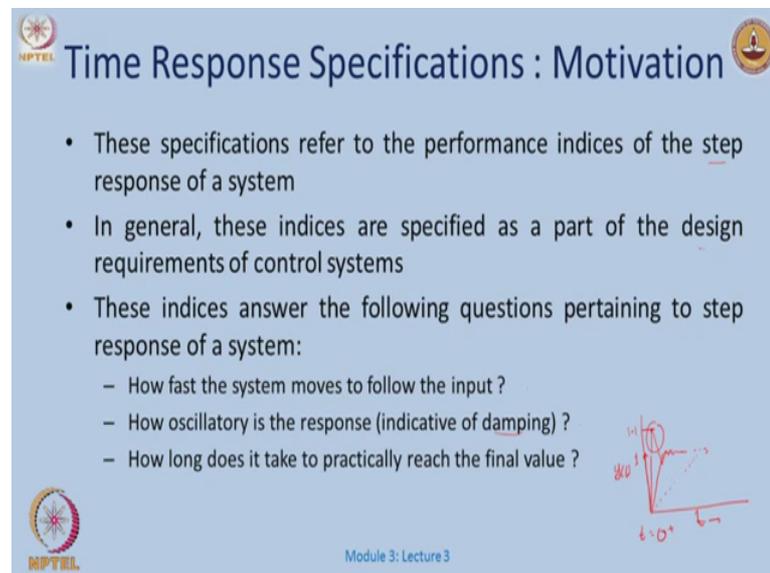


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**Module – 03**  
**Lecture – 03**  
**Time Response Specifications**

So what we will look at today is something called the time response specifications. So, last time what we had done was given a first order or a second order system.

(Refer Slide Time: 00:32)



The slide is titled "Time Response Specifications : Motivation" and features the NPTEL logo in the top left and bottom left corners, and a small circular icon in the top right corner. The main content consists of a bulleted list:

- These specifications refer to the performance indices of the step response of a system
- In general, these indices are specified as a part of the design requirements of control systems
- These indices answer the following questions pertaining to step response of a system:
  - How fast the system moves to follow the input ?
  - How oscillatory is the response (indicative of damping) ?
  - How long does it take to practically reach the final value ?

On the right side of the slide, there is a hand-drawn graph in red ink. The graph shows a step response curve starting from the origin (0,0) and rising towards a horizontal dashed line representing the final value. The curve exhibits oscillations, with a peak and a subsequent trough. A vertical line is drawn from the origin to the horizontal asymptote, and a horizontal line is drawn from the peak of the first oscillation to the vertical line, forming a right-angled triangle. The angle of this triangle is labeled with the Greek letter  $\zeta$  (damping ratio). The horizontal axis is labeled  $t$  and the vertical axis is labeled  $y(t)$ .

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We just saw how it responds to different kinds of signals and in the first order system, we saw there was something called a steady state error we will formalise that in a better way today. Second order systems we saw how the response of the varies with changing in the damping term where, where we saw for a under damped system your response is just continuous oscillations still you have critically damped systems to over damped systems and so on.

So, what we will see now is given some design specifications, what is a impact of the time response. So, loosely speaking if I say that I would want to go from say you know some signals say  $Y$  of  $t$  I want it to take some value of say  $Y$  equal to 1, somewhere here. So, ideally what if I would want to design a system I would want this guy to go instantaneously to this point I just say  $t$  equal to 0 plus, that I switch on my air

conditioner and immediately my temperature goes to the desired 24 degrees or 20 one degrees.

Sometimes might be difficult to do it instantaneously, but it could be it could be possible to do it a little you know something like this right or else something like this right. So, not exactly like this, but we could you know do you know in this time or this time, but what I would expect if I you know want to run from this point to this point. And do very fast then I just cannot stop here right I have to go little further right before then I can I can come back and I run a little faster then I go here and so on.

However, if I would not want these things to happen I would have to run a little slowly. So, that I just go here and stop right. So, these are the kind of behaviour we will analyse is to do with how fast can my system respond or how fast can it reach the given the desired value, but if it runs very fast are there any over shoots and if there is any over shoot is there a limit on the over shoot or is can I really say how say what is the value of Y at this point given this guy to be 1. Or if you know when does it really reach the desired value right and so on that is what we will try to you know quantify in this lecture. So, and we will restrict ourselves to the step response of the system and their analysis becomes much easier and it is also you know quite intuitive too.

So, in general all these specifications of how fast I go how fast I settle down are a part of the design specifications, if someone gives me a problem would say design me something which reaches the steady state value in so and so time that it should not over shoot beyond a certain limit is and this could be just point one and I would say I would not want to over shoot 1.1 right, for example, and this indices answer the following question. So, the based on this are humans what are we interested in is how fast this system moves to a give input is the response oscillatory and how oscillatory is the response which is again we know that it is depending on the damping that under damped systems will have an oscillatory response and how fast does it reach the final value.

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**Time Response Specifications**

- **Delay time  $t_d$ :**
  - Time required for the response to reach 50% of the final value at first instance
$$t_d = \frac{1 + 0.7\zeta}{\omega_n}$$
- **Rise time  $t_r$ :**
  - Time required for the response to rise from 10% to 90% of the final value for overdamped systems and 0 to 100% of the final value for underdamped systems, at first instance
- **Peak time  $t_p$ :**
  - Time required for the response to reach the peak value of time response

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So, here we will restrict ourselves to under damped systems right. And we will focus on the step response. So, the unit step response. So, to quantify this we will define for our self few things right, first is the delay time the time required for the response to reach 50 percent of the final value at the first instance. And I could calculate this based on the formula which I had derived earlier, I will come to this we just first define what these are and then go to see how this numbers are achieved.

So, the delay time is a time required for the response to reach 50 percent of the final value at the first instance you could have that in an oscillatory response or it could reach the 50 percent you know several times. The rise time is usually defined as a response to rise from 10 percent to 90 percent of the final value for over damped systems and since we are here looking at under damped systems we are we will look at what time does it reach the peak the desired value the first time or from 0 to 100 percent of final value at the first instance. For example, if I just were to plot it. So, this till this and I say this is my desired value these things right. So, here well this is what is called as the rise time right. So, to where it reaches the value of one the first time. This is where you get this is value here of one here and so on it keeps on you know oscillating, until it settles down to it is value final value the delay time is this time right. So, this is at  $t_d$  where my response takes the value of 0.5. So, this is my time this is my response  $Y$  of  $t$ .

Similarly, the peak time is the time required for the response to reach the peak value. Somewhere here. So, this is my  $Y$  peak time. Then I will also quantify if I reach a peak if I am going from the desired value to some other value, how much is this over shoot.

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**Time Response Specifications**

- **Peak overshoot  $M_p$ :**
  - It is the normalised difference between the peak value of time response and the steady state value
$$M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\%$$
- **Settling time  $t_s$ :**
  - Time required for the response to reach and stay within a specified tolerance band of its final value or steady state value
  - Usually the tolerance band is 2% or 5%

Note:  $t_p$  and  $M_p$  are not defined for overdamped and critically damped systems

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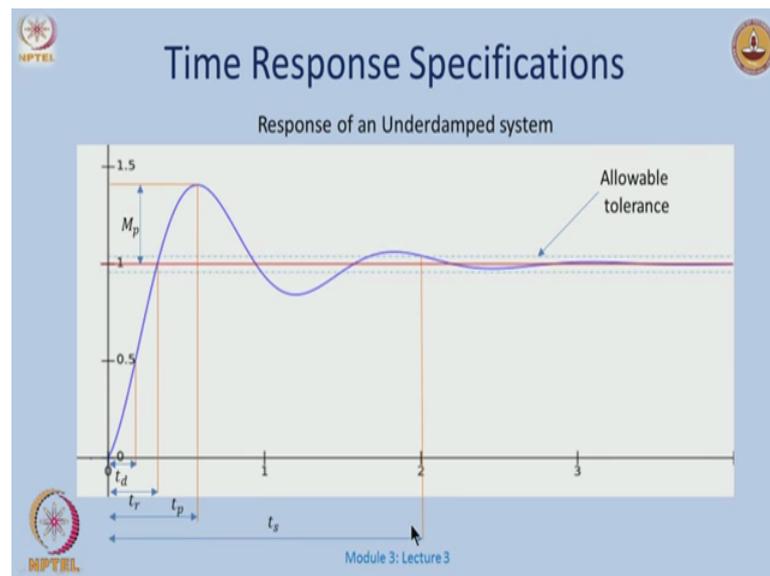
So, this is quantified or called as the peak over shoot and this is just completed as the percentage of over shoot like. So, this how much I deviate from the steady state value at the first instance. So  $Y T_p$  which is what I over shoot here, this one this is the value of  $Y$  at  $T_p$  minus, the steady state value and then you just calculate that it in percentage of deviation from the steady state value.

The settling time. So, we had weekly defined this last time when we were dealing with first order systems, and we will we will recall that all over again the settling time was defined as the time required for the response to reach and stay within a specified tolerance band. So usually I would say well I am I am this system has reached some kind of a steady state value, if it is using 2 percent of it is final value sometimes even 5 percent depending on my tolerance levels. So, if I in this case where the steady state value is one I would say the settling time is when it reaches 2.98.

So, again all these things; so settling time we said we thought or we had derived the settling time concept earlier for first order system right. And this concept the peak time and the peak over shoot are not defined for over damped and critically damped systems. That is because let us start with the response of. So, this is the under damped response

the critically damped response would be something like this it will never cross this value of one, or the desired value this is a critically damped system. So, this is the under damped system this is under damped this is critically damped. So,  $T_p$  and  $M_p$  are typically defined only for under damped systems. And that is what we will focus most of our analysis on.

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So, this what it looks like. So, typical response of a first order system this we derived last time it looks like this is like has some kind of damped to oscillations.

So, again I have the delay time the rise time the peak time and the settling time right. So, the first time when it reaches this guy. So, this could be within 2, 2 percent could be 0.98 or even 1.02, but once with it is here it always remains within that 2 percent value right, it will never go down you know below 0.98, because the oscillations are continuously being damped. So this is these are the values which we will which we will focus on  $T_p$   $t_r$   $M_p$  and or this settling time.

Now, the idea would be can I quantify this based on my system parameters right.

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## Expression for Rise Time

- Consider a 2<sup>nd</sup> order underdamped system
- Rise time  $t_r$  is the time taken by the step response to go from 0 to 100% of the final value i.e., one

$$y(t_r) = 1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) \text{ where } (\theta = \cos^{-1} \zeta)$$

$$\Rightarrow \sin(\omega_d t_r + \theta) = 0 \Rightarrow \omega_d t_r + \theta = \pi$$

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \cos^{-1} \zeta}{\omega_n \sqrt{1 - \zeta^2}}$$

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$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$   
 $\zeta < 1$

$\omega_d = \omega_n \sqrt{1 - \zeta^2}$

So, start with the second order under damped system with transfer function which we are derived which we had said last time was has a typical transfer function if I would could call it  $g$  of  $s$  was  $\omega_n^2$  over  $s^2 + 2\zeta\omega_n s + \omega_n^2$ . And the response of this system and we will do this  $\zeta$  is less than 1 and of course, greater than 0.

So, the rise time, so this system subject to a step input had a solution like this and with we derived this last time at with  $t$  no not replaced by  $t_r$  right. So, this  $t_r$  is the reason we have  $t_r$  is. So, the rise time is a time taken by the step response to go from 0 to 100 percent. So, this is the 100 percent this is the solution of the equation and I want to calculate the time  $t$  at which it reaches this value of 1 and therefore, I replace this  $t$  by  $t_r$  in the in the solution  $t$  that equation.

So, I do this one and one goes away. So, I am I am left with  $\sin \omega_d t$  this is the damped natural frequency if you remember was related to the under damped frequency in this way this was this was  $\omega_d$ . So, when this goes to 0 for the first time. So, which means  $\omega_d t_r + \theta = \pi$  or  $t_r$  is just this  $\frac{\pi - \theta}{\omega_d}$  and what about  $\theta$ . So,  $\theta$  is written here  $\theta = \cos^{-1} \zeta$  from the damping coefficient right and then this relation between  $\omega_d$  and  $\omega_n$ , but damped natural frequency with the natural frequency of the system.

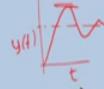
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## Expression for Peak Time



- Peak time  $t_p$  is the time taken by the step response to reach the peak value
- At peak, the time derivative of response is zero



$$\frac{dy}{dt} \Big|_{t_p} = 0 = \frac{\zeta \omega_n e^{-\zeta \omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \theta) - \frac{e^{-\zeta \omega_n t_p}}{\sqrt{1-\zeta^2}} \omega_d \cos(\omega_d t_p + \theta)$$

$$\Rightarrow \zeta \sin(\omega_d t_p + \theta) - \sqrt{1-\zeta^2} \cos(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\omega_d t_p + \theta) \cos \theta - \cos(\omega_d t_p + \theta) \sin \theta = 0$$

$$\Rightarrow \sin(\omega_d t_p) = 0 \Rightarrow \omega_d t_p = 0, \pi, 2\pi, \dots$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

(corresponding to first peak)


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Similarly, I can calculate the peak time is the time taken by the step response to reach it is peak value. Now how do I find what is the time right. So, if I find here I am looking at this value right. So, this is this is a function if I just. So, Y is a function of t. So, it starts here it goes here and again comes down right. So, this is the maximum value that that the function can take. And from calculus I know that the maxima can be computed by taking the derivative of Y. And then setting this to 0, is a limit I take d Y by d t I set it to 0 and I compute the time at which this happens right, so 0 the time over here.

So, I do all these computations. So, take the derivative of this. So, this one disappears, and I have is this one 0 is on this guys. And then I solve this step by step to get this value right omega d time Tp is this one, and therefore, again I am just interested in this first instance right. So, this is one peak and then d Y by d t would go to 0, here d Y by d t could go to 0 and several points right here and so on, but I am interested only the first time and this is the value which I would be interested in right.

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**Expression for Peak Overshoot**

- As per definition, peak overshoot for a step response,

$$M_p = \frac{y(t_p) - 1}{1} = y(t_p) - 1$$

$$y(t_p) - 1 = -\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \theta) = -\frac{e^{-\frac{\zeta\omega_n \pi}{\omega_d}}}{\sqrt{1-\zeta^2}} \sin\left(\frac{\omega_d \pi}{\omega_d} + \theta\right)$$

$$\Rightarrow M_p = \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(\theta)$$

Handwritten notes:  $\theta = \cos^{-1} \zeta$ ,  $\cos \theta = \zeta$ ,  $\theta = \theta$

$$M_p = 100 e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \%$$

(corresponding to first peak)

Handwritten notes:  $\xi = 0$ ,  $M_p = 100\%$ ,  $\xi \uparrow M_p \downarrow$

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So, this  $T_p$  would be simple  $\pi$  over  $\omega_d$ . So, how do I express this overshoot as a percentage right. So, I calculate the peak value and then the steady state value is 1 that is as steady state and here. In this case it will just be  $M_p$  would be  $T_p$  minus 1 right. So, how much I overshoot from here this number from here till here I can just substitute  $Y$  of  $T_p$  here. And I just  $Y$  if  $T_p$  would be just I know the value of  $T_p$  that is  $\pi$  over  $\omega_d$  I plus it into this solution is the solution over here and I just see how much it deviates from the steady state value of 1.

So,  $Y$   $T_p$  substitute for  $T_p$  in the solution of  $Y$  of  $t$  and this with all these computations would be something like this. So,  $M_p$  is  $e$  power blah and then this term gets cancelled because of  $\cos$  of  $\theta$ , if you see here is defined this, this  $\theta$  is  $\cos^{-1}$  of  $\zeta$   $\theta$  is  $\cos^{-1}$   $\zeta$  or  $\cos$  of  $\theta$  is  $\zeta$  and then I just use  $\sin^2 \zeta + \cos^2 \zeta = 1$  and this guy is still disappear. So, the peak overshoot in percentage is  $e$  to the power you know this guy. Some properties of this one right. So, if I say well what are the limiting cases here right. So, we go from say  $\zeta = 0$ , when there is no damping what we would expect is that  $M_p$  would be the highest.

So, if my steady state value is 1, then  $M_p$  here would be substitute for  $\zeta = 0$  and so  $M_p$  would be 1, which means my peak overshoot would go from this is a value is 0 to one this is the desired value it will go all the way till 2 and then come back and this keep

oscillating like this. This is the standard response which we saw it is almost it is like hundred percent over shoot.

Now, if zeta keeps on increasing I see that my over shoot keeps on decreasing right, if I just were to plot the peak over shoot, versus zeta this zeta going from 0 to 1 see that this will go from 1 and all the way till 0 this is very important right. So, one of the designed parameters here could be if I were to restrict my peak over shoot then my problem would be to have to design my system with the appropriate zeta.

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The slide is titled "Expression for Settling Time" and contains the following content:

- Time required for the response to reach and stay within a specified tolerance band (take 2%) of its final value or steady state value
- Equation: 
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$
- Graphs showing the response  $y(t)$  oscillating around a steady state value of 1. Handwritten annotations include:
  - Peak time:  $t_p = 4T(2\%)$
  - Settling time:  $t_s = 3T(5\%)$
  - Envelope curves:  $1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$  (top) and  $1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$  (bottom)
  - Another graph shows the exponential decay  $1 - e^{-t/T}$  with points marked at  $t = 3T \rightarrow 95\%$  and  $t = 4T \rightarrow 98.2\%$ .

So, similarly for settling time the settling time is the time required for the response to reach and stay within say at 2 percent band of it is final value. So, how do we compute this? So, let us say some pictures here that I am here and my response is again something like this right. And this is my steady state value let us say this is one, we have to go little further up here.

So, if I look at this response here let us say it is actually bounded from below by a curve like this and bounded from above by another curve like this. Now what are these curves, these are simple this exponential curves here the guy on the top would be 1 plus e power minus zeta omega n t. So, look at this, this I would just go over here this will just be a standard t over square root of 1 minus zeta square it is the curve on the top, and this is the curve on the bottom this is 1 minus e power minus zeta omega n t over square root of 1 minus zeta square and then so this value here would be 1 minus 1 over square root of 1

minus zeta square, and the value here would be  $1 + \frac{1}{\sqrt{1 - \zeta^2}}$ . So, this is all good right. So, if I say well when does this curve insight this solid reaches within 98 percent and stays there well the answer could be that whenever this curves you know this, this guy when whenever this reach the within the 98 percent and then stay there because these are enveloped by this curves right this the guy one the solid line is just enveloped by these 2 curves.

So, I can just now look at the response of this system right. So, I am just say let us take the take the bottom one, for example,  $1 - e^{-\zeta \omega_n t} / \sqrt{1 - \zeta^2}$ . And this if you remember this looks like the response of a first order system right, where we had the other day that the response was  $1 - e^{-t/\tau}$  right, and if you just look at that response we had some things defined like this. This is my one this will be my response  $Y$  of  $t$  and this kind of looks similar.

So, this is a time constant of  $t$  and time constant  $1 / \zeta \omega_n$  and with  $t$ . So, this guy reaches  $t$  at  $t$  at the first time constant, it reaches the value of 63 percent and then at  $t$  equal to 2 times the it reaches some value of 0.8 right and then at  $t$  equal to 3  $t$  it reaches it reaches 95 percent and at  $t$  equal to 4 times the time constant it reaches 98.2 percent right. Somewhere here with this we had we had done this last time this 2 times  $t$  this is 3 times  $t$  this is 4 times  $t$ .

Now, if this guy reaches settling time at 4 times the time constant, this guy would also be you know reaching in the same time right because this is just enveloped by this by the curve side right. So, now, my settling time can be defined directly in terms of the time constant right. This is for the 2 percent tolerance band and settling time  $t_s$  and then  $t_s$  would be 3 times this settling time when I am at the 5 percent tolerance band.

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The slide is titled "Expression for Settling Time" and contains the following content:

- Time required for the response to reach and stay within a specified tolerance band (take 2%) of its final value or steady state value

Handwritten notes on the slide include:

- The transfer function:  $1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$
- Settling time for 2% criterion:  $t_s = \frac{4}{\zeta\omega_n}$  (2%)
- Settling time for 5% criterion:  $t_s = \frac{3}{\zeta\omega_n}$  (5%)
- Time constant:  $T = \frac{1}{\zeta\omega_n}$
- Peak overshoot:  $M_p \propto f(\zeta)$  or  $f(\zeta)$
- Notes: "a smaller  $\zeta$  a smaller  $t_p$ " and " $\zeta \rightarrow M_p$  will be large"

A graph on the right shows a damped oscillation with a steady-state value. The settling time  $t_s$  is indicated as the time for the response to stay within a 2% tolerance band. The graph also shows that as  $\zeta \rightarrow 0$ , the response becomes more oscillatory, and as  $\zeta \uparrow$ , it becomes more damped.

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So, how do I write this? So,  $t_s$ , if I again look at my curve the time constant there was say I just take the take the bottom curve  $1 - e^{-\zeta\omega_n t}$  over  $1 - \zeta^2$ . So, this has a time constant of  $1/\zeta\omega_n$ , and therefore, my settling time would just be  $4/4$  times the time constant and the  $4/\zeta\omega_n$  or the settling time this is for the 2 percent criterion and this would be  $3/\zeta\omega_n$  for at the 5 percent criteria.

So, what does this mean that the settling time is inversely proportional to zeta and  $\omega_n$  again let us say the eliminating case let us say the eliminating case of zeta being 0 my settling time would be infinity this is not surprising because if my settling time is infinity or when zeta is 0, I am just having like an oscillatory response right. My system will never go anywhere within the 2 percent or even the 5 percent or even the 50 percent.

So, settling time would be infinity. So, as zeta increases the settling time tends to reduce right. So, based on this formula here zeta increases and then the settling time reduces. So, I do not want to draw graph for that at the moment, but we will just understand that the increase in zeta will cause a reduction in the settling time. Now what is that sufficient to increase zeta? So, my peak over shoot is also some function of zeta. In such a way that if I have a smaller zeta, I have a smaller peak time or then even a smaller rise time right, if I want a faster response my zeta is smaller, but then my over shoot will be larger right. So, usually in design specifications my peak over shoot would be specified, and my zeta

would be calculated based on the peak over shoot. And therefore, if I were to look at settling time it just depends on the natural frequency of the system.

So, to write the settling time I can play around with this guy  $\omega_n$  right the  $\omega_n$  also over here. Because zeta is already defined by the performance specifications of the peak over shoot. So, when this is phase I am may not be able to change it here again because you know the smaller zeta would may be increased in a bigger over shoot which may not be very desirable. So, I can just play around with this  $\omega_n$  that is only we will see now what how we could smartly design these things. So, that I have not a big over shoot, but also a smaller settling time and so on.

(Refer Slide Time: 23:18)

The slide is titled "Application of Damped Systems" and is part of "Module 3: Lecture 3". It lists three types of damped systems with their applications and includes handwritten red annotations:

- Overdamped systems:**
  - Push button water tap shut-off valves (with a red sketch of a tap handle)
  - Automatic door closers (can be critically damped also)
- Critically damped systems:**
  - Elevator mechanism
  - Gun mechanism (returns to neutral position in shortest possible time)
- Underdamped systems:**
  - All string instruments, bells are underdamped to make sound appealing (with a red sketch of a bell)
  - Analog electrical or mechanical measuring instruments (with a red sketch of a measuring instrument)

So, ideally what we would like is to have faster response and a smaller settling time. So, what could be application of damped or this several kind of systems right. So, let us start with over damped systems, now this is push button water tap shut off valves. So, such as before we started this according my student was telling me a nice analogy of this right. So, originally if you look at the trains the taps in the train we would have normal taps which people would just forget to turn off and then we could waste water, then later on they had these things you know I do not know if I could draw a picture of this right. So, it is like this and then you will have a tap and then the water flows through here right, and then you keep on pressing this and then once you release this the water will go will not come anymore.

So, what they do now is that you just press it. So, the system will you know give us give us water for some time and eventually the valve will close well that is like example of an under damped system like the emergency in technology is can be seen you know when you travel by train right. I do not know if the trains still have this kind of smart things or not, but this is in more sophisticate thing you have this sensor you just put the hand and then you just (Refer Time: 24:30) this and then water just you know the valve just close of closes off gradually same thing you can see in this automatic door closes right there do not really be they cannot really have a fast response and then bang on to the wall or to the stopper.

So, these are automatic door closers are also, typically over damped systems critically damped systems you would want them to have an elevator mechanisms or even gun mechanisms right. So, even the elevator to have a faster response, but you do not want it to over shoot. So, we talked of this last time as well and of course, under damped systems all string instruments would do that because you know if you if you pluck a string they have a guitar I pluck the string and usually it has this nice wave kind of formation right, but if you if you pluck the string from here it goes here and again comes back here it may not generate the sound which you want.

Similarly, all electrical or mechanical measuring instruments, right. So, you take a old meter if it if it were to measure via a scale like this. So, if this is a 0 position it will typically go to the desired value and then come back and these are usually under damped systems right all this is moving coil or moving measuring instruments.

(Refer Slide Time: 25:41)

### Steady State Error

- It is the error between the actual output and the desired output as  $t \rightarrow \infty$

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} (r(t) - y(t))$$

By final value theorem,

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

$$E = R - Y = R - \frac{GR}{1+G} = \frac{R}{1+G}$$

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1+G(s)}$$

Unity feedback system

*$\frac{G(s)}{1+G(s)}$*

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So once we do these things we are also interested does my output follow the reference all the time.

So, we saw last time when we were doing the response for a first order system, in some cases we saw that there is a steady state error which means the reference is not tracked directly by the output, but it is with some error which was defined by a time constant right. So, just try to formulise that that a little more and then we will revisit the original examples which we had. So, the steady state errors. So, a typical feedback loop looks like this you have the reference signal the output the feedback loop and that the difference between the reference and the output is termed as the as the error signal. And this error I would typically want to go to 0. Now given this characteristics of the system the  $g$  and this this one, it will also be h nothing would change in the analysis. So, given this  $g$  can I find what is the error right depending on the nature of the signal here.

So, what is the steady state error it is error between the actual output and the desired output as time goes to infinity right. So, we know from final value theorem if I call this signal as  $e$  of  $t$  that  $e_{ss}$  is limit as going to 0  $s$  times  $e$  of  $s$  in the laplacian domain. Now this  $e$  is  $r$  minus  $Y$  and  $Y$  I know is. So, this transfer function here is  $g$  of  $s$  over  $1$  plus  $g$  of  $s$ . So, this  $Y$  given this  $r$ . So, I can write this as  $r$  minus  $g$  times  $r$  over  $1$  over  $g$  and I get this one. So, this is this is my error signal.

So, in this thing the steady state error the limit s going to 0 s times e of s would be limit as going to 0 s times r plus 1 plus g. Now we will make use of this little thing here and it will give us lots of information of the system subject to several inputs.

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**Steady State Error for Standard Inputs**

- Unit step input:**  $R(s) = \frac{1}{s}$   

$$e_{ss} = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{1}{1 + G(s)} = \frac{1}{1 + K_p}$$

where  $K_p = \lim_{s \rightarrow 0} G(s)$  is called position error constant
- Unit ramp (velocity) input:**  $R(s) = \frac{1}{s^2}$   

$$e_{ss} = \lim_{s \rightarrow 0} \frac{1}{s + sG(s)} = \lim_{s \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{K_v}$$

where  $K_v = \lim_{s \rightarrow 0} sG(s)$  is called velocity error constant

**Note:** Velocity error is not error in the velocity but it is error in position due to ramp input

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*Handwritten notes on the slide include a block diagram of a feedback system with input R(s), a summing junction, a controller G(s), a plant 1/s, and output Y(s). Calculations show the limit of sR(s) as 1/s and the limit of sG(s) as K\_v.*

So, let us start with a step input right. So, this will be phase of the time. So, for the step input r of s is 1 over s, and what is the steady state error e s s is limit s going to 0 s times r of s and this r of s is 1 over s right. So, this s s and s disappear and I am left with this. So, this limit would be 1 over 1 plus Kp right. So, this Kp would be limit as going to 0 G of s and this limit will always exist, because we are we are looking at coaxial system.

So, well with this limit is always exceeds we do not really care about if this might block to infinity or not. And this also we with this slide the (Refer Time: 28:49) Eigen will always exists. So, the Kp is computed as limit s tends to 0 g of s is called the position error constant right. So, I am just tracking a position of fix position 1 over s. Now similarly if I go to an input which is ramp I am tracking a ramp. So, this steady state error is again I do all the computations. One of the s disappears and I am just left with this and this this guy goes to 0, let us let us compute that. So, I have s limit s going to 0 s 1 over s square 1 plus g s. So, this would be limit s going to 0, 1 of the ss would disappear and I have 1 over s plus s times g of s. So, now this would go to 0 this s. And what I am let with is limit s going to 0 s times g of s and I call this the k v. So, when Kv is limit s going to 0 s times g of s and I call it the velocity error constant.

So, if there is if I have a feedback loop, I will just draw it here again so one of  $f$  ones plus minus  $Y$  of  $s$  and so on right. So, if the input or the reference is a step, then there will always be a error. So, the response, so  $Y$  of  $s$ , if I just take an under damped whatever kind of system and if I just look at the steady state curve. So, let us say this is my desired value my actual value will might just be somewhere here right. So, this is the steady state error.

So, we might think of if this is a position error, we might sometimes think that the velocity error, is actually that error in the velocity, but it is not true it just a velocity this is a error in it is not the error in the velocity, but error in position due a ramp input or a due to due to velocity input, for example, I am you know may be chasing a car here right, is say a car  $c_1$  and then this is my left hands car and I am just I was just want to make it right and at  $c_2$  this is my car.

So, as time goes to infinity. So, we are always at say a distance of say one meter from this right. So, this is just error in the position, we are not having a error in the velocity even though we are just we could just be moving at constant velocity right. So, this could be some velocity  $v_1$  this will also be a same velocity  $v_1$ , but I will just have a error in the position. So, just you should not be confused with error in the velocity similarly if I were to say well give me a temperature of 24 degrees I just go to 23 the steady state error here is one right. So, I am just tracking a position right I just want to go to 24.

But; however, if I say keep on increasing my temperature profile with time in this way. So, say this is it increases let us say one times  $t$  this is my  $t$  and this is my temperature the reference temperature right, with time it keeps on increasing. So, if I say that the velocity is you know there is a velocity error I will start from here and I will just be here. So, if this has to increase constantly at  $t$ , well I will always be behind by say one degree. So, at say at some  $t$  equal to 50, I might just be at 49 at  $t$  equal to 100, I am at just be at 99. So, it just the little difference in the position not really the rate of the rate of my car here or the rate of change of temperature here.

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## Steady State Error for Standard Inputs

- **Unit parabolic (acceleration) input:**
$$R(s) = \frac{1}{s^3}$$
$$e_{ss} = \lim_{s \rightarrow 0} \frac{1}{s^2 + s^2 G(s)} = \lim_{s \rightarrow 0} \frac{1}{s^2 G(s)} = \frac{1}{K_a}$$
where  $K_a = \lim_{s \rightarrow 0} s^2 G(s)$  is called acceleration error constant
- The error constants  $K_p$ ,  $K_v$  and  $K_a$  describe the ability of a system to reduce or eliminate steady state errors
- These values mostly depend on the type of the system
- As the type of the system becomes higher, more steady-state errors are eliminated



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Similarly, if I have parabolic input for which the Laplace transform looks like 1 over s<sup>3</sup> I can do all the math and I say that the steady state error is 1 over k<sub>a</sub> which k<sub>a</sub> defined in this way. And this guy is called the acceleration error constant and I just you know this has an extra s over here right. So, the error constants this 3 things and these are the standard test signals which we also used in the earlier lecture to study response of systems.

So, the error constants K<sub>p</sub>, K<sub>v</sub> and K<sub>a</sub> they describe the ability of a system to reduce or eliminate steady state errors. For example, if I will come to that right. So, and then these values depend mostly on the type of the system why this type is important and as a type becomes higher more the steady state errors are eliminated we will just define this type in the next slide.

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## Features of Steady State Error

- Steady state error is a measure of system accuracy
- In an ideal scenario, the system should match the reference input as time progresses
- It means the steady state error should be as low as possible and hence it is an important performance measure
- Steady state errors depend on two factors:
  1. Type of the reference input  $R(s)$  – step, ramp or parabolic
  2. Type of the system  $G(s)$
- Steady state errors are calculated only for closed loop stable systems



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## Type of a System

- Consider following pole-zero form of open loop transfer function of a system:
$$G(s) = \frac{K'(s + z_1)(s + z_2) \dots}{s^n(s + p_1)(s + p_2) \dots}$$
  - Term  $s^n$  in the denominator denotes the number of the poles ( $n$ ) at origin
  - System with  $n$  poles at origin is called Type- $n$  system
  - $n$  also indicates the number of integrations  $\left(\frac{1}{s}\right)$  in the system
  - As  $s \rightarrow 0$ ,  $s^n$  term dominates in determining the steady state error



Module 3: Lecture 3

So, I will I will come back to this slide little later, but just the type of a system is defined here by the number of poles at the origin. So if I have a system in the pole 0, configuration  $G$  of  $s$  could be typically some gain with a set of zeros with the set of poles and additionally some set of poles at the origin.

So, if there are  $n$  poles I will call it a type  $n$  system is  $n$  is 0 I will call it a type 0 system and so on right. And we will see how does this type actually now influence the way I compute the steady state error. So, that is a type of system. So, the steady state error is a

measure of how accurate my system is right, it does it really track my reference accurately or is there. So, the accuracy should. So, in an ideal scenario I would always want it to track it to its desired value right.

However, there could be situations where it may not be possible to do. So, in that case I would like the error to be as small as possible. And that is a very important performance parameter right I do not want a desired temperature of 24 and end up at 12 or 31 for example, right. So, this steady state error now depends on 2 factors. And it is the nature of the signal which I am tracking and also what is the nature of the physical system or the type of the system right. And of course, all these things we will calculate for closed loop systems. Of course, I will calculate for closed loop systems only based on the open loop information I do not really need to see what is happening within the loop and so on.

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**Steady State Error for Different Systems**

- **Type-0 system:**

$$G(s) = \frac{K'((s + z_1)(s + z_2) \dots)}{(s + p_1)(s + p_2) \dots}$$

Handwritten notes:  $K'$  and  $s$  are written in red next to the numerator and denominator respectively. A red graph shows a step response with a steady-state error.

  - $e_{ss}(\text{position}) = \lim_{s \rightarrow 0} \frac{1}{1+G(s)} = \frac{1}{1+K_p}$  (Some error)
  - $e_{ss}(\text{velocity}) = \lim_{s \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{0} = \infty$
  - $e_{ss}(\text{acceleration}) = \lim_{s \rightarrow 0} \frac{1}{s^2G(s)} = \frac{1}{0} = \infty$
- Constant position error, infinite velocity and acceleration errors at steady state

Module 3: Lecture 3

So, we will see via how we will do that. So, start for a type 0 system this  $G(s)$  is again  $K$  set of zeros and there are no poles at origin right. So, this  $n$  in the previous slide this guy would be 0. So, I just have these poles.

So, how does it respond to a step signal right, to for a step signal I have this guy.  $1$  over  $1 + K_p$ . So, that there is always be some constant of course, some constant error (Refer Time: 36:09) some constant error right. As a steady state right it can track positions signal, for example, this is my position signal this guy can track well possibly I would take it as the response, but it could be something like this, and may be it will go and there

will be some error right. So, there will be some steady state error here when you tracking this step.

Now, this system can never track a velocity signal. Why because the steady state error is computed in this way limit as going to 0 1 over s times g s, this is infinity for steady state it will never be able to track a signal which is like this and the error will keep on increasing right. So, this, this guy as we keep on moving further the error will keep on increasing. Similarly, here it will also never be able to track a parabolic input. So, the acceleration error will also be infinity.

So, to summarise a type 0 system has a constant position error and infinite velocity and acceleration errors all at steady state. So, all these are related to this steady state behaviour at the moment, I am not worried what is a peak over shoot what is a rise time and so on I am only interested in what or how my system behaves at the steady state.

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## Steady State Error for Different Systems



- Type-1 system:
 
$$G(s) = \frac{K'((s + z_1)(s + z_2) \dots)}{s(s + p_1)(s + p_2) \dots}$$



  - $e_{ss}(\text{position}) = \lim_{s \rightarrow 0} \frac{1}{1+G(s)} = \frac{1}{1+\infty} = 0$
  - $e_{ss}(\text{velocity}) = \lim_{s \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{K_p}$
  - $e_{ss}(\text{acceleration}) = \lim_{s \rightarrow 0} \frac{1}{s^2G(s)} = \frac{1}{0} = \infty$
- Zero position error, a constant velocity error and infinite acceleration error at steady state

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Type one system well I just do all the computations again I have one my limit s going to 0 1 plus g s and since this has a pole at 0 this guy will go to infinity right the Kp.

Let us I can go back to see how we had defined Kp Kp was defined at limit s going to 0 g times s. So this guy since s has a pole at the origin this will go to infinity and the error will go to 0, similarly the velocity error would be limit s going to 0 1 over s times g of s. So, this s and this this s would cancel out and I have a constant here. And the

acceleration would go to infinity. So, if I take a type one system and I ask it to follow a step input, then it will at steady state there will be no error it will actually follow this input very nicely.

However, if it take the same system I ask it to follow a ramp it might have some steady state error right I just go there response could be something like this and eventually go to the constant error here like a straight line. Similarly, acceleration it will never be able to track a signal which is which is parabolic. So, it will have a 0 position error a constant velocity error and infinite acceleration error steady state.

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## Steady State Error for Different Systems



- Type-2 system:
 
$$G(s) = \frac{K'((s + z_1)(s + z_2) \dots)}{s^2(s + p_1)(s + p_2) \dots}$$
  - $e_{ss}(\text{position}) = \lim_{s \rightarrow 0} \frac{1}{1+G(s)} = \frac{1}{1+\infty} = 0$
  - $e_{ss}(\text{velocity}) = \lim_{s \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{\infty} = 0$
  - $e_{ss}(\text{acceleration}) = \lim_{s \rightarrow 0} \frac{1}{s^2G(s)} = \frac{1}{K_a}$
- Zero position error, zero velocity error and a constant acceleration error at steady state

Module 3: Lecture 3

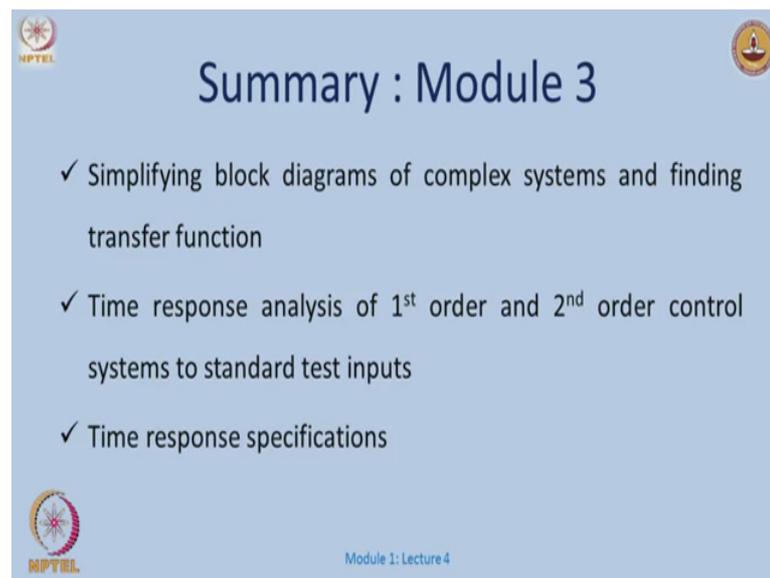


Now, type 2 system right. So, nothing will change here the position error will still be 0 because I have this s in the in the denominator I put it to 0 g of s goes to infinity and 1 over infinity goes to 0. Similarly, the velocity error. So, I do s times g s 1 1 s still remains in my denominator. So, this still be 1 over infinity and the velocity error goes to zero; however, my acceleration error constant now has some number here it was infinity in the previous 2 cases. So, I could summarise by saying that if I take a type 2 system, I ask you to track position or a step it will track it perfectly right. There will be steady state error if I ask you to track a ramp it will also do it perfectly. If I ask you to track a parabolic signal it will do it, but there will actually be a small error determined by this acceleration error constant.

Now, we see that you know if I say it will give a little you know problem saying take a type 0 system and make the position error to be 0. What would I do I just say is this look same over this theory now and I say see just put a integrator right or add a pole at the origin. I can always do this is my system which has more poles and zeros I can possibly always realise this, and I can just put it here right. So, this is can this be done can this not be done or can this be done with some being a little careful we will see all those things, but these are the design specifications which would be given to us and then we know we can see that these errors can be quantified just based on the type of signals which we are tracking and the type of the system.

So, having an integrator in my system or having a pole at the origin gives a little more hope to me in designing the system.

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The slide is titled "Summary : Module 3" and lists three key topics covered in the module. It features the NPTEL logo in the top left and bottom left corners, and a small circular icon in the top right corner. The text is centered on a light blue background.

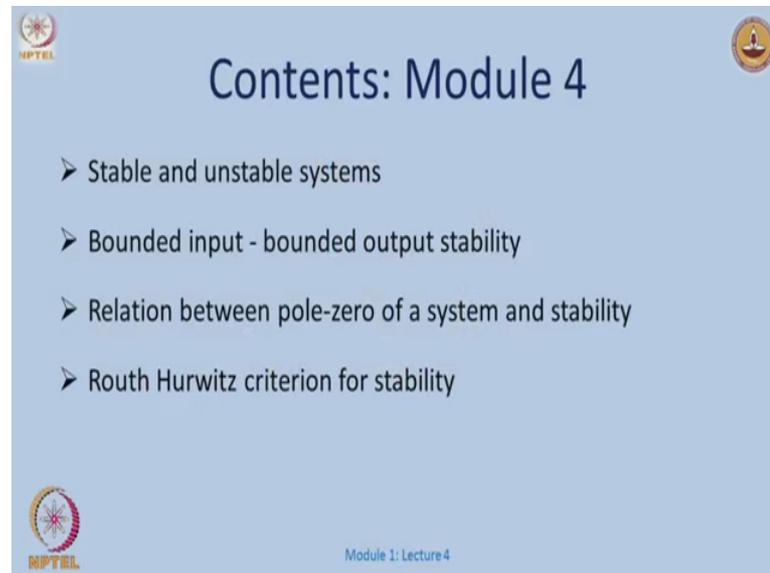
- ✓ Simplifying block diagrams of complex systems and finding transfer function
- ✓ Time response analysis of 1<sup>st</sup> order and 2<sup>nd</sup> order control systems to standard test inputs
- ✓ Time response specifications

Module 1: Lecture 4

So, in this entire module what we started with block diagram representation signal flow graphs and then reducing the complexity or finding the transfer function by reducing the complexity of the block diagram or simply using the signal flow graph formula to find the transfer function, then we also saw the time response of first and second order systems, with standard inputs and we also justified why we were using the standard test inputs because in real time we can see all these signals the real time signals could be a combination of all these test signals. And then we actually quantified our performance in

terms of the steady state error in terms of how much do I over shoot what is a rise time and so on and we had expressed it formulas for those right.

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And in the next lecture we will and so far what we assumed that the system is stable and we had a very crude version of the definition of stability saying, if it is start with 0 initial conditions it should come back to it is original configuration. So, we will have notions of define these notions formulae of stable and unstable systems. And see how can we define these things directly by the transfer function looking at the pole and zeros, just by looking at the poles and zeros can is say this system is stable or not. Or another notions were you know what is called as the bounded input bounded output stability and then given a transfer function of fairly you know complex level, which has lots of poles and zeros. I will just teach you how we use the Routh Hurwitz criterion for the stability of to determine this systems as stable or not.

Thank you.