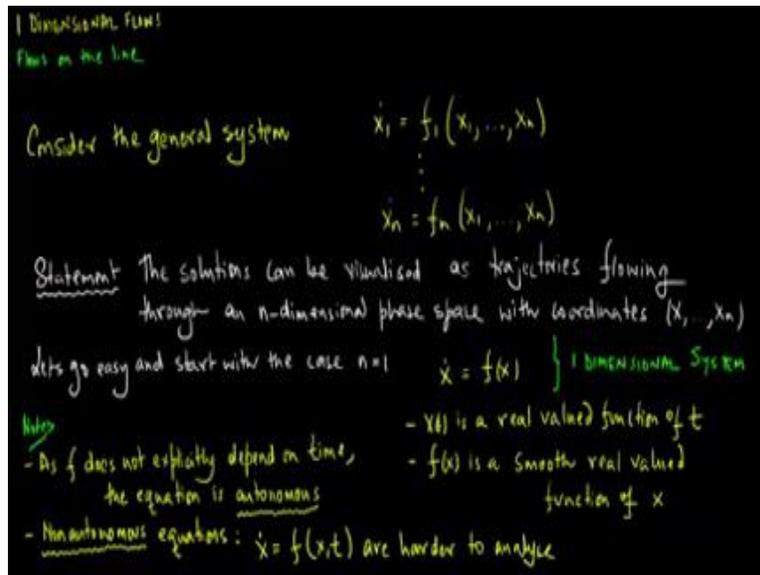


Introduction to Nonlinear Dynamics
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Module -02
Lecture - 03
1-Dimension Flows, Flow on the line, lecture 1

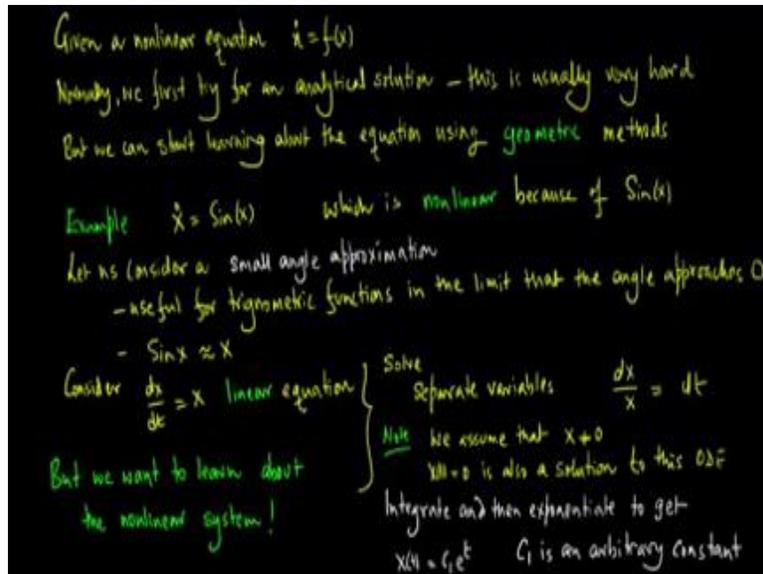
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We start with one dimensional flows and we focus on flows of the line. Consider the following general nonlinear system, \dot{x}_1 is equal to f_1 is the function of x_1 to x_n all the way to x_n dot is equal to f_n is the function of x_1 to x_n . Now let us make a statement about the above system, the solutions can be visualised as trajectories flowing through an n dimensional phase space with coordinates x_1 all way to x_n .

Now let us actually go bit easy and start with the case n is equal to one, that is \dot{x} is equal to $f(x)$ where $x(t)$ is the real valued function of t and $f(x)$ is a smooth real valued function of x . So this is an example of a one dimensional system. Now here are some notes as the function f does not explicitly depend on time the resulting equation is autonomous. Non-autonomous equations are the equations of the form \dot{x} is equal to $f(x,t)$ and are in general much harder to analysis.

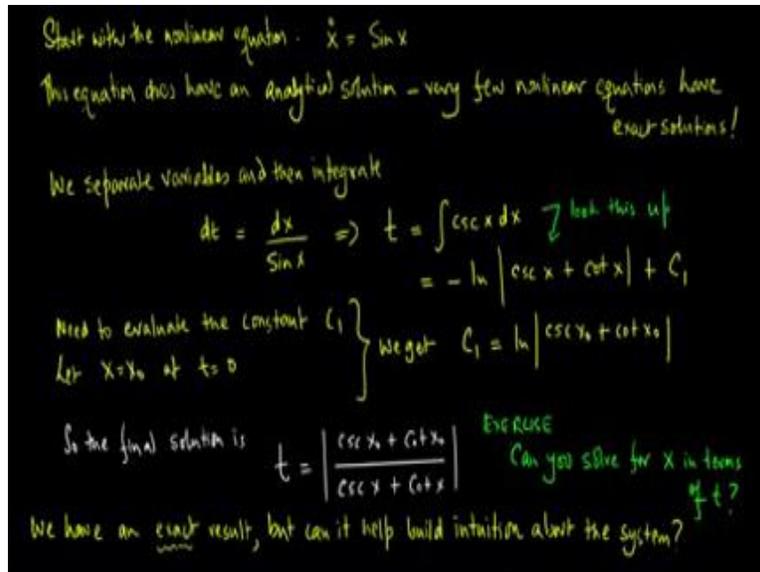
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Now given a nonlinear equation \dot{x} is equal to f of x , normally we first try and look for an explicit analytical solution. This is usually very hard, but we can start learning about the equation using geometric methods. Here is an example, \dot{x} is equal to $\sin x$, which is nonlinear because of the $\sin x$ term. Now let us actually first consider a small angle approximation, which is useful for trigonometric functions in the limit that the angle approaches zero so $\sin x$ is approximated as x .

So now consider $\frac{dx}{dt}$ is equal to x , which is a linear equation. So to solve it we separate variables to get $\frac{dx}{x}$ is equal to dt . Note that we are assuming that x is not equal to zero. However the x of t is equal to zero is also a solution to this ordinary differential equation. So we integrate and then exponentiate to find a solution x of t is equal to $C_1 e^t$ where C_1 is arbitrary constant. But what we really wanted to do was to learn about the original nonlinear system.

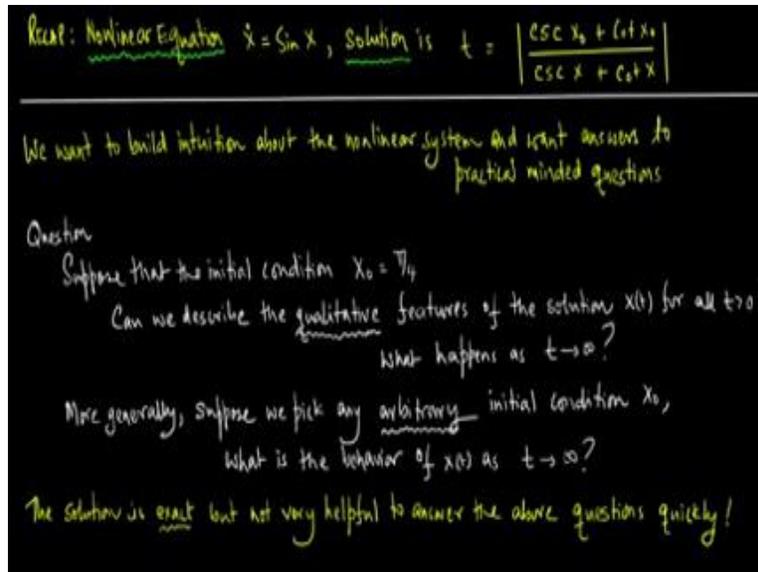
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Now let us actually start with the nonlinear equation \dot{x} is equal to sine x . Now this equation actually does have an analytical solution, but very few nonlinear equations actually admit exact solutions. So to solve the equation we separate the variables and then integrate, so dt is equal to dx by sine x which gives us t is equal to integral of cosecant of $x dx$ evaluating this integral we get minus log of cosecant of x plus cotangent of x plus a constant.

We suggest that, the look this integral up, we need to evaluate the constant C_1 , so let x is equal to X_0 at t is equal to zero and we get C_1 is equal to log of cosecant of X_0 plus cotangent of X_0 . So the final solution turns out to be t is equal to the cosecant of X_0 plus cotangent of X_0 divided by cosecant of X plus cotangent of x . So here is an exercise can you solve for X in terms of t you should do it on your own. Now we have an exact result but can it actually help to build intuition about the original nonlinear system.

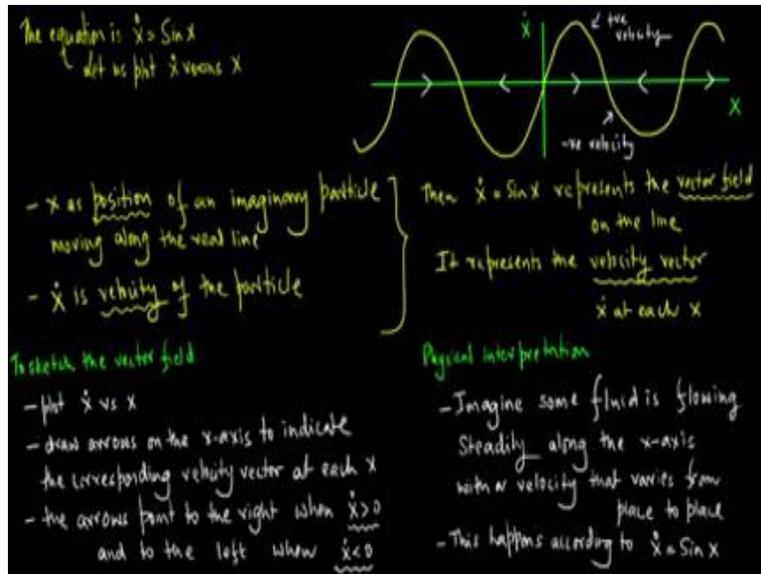
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Now, let us just recap what we have, we have a nonlinear equation \dot{x} is equal to sine x and using some analytic methods we were able to obtain an exact analytic solution to this particular equation. Now what we really want is to build intuition about the nonlinear system and be able to answer practical minded questions.

For example suppose that the initial conditions is X_0 is equal to $\pi/4$ and can we describe the qualitative features of the solutions $X(t)$ for all t greater than zero. What happen as t tends infinity more generally, suppose we pick any arbitrary initial condition X of 0 , than what is the behaviour of X of t as t tends to infinity. Now the explicit solution is exact, but not extremely helpful to answer the above questions quickly.

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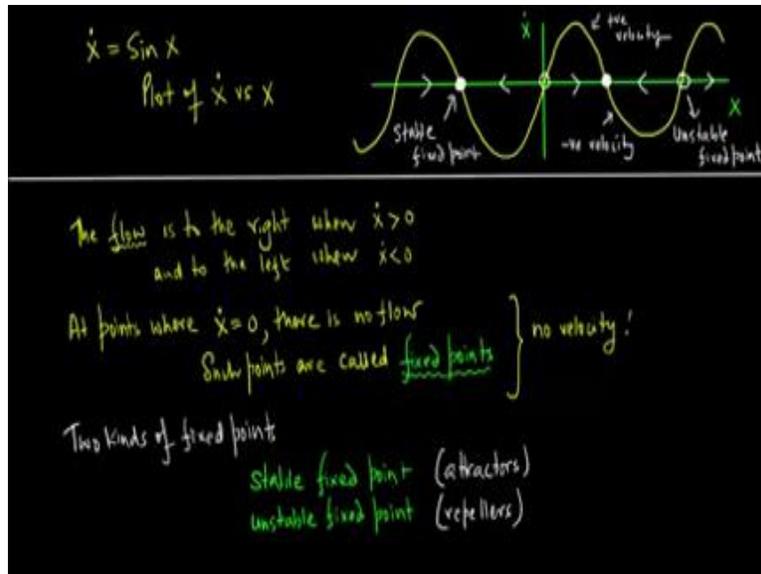


Now the nonlinear equation at hand is $\dot{x} = \sin x$, so let us actually plot \dot{x} versus x . Here is slightly simple minded plot of \dot{x} versus x . Now think of x as the position of an imaginary particle, that it is moving along the real line when \dot{x} is the velocity of the particle then $\dot{x} = \sin x$ represents the vector field on the line. Now, essentially it represents the velocity vector \dot{x} at each x .

Now to plot the vector field we do the following plot \dot{x} versus x . Draw arrows on the x axis to indicate the corresponding velocity vector at each x , the arrows should point to the right when \dot{x} is greater than zero and they should point to the left when \dot{x} is less than zero. Now, please pay very close attention to the plot of \dot{x} versus x .

We highlight the region of positive velocity where the arrows point to the right and note the region of negative velocity when the arrows when point to the left. Now let us actually offer some physical interpretation, imagine that we have some fluid that is flowing steadily along the x axis with a velocity that actually varies from place to place then this is essentially just happening according to $\dot{x} = \sin x$.

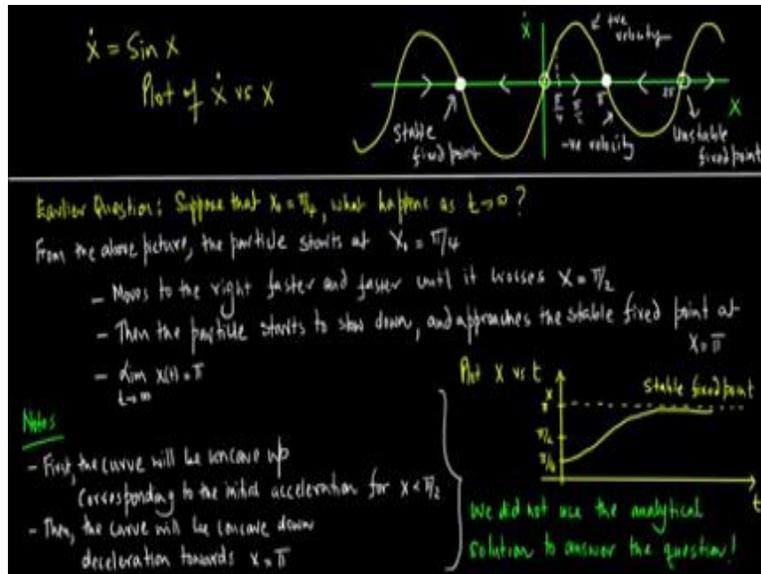
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Now the flow is to the right when \dot{x} is greater than zero and the flow is to the left when \dot{x} is less than zero at points where \dot{x} is equal to zero there is actually no flow and such points are called fixed points this are points there is no velocity. Now there are two kinds of fixed points, stable fixed points also referred to as attractors and unstable fixed points referred to as repellers.

Now look at the diagram the closed circles are the stable fixed points and the open circles are the unstable fixed points. Note that on the stable fixed points the flow is getting attracted towards them and on the unstable fixed points the flow is getting repelled from there.

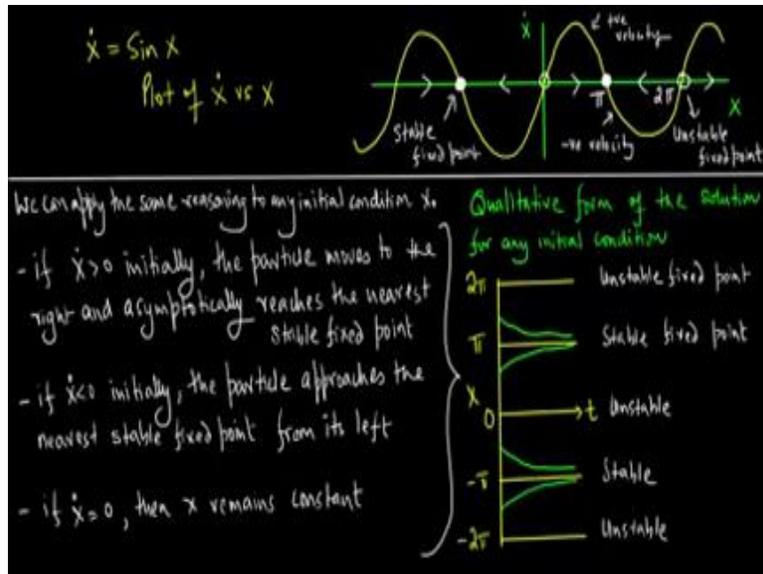
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What we really want to do now is get some additional insight into this nonlinear system that we have now recall the earlier question suppose that x_0 is equal to π on four then what really happens as t tends to infinity. Now from the above picture the particle starts at x_0 is equal to π on four and moves to the right faster and faster until it cross X is equal to π on two then the particle slowly starts to slowdown and approaches the stable fixed point at X is equal to π .

So the limit as t tends to infinity $X(t)$ is equal to π if the particle actually as started at π on four. Here are some notes, first the curve will be concave up corresponding to the initial acceleration for X less than π on two. Then the curve will be concave down highlighting deceleration towards X is equal to π . Now let us go ahead and plot X versus t . We first highlight X is equal to π which is our stable fixed point, highlight π on two π on four which was our initial condition and that is the curve of X versus t . Observe that we did not use the analytical solution to actually answer the above question.

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We can apply the same reasoning to any initial conditions x_0 . If \dot{x} is greater than zero initially the particle moves to the right and asymptotically reaches the nearest stable fixed point. If \dot{x} is less than zero initially, the particle approaches the nearest stable fixed point from its left and if \dot{x} is equal to zero then x remains constant.

The qualitative form of the solution for any initial condition can actually be plotted using the rules that we have identified on the left. Here is the snapshot of what the solution should look like. Note that the trajectory are converging towards x is equal to π and x is equal to minus π and the reason is because these are the stable fixed points.

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Ok so now let us just rap up this lecture with some concluding remark. We started our study of nonlinear system with equations of the form \dot{x} is equal to f of x . These were one dimensional flows and in particular, we were rather keen to understand flows on the line. Now when you given a nonlinear equation the first thing you normally try, is to try and get an explicit analytical solution. A first thing to remember is that explicit analytical solutions are usually very, very difficult to get a nonlinear system ok.

So we have a particular example we said let \dot{x} is equal to $\sin x$. Now in this particular case we were able to get an explicit analytic solution we separated the variables and we integrated and the integral work out explicitly. And in this case, we were able get explicit analytic solution. However, the form of the solution was not very easy to understand it was not very easy to absorb its certainly was not easy to develop intuition about the form of the solutions.

So what we did was we went on to look at some geometric reasoning. Now essentially, all we did was we plotted \dot{x} versus x , for this particular equation \dot{x} is equal to $\sin x$. Now as soon as we did that we found that there were interesting cases that showed up when \dot{x} was greater than zero, when \dot{x} was less than zero and when \dot{x} was actually equal to zero, we encountered fixed points for the first time. Now such fixed point we found either be stable or they could be unstable.

And now with this geometric form, we were able to ask and start answering questions of the form that if you choose a particular initial condition, what would be the solutions look like as t tends to infinity. So in particular, we just picked X_0 is equal to π on four as initial condition and ask what happens as time carried on. So the lesson learnt from there was that geometric reasoning can certainly complement analytical solutions within the case.

That we can find analytic solutions and in the case, that we actually cannot find the analytic solutions is a good way to actually start yah. And but we have to be very careful that geometric reasoning may not be able to give all the answers we want. For example, if we were ask after a particular quantitative question of the form, what is the time at which \dot{x} is equal to $\sin x$ has

the greatest speed then geometric reasoning will not be in a position to answer such form of quantitative questions.