Control Engineering

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Module – 04
Lecture – 01
Stability

Hello everybody. So, far in the lectures or in the previous module what we saw was a typical response of a first order system to various test inputs. We also justified why we had those typical test inputs like an impulse step ramp and so on. We also saw the notion of steady state error while we are just introducing response of first order systems. We even looked at response of second order systems and we saw there are lots of other things to take care of while designing a second order system like the peak overshoot like the rise time and then you had the notions of setting time and so on. And these were inherently related to the systems damping and also the natural frequency.

Later on we characterized steady state errors of the systems, and we saw how addition of an integrator or a pole at the origin helps in achieving a good steady state behavior right. Only in terms of error it not necessarily with reducing the settling time. And in all these analysis what we had assumed was at the system was inherently stable. And for that analysis purpose we had just defined stability as a property where if the system is subject to some initial condition, it will regain it is equilibrium position or it is original configuration.

So, what we will do in this lecture is to first formulize the notion of stability of a system. Because before we even try to go for an analysis of steady state or even the peak overshoot, what is important before any design is carried out on the system is at the system should be stable.

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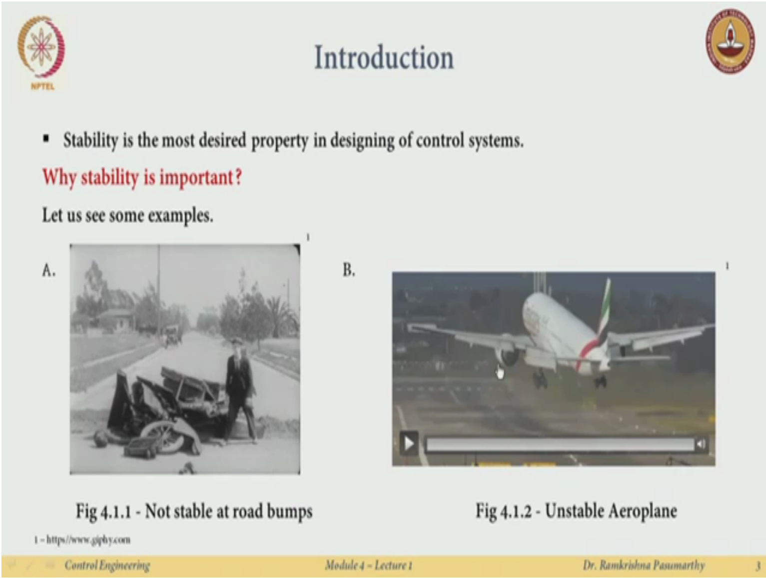

Now, what is this stability? So, so this is a very important property before we design a control system right. So, let us see a couple of videos to just get a little idea of what this stability business is all about and of course; so videos from these websites. So, there is a guy driving and oops. So, so nothing is left for him right and you have this this aircraft which. So, by the way this is not an animation video this is something which could happen in real time let us say you have big turbulence and then your aircraft behave in a very abnormal way. So, these are not just the 2 definitions of these are not just the 2 pictures which define instability.

So, how is this characterized by the system? So, for by the system we had so far said that entire course would be on linear time in variant systems. Typically with single input and single output and what we had as a information of the system was essentially the transform system. Now we will see how the transfer function gives us some information on the stability of the system.

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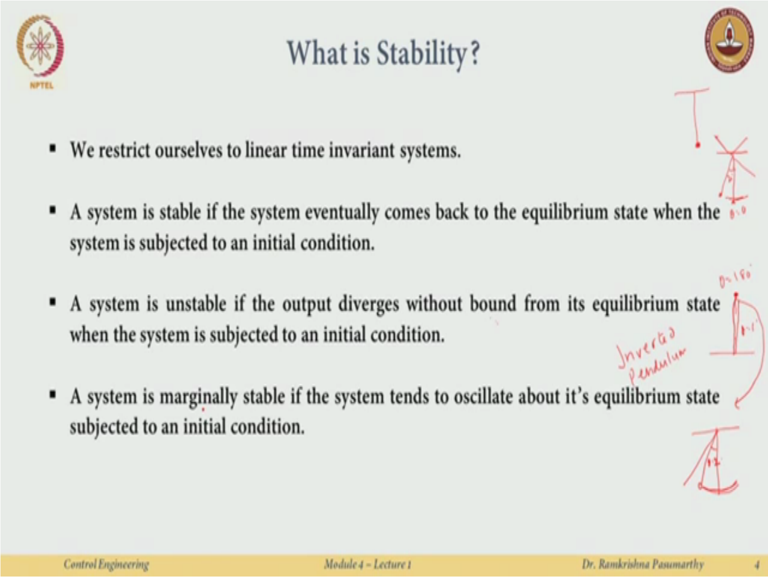

So, the first definition would be a system is stable, if the system comes back to it is equilibrium state, sometimes even called the 0 state when the system is subject to an initial condition. And I can the simple example which we saw. So, far is the simple pendulum with the certain damping. So, if the pendulum is at the equilibrium position it will always remain at equilibrium position.

However, if I start at some position say which is somewhere here say, theta equal to some 20 degrees or whatever then I just. So, this is my initial theta, I just leave it will just oscillate and then come back to it is position which is theta is equal 0 right. So, this stable behavior when subject to an initial condition of theta equal to 20, the system regains it is equilibrium configuration same thing happens, when theta is essentially somewhere here or here or here and so on.

Now, what is the meaning of an unstable or what is the definition of unstable. The system is unstable if the output divergence without bound from it is equilibrium state when the system is subject to an initial condition. So, let us do the reverse of the pendulum, right see here. So, I have this guy. So, this is this is my pendulum not a very good drawing, but as long as it is in the upward position it will always remain in the upward position and then physics tells, once you are in the equilibrium you are always in the equilibrium position.

Now, as a I do a very small perturbation or subjected to initial conditions say theta equal to one degree for example, right then what would you expect is that the system does not go back to it is original configuration, is just open of system there is original control nothing. So, what it does it is goes down right. And then this is energy increases quite a bit right. So, most systems it does not really go out of bound because of this physical limitation here. Right, but mathematically we will say where the system is kind of going out of bound.

So, this system with what we call as the inverted pendulum system is unstable in a way there. So, if this is starting with this notion theta equal to 180 degrees this is also an equilibrium position, but this is unstable behavior. And the system is marginally stable if it is tends to oscillate about it is equilibrium state, when subject to initial condition. Or again consider this simple pendulum without damping. Now say theta equal to 20 degrees right. So, what will be accept it will go to say minus 20 here and then come back again this keep on oscillating like forever.

So, here the output or the system does not really go out of bound. It just remains within this nice bound of theta equal to 20. If a sided theta equal to 40 it will just keep oscillating between plus 40 and minus 40 and so on. So, these are systems which are called marginally stable.

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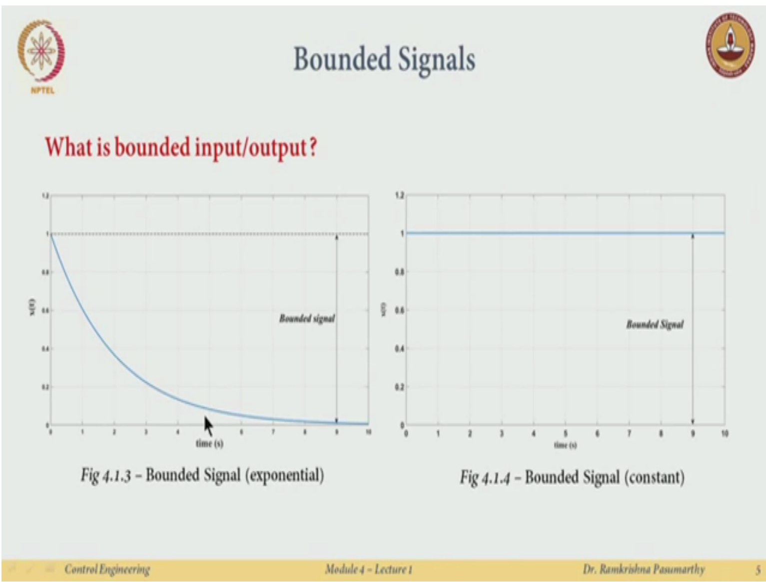

So, stability also depends on the nature of input and how much system behaves to that certain input. So, what we in a input output notion stability is defined also as small changes in input should not lead to very large changes in output. Small changes in input could lead to small changes in output that is acceptable.

For example, here if I small change in initial condition could be well I am going from 20 to 40, but then my output does not really good go like you know the behavior does not go like see 400 degrees or. So, something even here if I start from initial condition of 22 well the output does not change really much. It will very much within the desired behavior right. So, how did this bounded signal look like well. So, this is one the exponentially decaying signal a is an example of a bounded signal. So, this kind of bounded in values see have. So, have a constant input this guy is also bounded a sinusoid is also a bounded signal.

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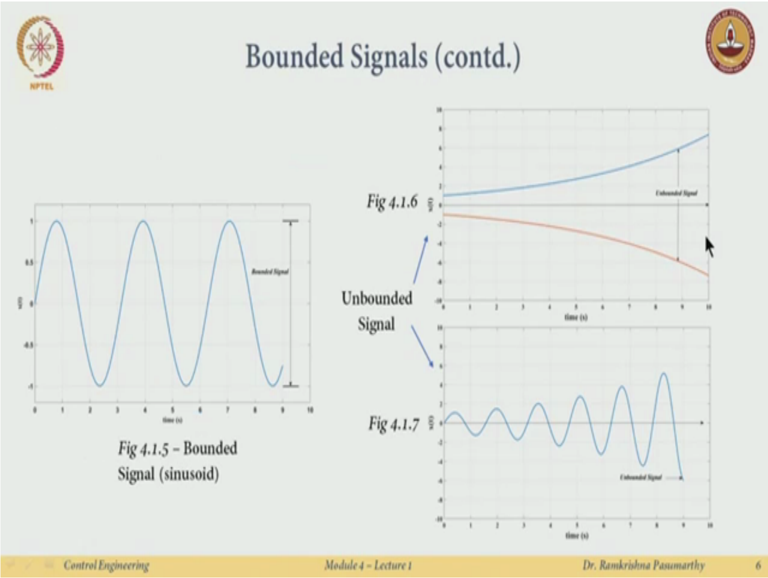

So, it does not it does not really go up and then unwanted signals are one which is increased very large with time. Let us say these guys just keep on increasing similarly this oscillation said this is the amplitude just goes on increasing with time and these are essentially unbounded signals. So, so they keep on attaining very large values for large values of time.

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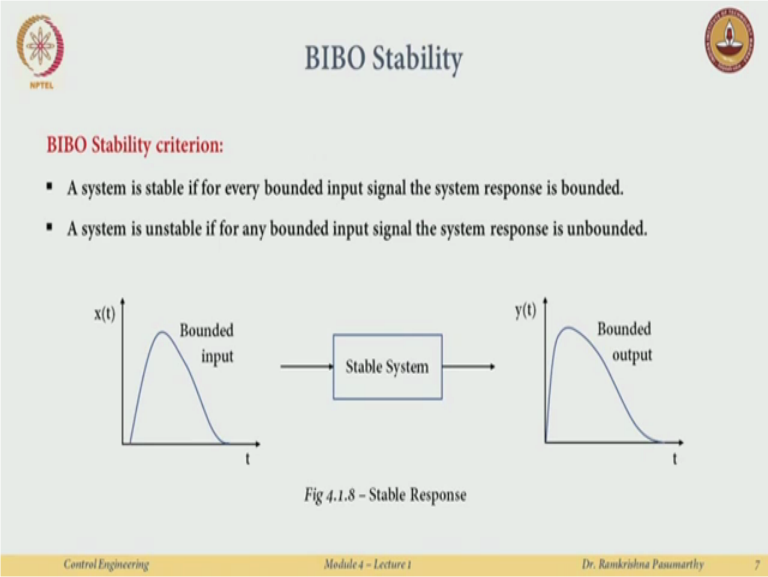

So, in the case of input output what we call as bounded input bounded output stability or in short BIBO stability, a system is stable if for every bounded input signal the response should be bounded right say for example, like have a bounded input here the response. So, just be bounded.

The response goes like something like this then it unbounded. So, how will we define unstable systems in the input output settings? So, if I find any one signal which is bounded for which the output goes unstable, this or output goes unbounded the system is unstable. There could be say 25 inputs for which the system behaves like this, but for the 26th bounded input if it goes if the output goes unbounded that is the system I would still call unstable. So, system is unstable if for any bounded input signal, the response goes unbounded.

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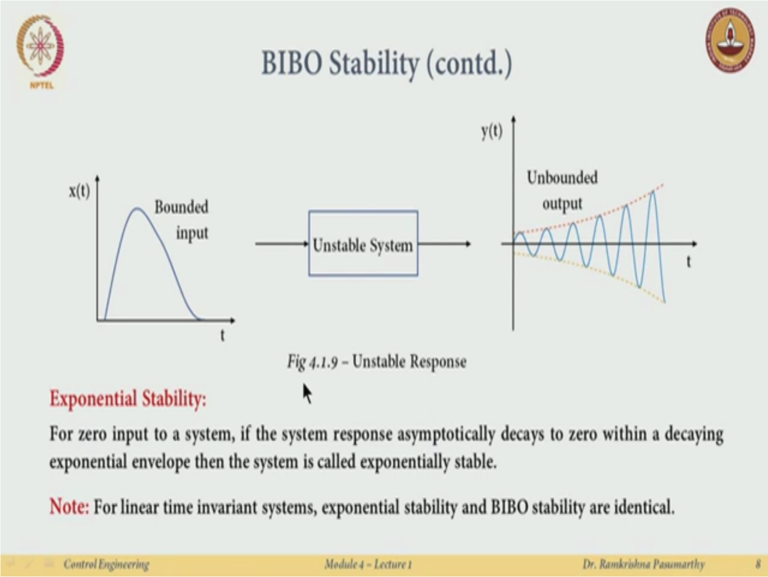

So, this is how it looks. So, I have bounded input I pass it through an unstable system and I get a response which is unbounded.

So, there is also a notion of exponential stability in linear systems. So, what does it mean that in the absence of input or if the input is 0, if the system response asymptotically decays to 0. Again within an envelope then the system is called exponentially stable. For example, are response of say e power minus t is an exponential stable system. So, we will not go into details of this, but just to make a side note that if I take a linear invariant system, and if I say that the system is exponentially stable, and then it is also BIBO stable.

So, just the definitions of stability what are we how do we how did we define stability first is that the system, if it is subject to a non 0 initial condition will eventually go back to it is original configuration or the equilibrium position. The second notion in terms of input and output was if the system is subject to small changes in input it will result in small changes in output. Or more formally if the input is bounded the output should be bounded and if the input is bounded output goes bounded I call it an unstable system.

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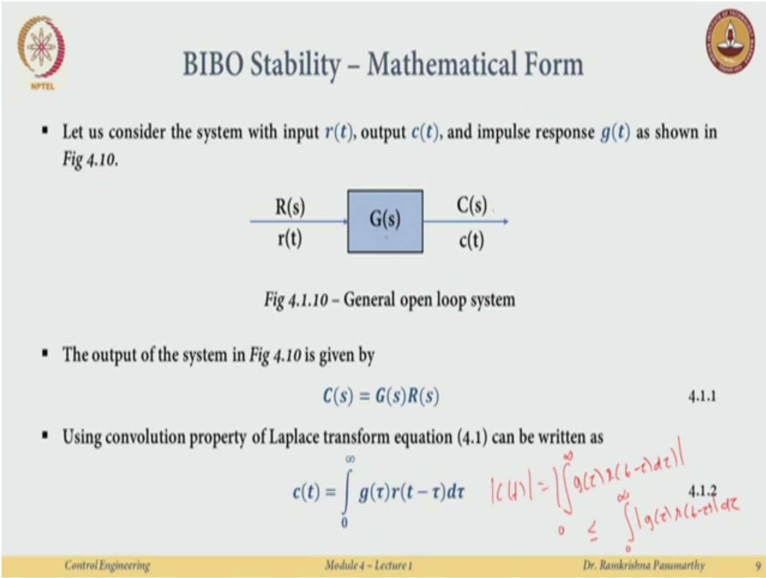

So, let us see what that means in terms of what we have learnt so far in terms of the transfer function or in terms of the block diagrams. So, let us take the standard you know I have a plant system here, I was certain input or a certain reference generating certain output C of s right and then the output we know in the Laplace domain is given by C of s is G of s times R of s. Or in the time domain we use the convolution property that C of t is integral G tau r t minus tau d tau.

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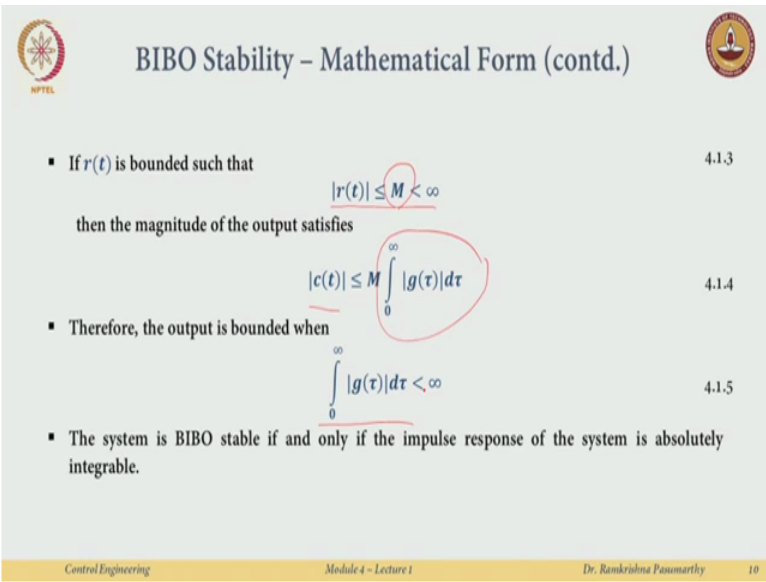

What do I know here is, that the input or the reference signal is always bounded right? So, this is kind of necessary right. So, the input is bounded and the magnitude of the output. So, so I could just say I just say. So, this c of t for just look at the absolute value of it C of t is integral 0 to infinity g tau r t minus tau d tau and let us just take the absolute value to the inside I get this one 0 to infinity g tau r t minus tau d tau.

Now, this r I know is bounded and let me assume that it is bounded by some number m and this absolute value of this output c of t then takes this form. Now I ask the question r of t is bounded when is c of t bounded will, the answer has got to do with this guy. Now because I already know that r is bounded by m. So the answer of when does a bounded input result in a bounded output a lies in here in this expression here. So, the output is bounded when this guy is also infinite or when integral 0 to infinity the absolute value of g is less than infinity or it is boundary.

So, and this guy is essentially the impulse response of plant. So, the system is BIBO stable if and only if the impulse response is absolutely integrable that the limit of this integral exists or it is a finite value otherwise it goes unstable.

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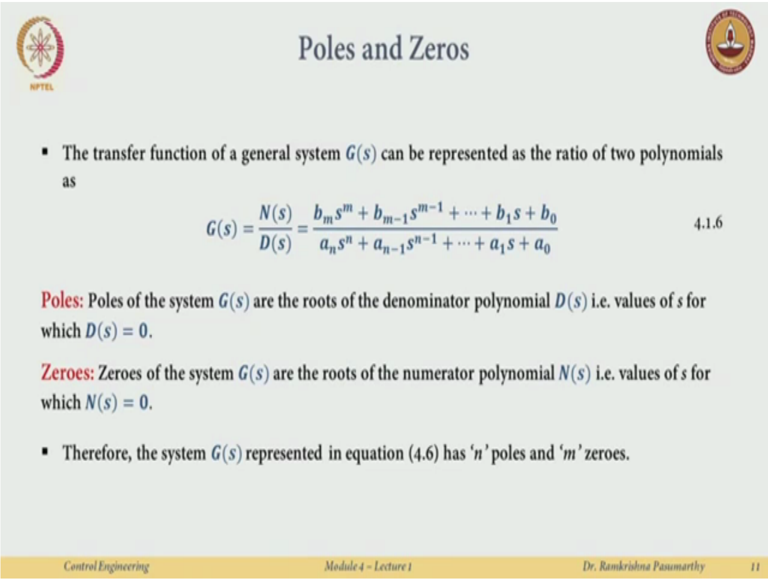

Now, in terms of the transfer function we know that the transfer function is represented as a ratio of 2 polynomials in s, you can also write this in terms of the poles 0 configuration right. So, how did we define poles of the system are the roots of the denominator polynomial, for which no Ds I equate Ds equal to 0 and find out what are the roots.

Similarly, for the zeroes, I equate n s equal to 0 and find out over the values of h for which this equation holds true right. So, this system here has n poles and m zeroes. And of course, the number of poles will always be greater than or equal to the number of zeroes is what we will follow throughout.

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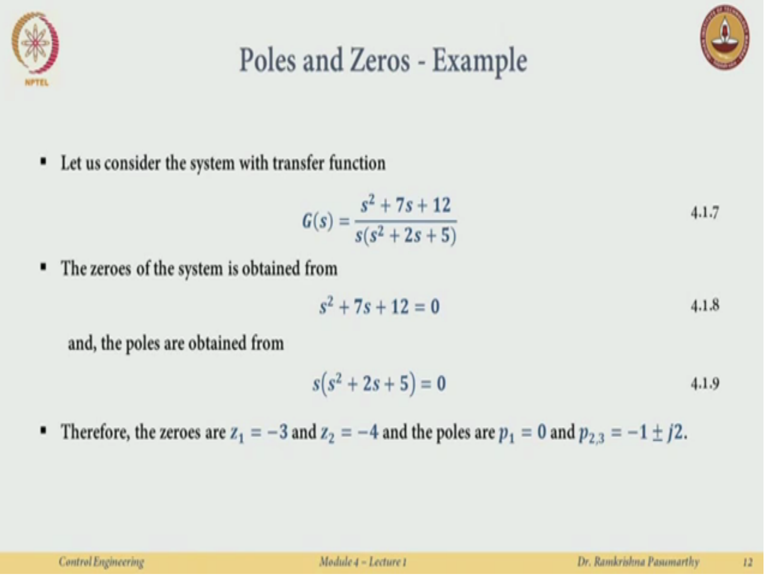

Let us see with the help of an example, I have the system which is like this. So, I just look at the numerator equated to 0 and I just get 2 zeroes z equal to minus 3 and z equal to minus 4. And there are 3 poles one at the origin and I have complex conjugate pair minus 1 plus minus j 2.

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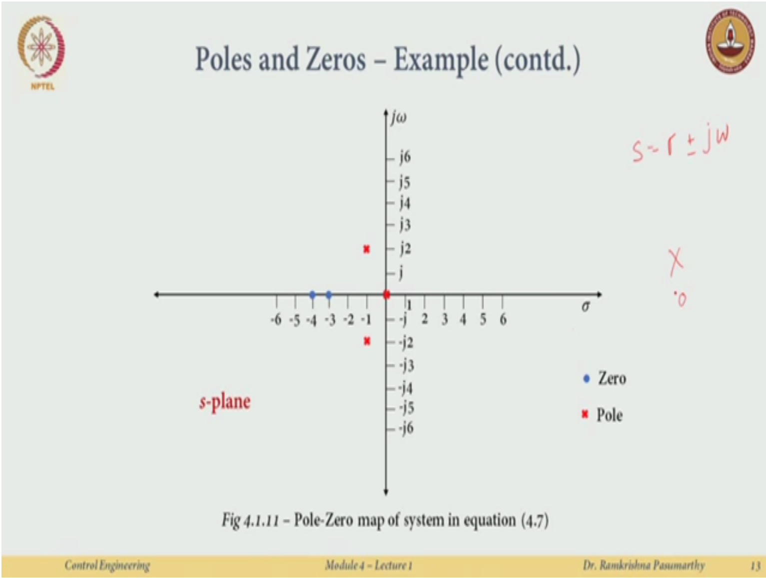

So, if I look at go to the s plane, with sigma s is typically like this kind sigma plus minus j omega right. So, if it has the real values and the imaginary values. So, the pole at the origin would be. So, this guy entirely was 0. So, I am here. So, pole is usually denoted by a cross. So, I have a pole here complex conjugate pores minus 1 plus j 2 minus 1 minus j 2 and 2 zeroes one at minus 3 1 as minus 4 and zeros are this is typically. So, this is the notion for 0.

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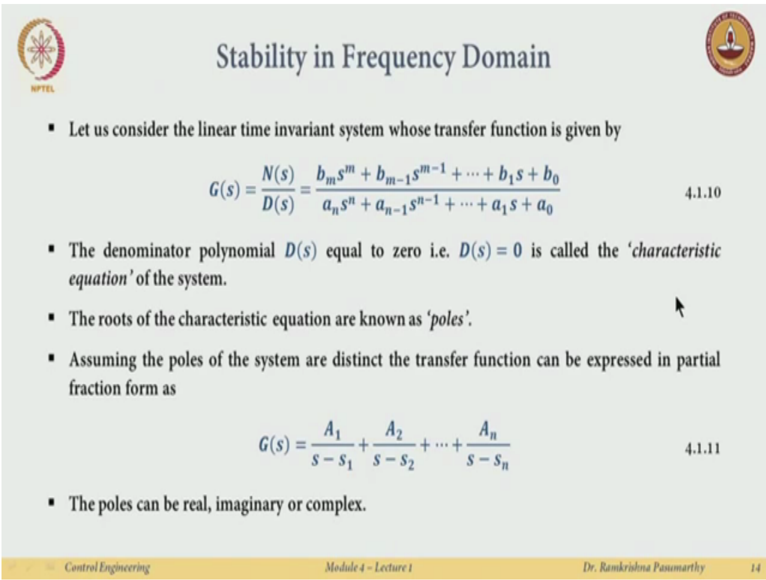

Now, we will see what is what to do with stability. What has this locations here 1 2 3 4 5 guys do they give us any information on stability right. So, that is what we will try to investigate; so again the roots. So, the denominator polynomial Ds going equal to 0 is called the characteristic equation of the system.

So, will do a little bit more of this later, but at the moment you can just assume that I am just interested in the solutions of the denominator equation Ds going to 0. And then the roots are called as the poles and if I assume for simplicity that the poles of the system are distinct. I can rewrite this from my laplacian theory as like this side, so s minus s 1 s minus s 2 and so on. And these poles can be real imaginary or complex just that they are not repeated we will come to the repeated cases little later.

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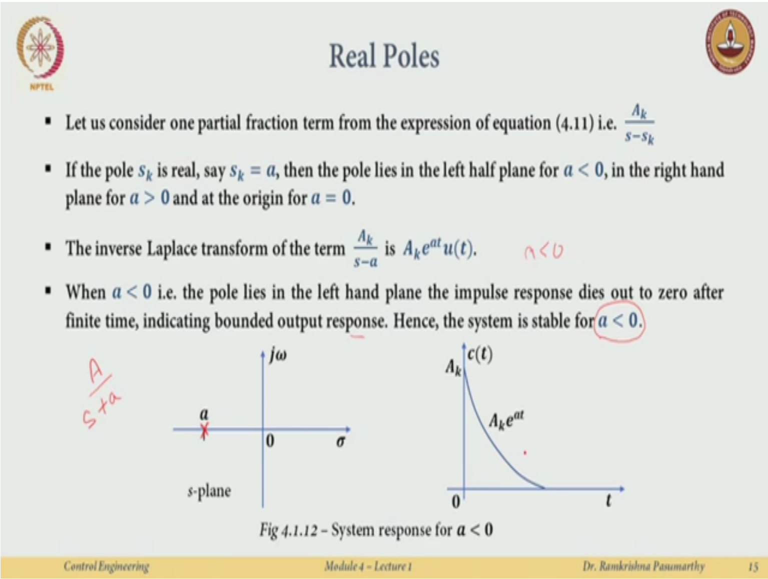

First: let us take some partial fraction expansion of like this. And say well I am just interested in this guy say the pole is real, say s k equal to a then the pole lies in the left half plane for a equal to 0 right. So, it will be somewhere here, here. And this negative line now the inverse Laplace of this guy we know is a power k e power a t u t. And therefore, when a is less than 0, the pole lies somewhere here and denoted by a cross. And then the impulse response I know is something like this right it just lies down to 0, after well as if sympotically it goes 0 and this actually a bounded output response. And hence the system I would call is stable for a less than 0.

So, my first observation I could make is if I have a transfer function of the form a over s plus a, and a lies in the left half plane. Then my impulse response of the system is bounded and therefore, by the first definition I could say that the system is stable.

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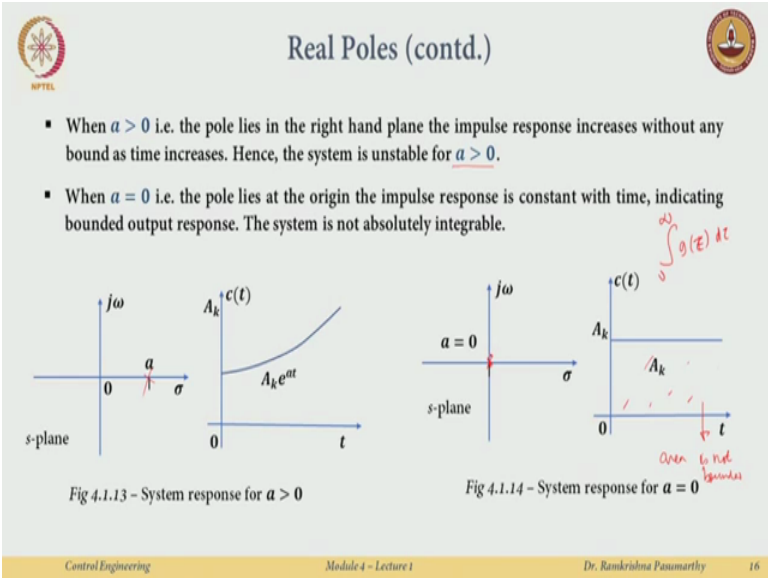

Now, similarly when a is greater than 0, when a is greater than 0 my pole is sitting over here. And the impulse response is some constant a e power a t, a is greater than 0 therefore, I would expect this guy to keep on increasing right. So, this is on bounded response. If t goes very large this guy assumes very large values. And therefore, I can say that the system if the pole is sitting on the right side then my system is unstable right second observation.

Now, third observation what are what if my pole is just sitting here. When a equal to 0 the pole lies at the origin. So, this guy put a cross here. And the impulse response is constant with time right. So, this the output does not increase with time, but; however, if I just integrate for this case g of t from 0 to infinity say g of tau d tau then I see that the area. So, g this is essentially the area, that this area is not bounded right the area is not bounded, but my signal does not really increased like it really increased this here right. So, this system the simple integrator or with the pole at the origin is not absolutely integrable and we can just call this as some kind of a marginally stable system.

So, now we saw 3 things what about pole on the left, a pole on the right, and at the origin and the nothing to do with zeroes so far.

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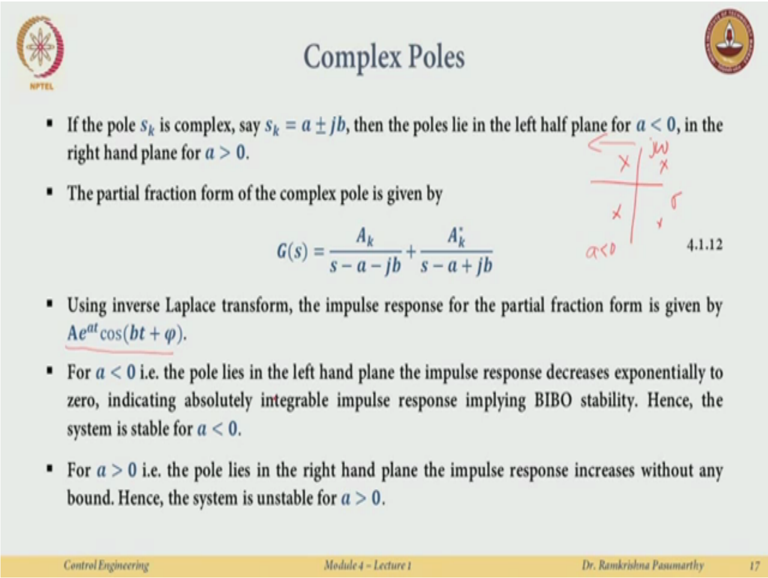

What happens if my pole is now complex conjugate? Well and say there could be 2 possibilities that the poles could. So, this is my sigma, this is the j omega the poles could either be here or be here. So, if a is less than 0 then these are my poles right this for a less than 0. And the poles are on the left half plane not just on the on the left of real line, but on the left half plane this entire thing. Now if I just do all these partial fraction thing then I get that the response of the system is given by e power a t cos b t plus 5.

Now, this is the general expression no matter what the sign of a is. Now for a less than 0 the pole lies the poles lie here. And the impulse response since this is e power minus a t is decaying. It decays exponentially to 0 indicating there is also absolutely integrable.

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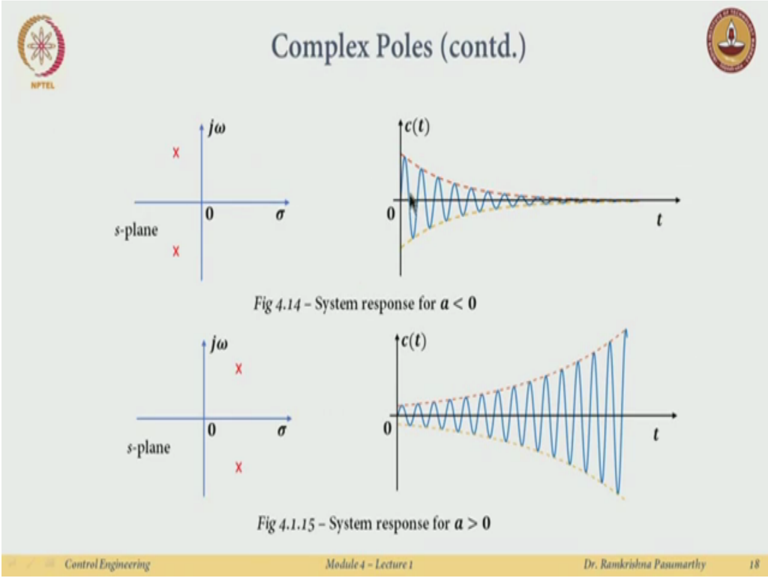

So, it will be something like this right this is integrable this is see the area is finite. The response is unbounded. And therefore, this system is BIBO stable. And of course, it is also stable by both the definitions for a less than 0, for a greater than 0 the poles are somewhere here on the right half plane and the impulse response increases without any bound.

So, it just does this. So, you see that the oscillations keep on increasing in amplitude right and this is an unbounded behavior.

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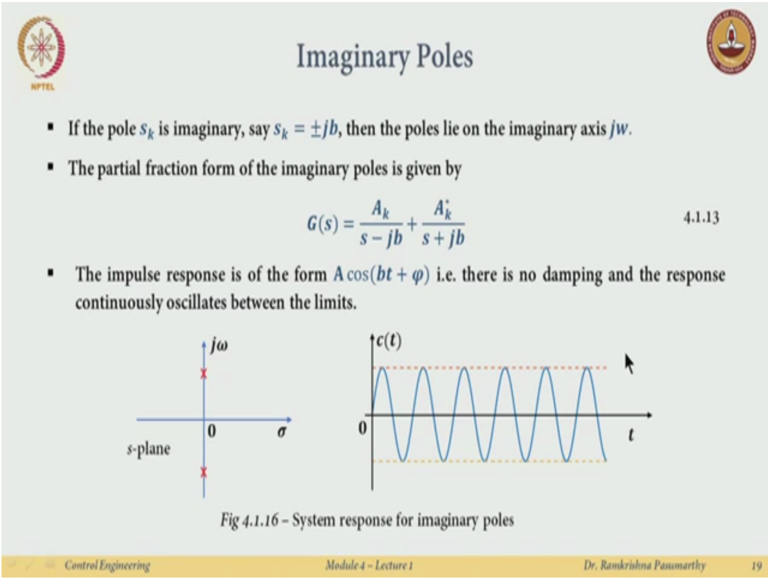

So, what if a equal to 0 again right if the pole s k is imaginary and say I just have here, so 1 plus j omega and minus j omega. Then the response is simply of the form a cos b t plus phi. So, the response I know just the just the response of a sinusoid, it is just be like this right it just have sustained oscillations, and therefore I call this system to also be marginally stable right. It neither increases unbounded in value nor do the oscillations died down to 0 as in the previous 2 cases. So, this response or this oscillatory behavior is what I would call as marginally stable systems.

So, far what we have seen is if there is a distinct pole like s equal to a, if a is less than 0 it is stable a is greater than 0 it is unstable, a being equal to 0 has some kind of a tricky response it output grow unbounded, but the system is not actually integrable, we could call this as a marginally stable system. If I have complex poles I am just interested in what does the real part do. If the real part is less than 0 then the system is stable, real part is greater than 0 we see unbounded response, and if the real part is 0 which means my poles are just on the imaginary access then I just see an oscillatory response.

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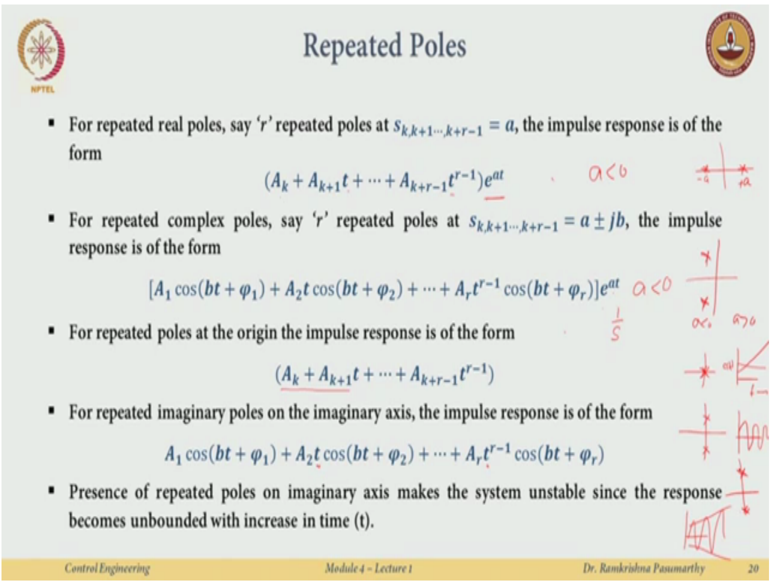

Let see what if now we have repeated routes, say I have say couple of roots at a at if I just to draw it here and this is my minus a or plus a right here. So, my response would be some constant a, plus some other constant times t e power a t and then again this is depends on what is the value of a. If a is less than 0 then the impulse response is stable and if a is greater than 0 then the impulse response goes unstable and that could computed quite easily. Similarly if there are repeated complex poles say there is a couple of guys sitting here and couple of guys at the same location here, then again depending on the sign of a my system is unstable or unstable. So, this region is again the stable region when a is less than 0 and a greater than 0 is always unstable.

So, the same thing holds right if the poles or the real part of poles are less than 0, I am still strictly less than 0 I am still stable. And if they are strictly greater than 0 then my system is unstable. Now earlier we saw something about the route at the origin something like one over s or when the pole is at the origin then I saw that my response was just some straight line right. It was bounded in value, but not integrable what if I have two of this. If I have 2 of this my response would look something like this right. It will just keep on increasing with time and thus because of this kind of thing it Ak plus Ak plus 1 some other constant time t, and if I just plotted with time this is my response it just goes unbounded with time.

Now, let us take these guys for their upward side, well I had a j omega, I had a minus j omega and I saw earlier that this response is just oscillatory. Now tricky things will happen when again I have multiple things here, and multiple things here. Then my response is A 1 this is response when there is only one pair of poles which are imaginary. And if a second guy comes and then I have this guy A 2 times t cos b t plus 2 and then t square and forms. So, these responses will actually keep on increasing like this this way and this is also an unbounded response.

So, these 2 guys as long as A is less than 0 a is less than 0 has the stable behavior whereas, these 2 guys has a strange kind of behavior right. If there is one pole at the origin then I am in some kind of a marginally stable behavior. If there are 2 poles at the origin, then I go just unbounded, similarly for 3 4 and so on. Also here if there is one pair of complex pole on the imaginary axis, then I have a some kind of a marginally stable or an oscillatory behavior. Whereas, this guy makes it unstable if there is more than one pair of poles on the imaginary axis.

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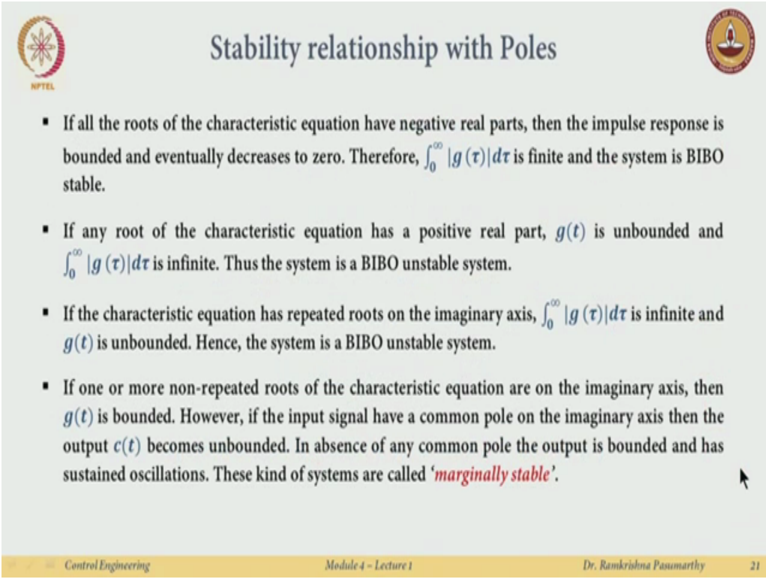

So, just to summarize, what is the relationship of the stability with the poles of the transfer function. If all roots of the characteristic equation have negative strictly negative real parts, then their impulse response is bounded and eventually decreases to 0. This also means that this integral which I was trying to compute is finite and therefore, the system is BIBO stable. If any root of the characteristic equation has a positive real part it could just be one it could be 500 roots of the system even if one is on the right of plane, then my system is unstable as g t is unbounded and this guy this integral is not defined.

Thus, the system will become an unstable system lastly if like if the characteristic equation as repeated roots on the imaginary axis right. So, again the same thing it could it results in a unstable system. So, lastly if one or more non-repeated roots of the characteristic equation or on the imaginary axis than g t is bounded, but the integral would not be bounded, so there is a we saw here this this integral here is unbounded this is one root at.

However, if the input signal have a common pole on the imaginary axis, then c t becomes unbounded or the output becomes unbounded. And there are also cases where in the absence of any common pole is output is bounded and has sustained oscillations like these 2 cases, here I have sustained oscillations or here is just this is a straight line and this kind of systems are typically referred to as marginally stable system.

So, we have categorized stable unstable and marginally stable systems. So, on the left of plane it is pretty clear on the right of plane it is pretty clear. On the imaginary axis if there is just one pair of complex pole then there is some kind of marginal stability, but if there are repeated roots, then I go to an unstable level.

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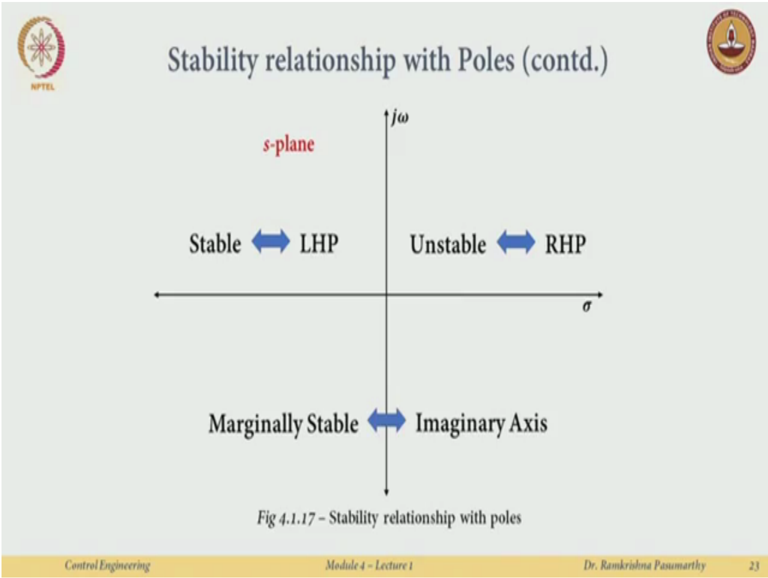

To just pictorically say what I have said. So, this is stable region left of plane unstable and this is imaginary thing. So, we are it could just be marginally stable again depending on how many roots or how many repeated roots are there.

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So, what are we learnt. So, far is to define the concept of stability first from a system response to a non 0 initial condition. We also talk define bounded signals and studied or defined what is called the bounded input bounded output stability, in which case we said if this input is bounded and the output is bounded then the system is stable, or if there is any one bounded input which results in an unbounded output the system is unstable.

However, what we need to be careful is what if the input is unbounded and the output is also unbounded, can I say something about the stability well the answer is not is kind of no. Because if I am tracking a ram as we saw in some of our previous examples when we are doing a steady state error my input is unbounded right, it just t which is increases with time and the output is also expected to increase your time because I am tracking this signal back, this is an unbounded input and an unbounded output.

So, there I just cannot categorize saying that input is unbounded output is unbounded and therefore, the system unstable no we have to be little more careful with that. So, what we did next was to relate stability to poles and zeroes, but typically to the poles and we said depending on the location of the poles, I can judge or I can say something about the stability or not of the system.

Now, next we will see is if I have given polynomial of say some order 10 or 15, how do I compute it is roots, or how do I at least say where the roots lie. I do not at some point of time of need to explicitly compute where the roots are as long as I know that they are somewhere in the left half I am happy. Or they say there is one guy in the right of pane I do not really know where he is need to know where he is sitting right even if it is at plus 1 or plus 100 it is still unstable.

So, we will just try to learn some tricks call the Routh Hurwitz criterion which will tell us about stability of a given transfer function.

Thank you.