 to 26 by 4, again answer is no; 26 by 4 to 10 again the answer is no. So, all there is no sign change, no sign change, no sign change and therefore, the system is stable.

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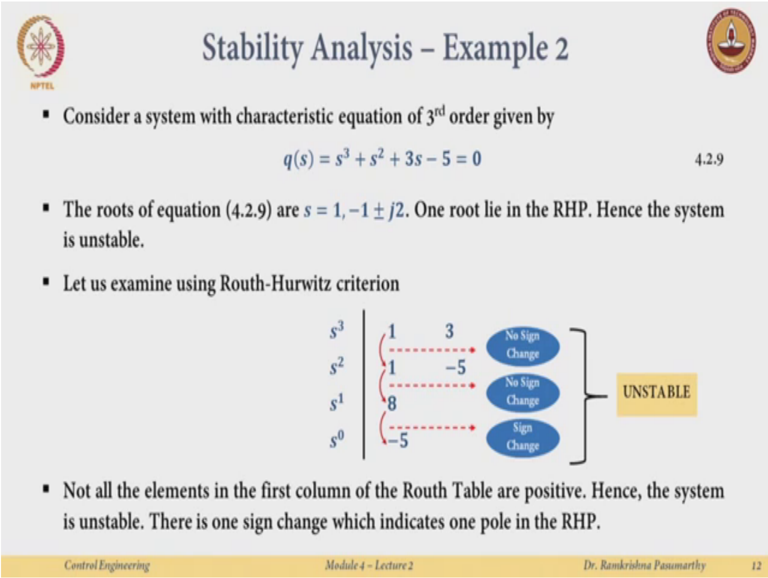

Now, in the case of unstable thing, so you have s cube and s square 3 s and minus 5; so this again by inspection is unstable. Now, let us just verify what the Routh guy tells you. So, I just do s cube and so o on and 3 then s square have 1 and minus 5, then I do all the thing and then I just find the sign change here right from 8 to minus 5. And therefore, the system is unstable you could even compute the roots exactly and say that. So, we have one this is the unstable, and then you could there are two stable, there is a one sign change. So, one pole on the right half plane, this is what this last statement tells you.

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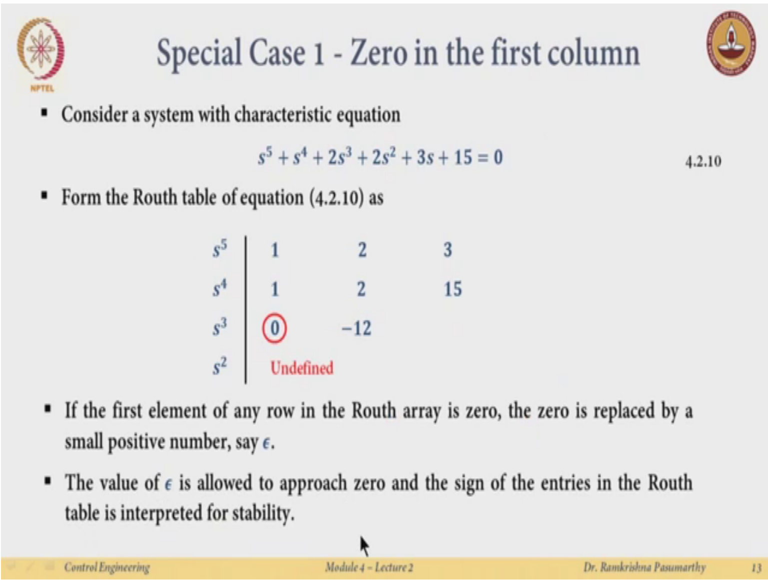

Now, let us do some other examples. So, I have s 5 plus s 4 2 s cube plus 2 s square 3 s plus 15 is 0. And my necessary condition here is met. So, again I start constructing the Routh array. So, s 5 is 1 s cube is 2 s power 1 is 3, and s 4 is 1 s square is 2 and s power 0 is 5. So, something strange happens here. If I look at this matrix when the determinant is 0, so something strange happens here and these are like the special cases where there is a 0 in the first column. What do I do, do I just leave it here and say well there is no answer to this problem, no, there is a there is way out of this. If the first element of any row is 0 then just replace this by a small positive number and call it epsilon and this epsilon value epsilon is allowed to approach 0; it is like very, very small number and then we compute the Routh table. Now, this now with this epsilon being a small positive number and then interpret the table for the stability how do we do this?

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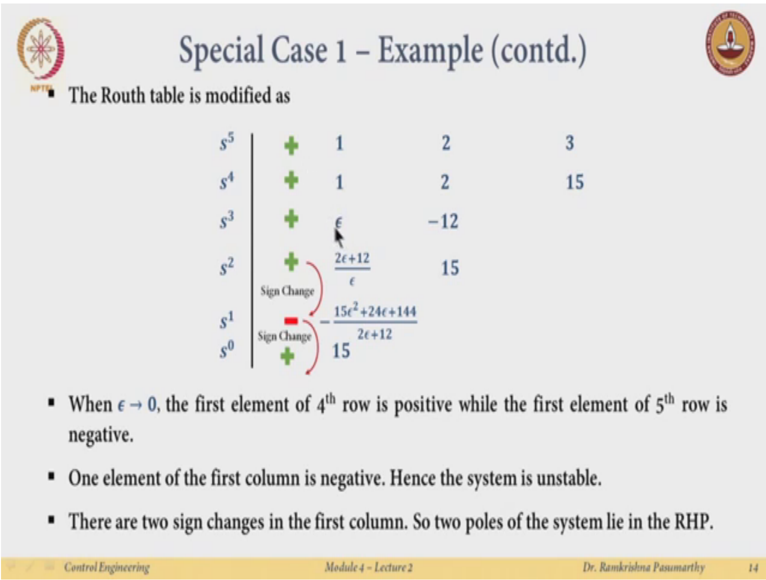

So, the way we compute is instead of the zero I substitute an epsilon which is considered to be greater than 0. Then this element is computed as twice epsilon the 12 power epsilon and I was 15 here and then the s power 1 term turns out to be this again doing this, this computations. And now closely look at this term, as epsilon becomes very small you could even take the limit as a epsilon goes to 0, this guy becomes a negative number, I just have 144 by 12. So, this and then the last number becomes 15. So, I just keep traversing here, I say will there is no sign change from 1 to 1, no sign change from 1 to epsilon as epsilon is assumed to be positive, since epsilon is assumed to be the positive this is also a positive number. Then I go here this guy becomes negative and again the last guy is positive.

So, I see there is a sign change and there is another sign change. So, since there is one negative element here, the system is definitely unstable. And then how many sign changes are there, this is a plus to minus, one sign change, and again a minus to plus. So, the system is unstable. Further, this table or this row tells me that there are two poles of the system in the right half plane corresponding to two sign changes.

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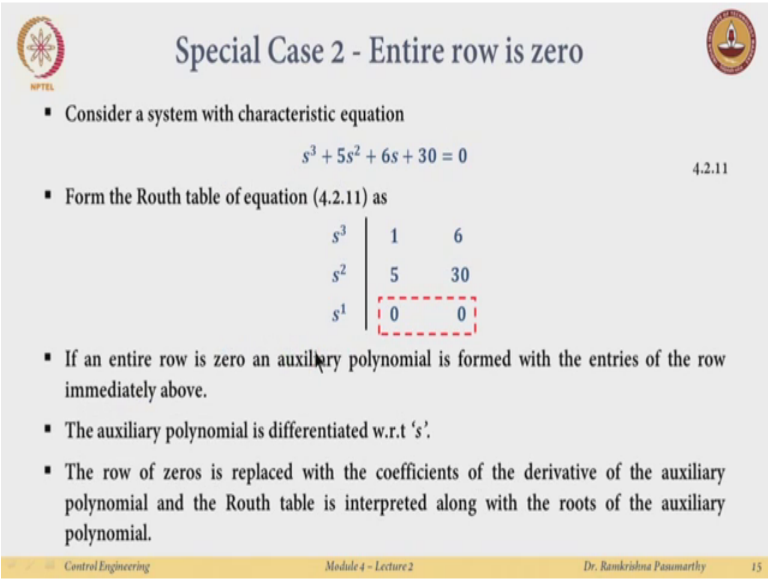

So, well let us see another case. So, have s cube 5 s square 6 s and 30 and then I put this in the table I pluck plucked this numbers in 1, 5, 6 and 30. So, there is some strange happening here right. So, the entire row is now going to 0. What do I do now where we substitute epsilon and epsilon again and do the analysis well the answer is no, because you still again get encounter 0 if you do like that. So, what do we do here we will be a little careful that.

If an entire row is 0, we form something called an auxiliary polynomial with the row immediately above this guy. So, this is s 1, I find I form a polynomial with this guy s square which has entries 5 and 30. And what I do is then this polynomial is differentiated with s and I will have then, so if this is a polynomial of order 2 then I will have something in order 1 here. The rows of z of this guy are in replace with the coefficients of this differentiated term of these differentiated auxiliary polynomial with respect to s.

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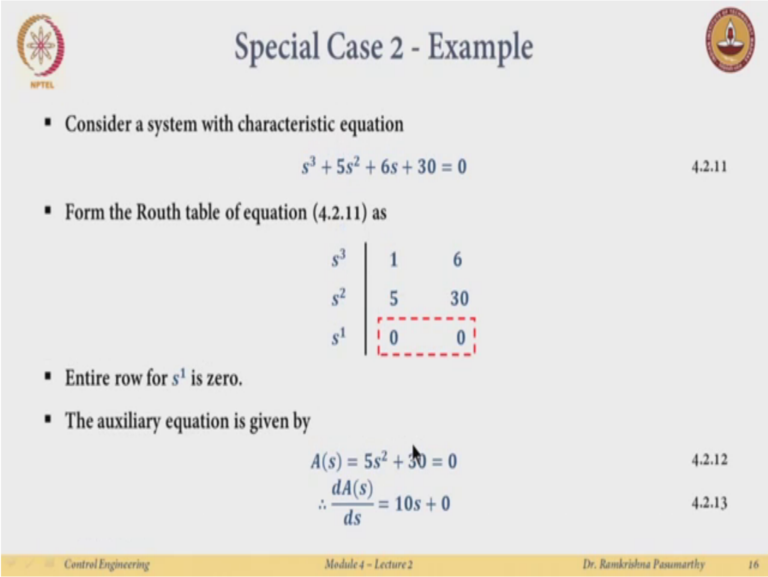

Let us see how this works. So, I have a 0 here, the polynomial just above this is 5 s square plus 30, I differentiate with s and I get this is 10 s plus 0. I take this 10 I plug in here I take the 0 and plug in here.

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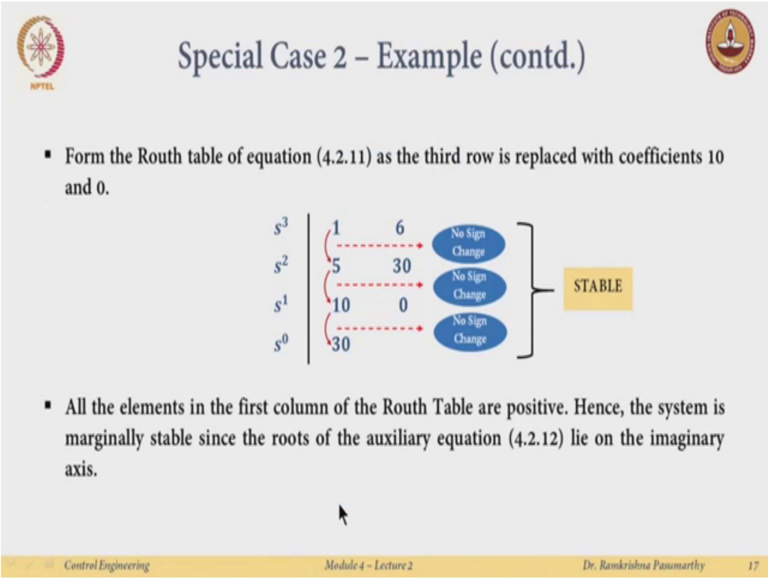

So, it will so my new or my new table would now look the like this and then Routh array. So, I differentiate I take 10 and 0 and then I proceed with the rest of the computations this. So, this guy will just be a 30. And then I again do the sign change test if there are sign change answer is no, no, no. And therefore, I can conclude that the system is stable because all elements of the first column are positive. There is something little more here right that since this guy is entire row is going to 0, there is some there is an existence of a pair of roots which are on the imaginary axis and that is simply given by solutions of this equation 5 s square plus 50 equal to 0.

You solve this; this will give you the exact location of the roots on the imaginary axis. So, whenever entire row goes to 0, then you detect a pair of roots on the imaginary axis. Now, this is if the conjugate pair.

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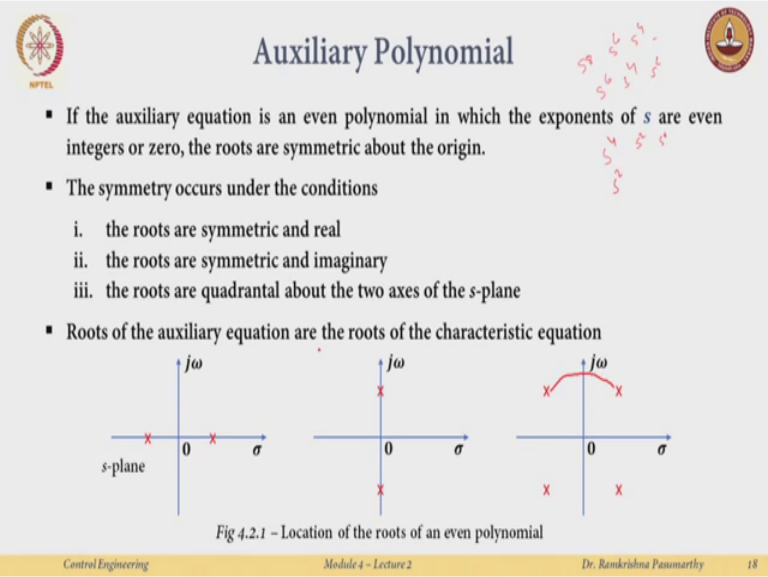

So, what if there are more of these auxiliary polynomials. So, the auxiliary polynomial is essentially it has the highest power is of course, and even number actually it could have s 4 s square and so on or we say s 6 or s 8, so this is highest power. And then you could have all the other powers could be s square, s power 0, s power 4, s power 2 and so on, it will be all just have even powers s 6, s 4 and so on. In this case, there are what we see is there are roots that are symmetric about the origin and symmetry could occur under several conditions.

Symmetry could be like here at some from here we have a minus 1 you could have a plus one here they could be symmetric just and lie just on the imaginary axis plus j omega and minus j omega there could be the something like this. Also you have plus 1 plus 0 omega minus 1 minus j omega you have sorry this is minus 1 plus j omega this, this is the symmetry from here till here. So, this is a minus 1 this will become a plus 1 if this is plus

1 plus j omega this will be minus plus 1 minus j omega, and here this is symmetry about this imaginary axis. So, if this is a plus 1 the real part will become minus 1 and so on. And the roots of the auxiliary equation are also the roots of the characteristic equation. This again appear only when you know when some these kind of things happen.

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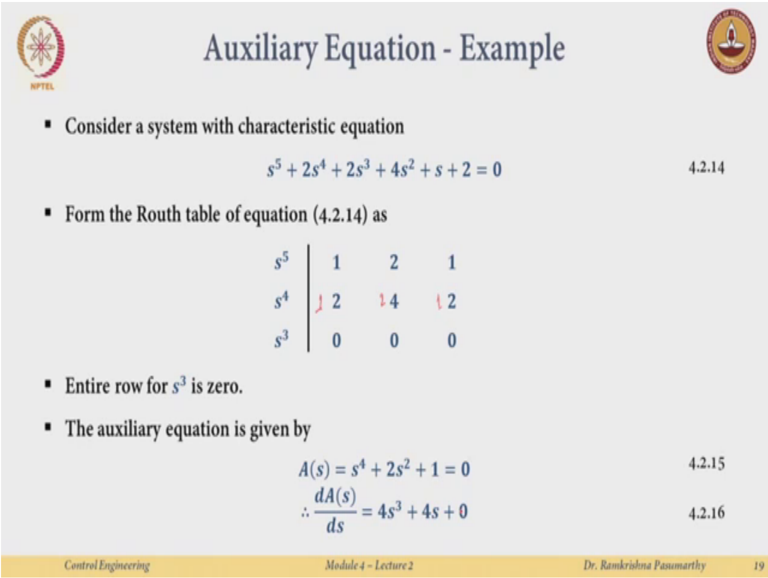

So, let us do another polynomial here, I have s 5 plus 2 s 4 plus 2 s cube 4 s square s 2 all be equal to 0. So, I construct my Routh table and I take s 5. So, I have one then s 3 the 2 goes here, 1 goes here for s 4, 2 sits in here, 4 goes here and the s goes here. So, whenever you have things like this you could just even replace this divide this by its common factor. So, it could even be 1, 2 and 1. So, you can divide by 2 divide by 2 divide by 2, it will still be the same analysis.

And you see that the rows are the same, and therefore this entire row of s cube goes to 0. What do I do now I first construct the auxiliary polynomial that is in s 4. So, you have s 4 plus 2 s square plus 1 equal to 0. And even will be the same, even if you write 2 s 4 plus 4 s square plus 2 equal to 0, I just take the 2 factor. And then I differentiate d A by d s would give me 4 s cube plus 4 s plus 0. Now, I go here right, I just substitute these entries 4 and 4 over here as.

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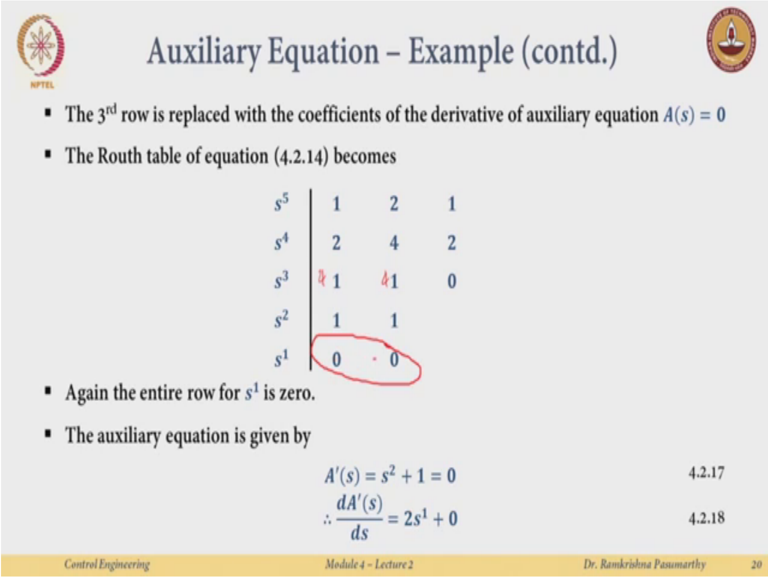

So, this would be 4, 4. And since 4 is common I just get rid of the four and I can just write equivalently 1 and the 1. Now, I compute s square and what I get is again a one and a one and not surprisingly because this is 1 1 this is also 1 1 there is another 0 entry here. Now, I construct further one more auxiliary polynomial. So, I keep on doing at each time, I encounter a row which has entries which are completely 0, and then I take this guy. The second polynomial I call it as A prime and s square plus 1 equal to 0, I differentiate I get 2 and then I get an s here. So, I just do this.

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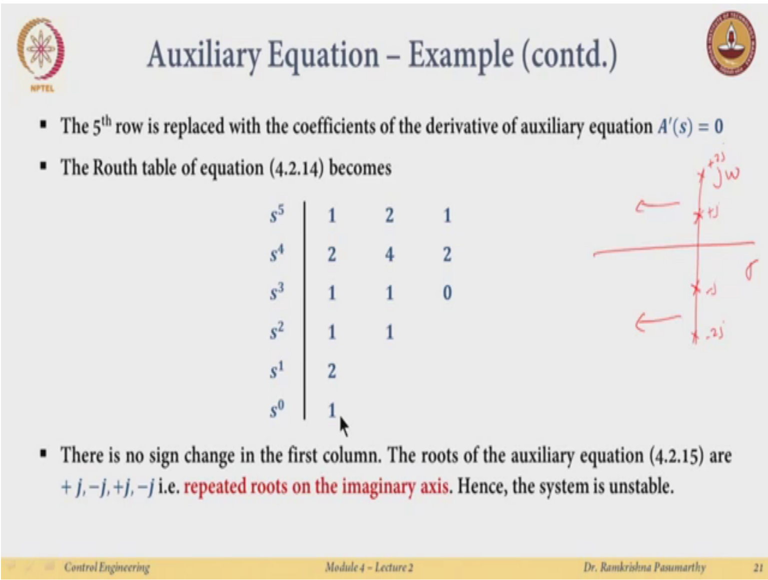

Now, there are no sign changes you can say well this system is stable. Well, from my stability theory we had learnt something that if let us just draw the complex plane here I have a sigma j omega everything here is stable. So, if I have a root where this is plus j minus j, this I called as marginally stable. If I had something like this, say plus 2 j minus 2 j this was also stable marginally even though marginally, but if I had repeated roots like this plus j, plus j, minus j, minus j then there was we derived that this roots actually lead the system to instability now where are this. So, I know that there are now four roots on the imaginary axis, two coming from, because of this row and two more coming because of this row.

So, let me calculate the roots. So, s square minus 1 equal to 0 would give me roots as plus minus j. And I even further here if I calculate the roots that will just give me another plus minus j. So, I have four roots which are plus j, plus j, minus j and minus j and these are repeated roots on the imaginary axis. And we concluded in this stability analysis class that when I have repeated roots on the imaginary axis then the system goes to being unstable. And therefore, we conclude that this system is unstable. So, the special cases which we need to be to be careful of while we are ca computing stability. So, we just need to remember all those cases which we analyzed during the stability class.

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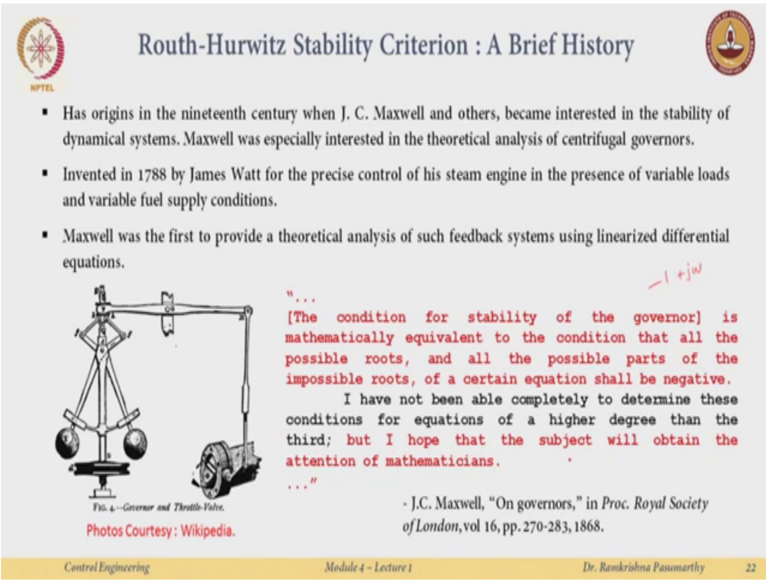

Now, bit of history here. So, where did all this all this start from. So, this had roots the Routh-Hurwitz stability criterion was developed somewhere in the 19th century when

Maxwell now the famous guy of the Maxwell's equations they became interested in stability of dynamical systems. And this Maxwell was motivated by this analysis of centrifugal governors which will essentially invented by James Watt for his steam engine. So, Maxwell was a first to make some initial guesses around this thing he was first to provide a theoretical analysis of feedback systems using some linear system, so linearized differential equations.

So, this is how the governors look and this picture which we got from Wikipedia. So, what did Maxwell say in his paper. So, this is from his paper in 1868 called on governors which appeared proceedings of the Royal Society you can still download it. So, what Maxwell said that the condition for stability of the governor. So, this guy here is mathematically equivalent to condition that all the possible roots thus did not defined this because they did not have a notion of right half plane left half plane you know stable roots unstable roots marginally stable roots and so on. Thus all the possible roots so instead that the stability was equivalent to condition that all the possible roots and all the possible parts of the impossible roots of a certain equation shall be negative.

So, this little miscellany because they did not have any formal terminology to define it, so essentially he meant that all roots must be negative. So, possible roots were all the real roots they shall be negative and then you could have impossible roots which were imaginary and you say the possible parts of the imaginary roots were the real parts. And this possible parts of the imaginary roots, so of the impossible roots say for example, minus 1 plus j omega was regarded as an impossible root, but this was the possible part all they should be negative. And then Maxwell could do it for a third order equation. So, he said that I have not been able to completely determine these conditions for equations of some higher degree than 3 and then he posed this problem to set of mathematicians.

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And then, this analysis by Maxwell lead to the subject for the Adams prize in 1877 with Maxwell was being as one of the examiners. And then the problem which was thrown at mathematicians was to find s general criterion for dynamic stability of systems right and then later on Edward Routh in his essay called A treatise on the stability of a given state of motion presented this condition which we which we learned just now from the Routh table.

So, parallely and not surprisingly from Russia there were also people working in similar areas and then the results again of a steam engine with governor these were used by a some guys to decide water turbine governors. And then Stodola which his collaborator at ETH, in Switzerland independently also arrived at the stability criterion and these two have actually evolved independently right using some different methods. And to honor this parallely evolved result. So, the stability criterion is now what we know as the Routh-Hurwitz stability criterion. And this is little interesting story of how this stability criterion evolved.

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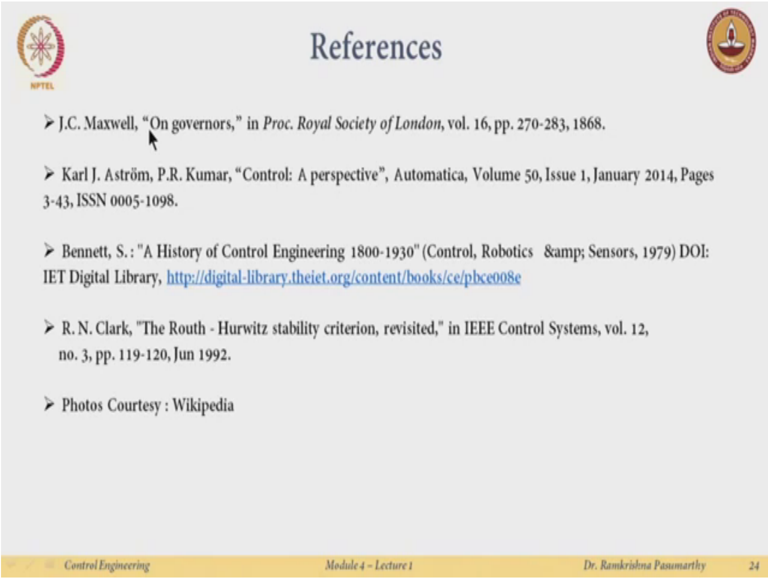

So, what we and these are the papers which you could which would could actually referred to see the history of all these all these analysis of stability of differential equations. So, what we learnt so far is the Routh-Hurwitz criterion some special cases of what happens when there is a 0 in one entry, what happens if there is a 0 in the entire row then did the analysis of the auxiliary equation.

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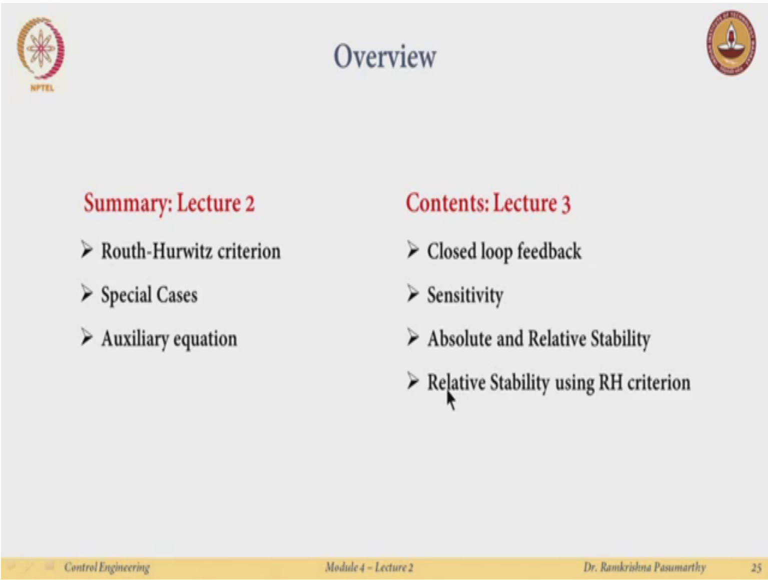

So, next week we will look at more, so at the moment we just looked this analysis only in terms of roots of a certain polynomial. So, next time we will do give it a little more control flavor analyze this in terms of feedback. Then we will do things related to sensitivity. We will also slightly introduce the concept of relative stability, what relative stability means. And we will see if we could solve some of that relative stability criterion or relative stability problems using Routh-Hurwitz criterion.

Thank you.