Introduction To Adaptive Signal Processing

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Lecture No # 13

PSD and Linear Time Invariant Systems

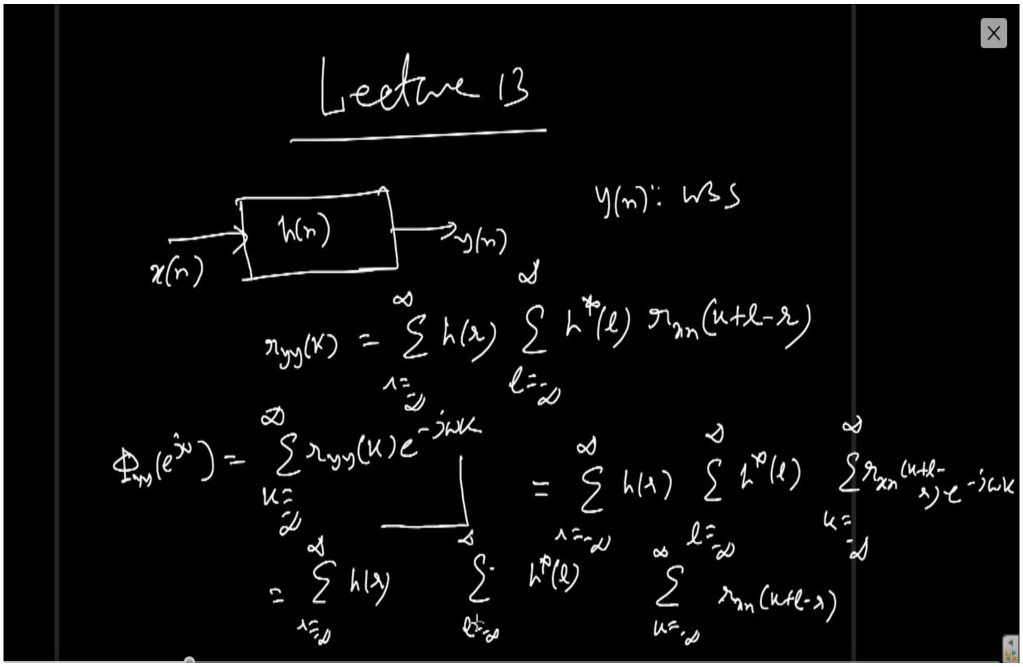
Ok, in the last class we have seen this, that if there is a WSS process random process Xn goes through a linear time invariant system that has impulse response hn and output is Yn then Yn WSS that is it has also mean which is independent of the index n there is same everywhere and its correlation depends only on the gap between the two samples not on their absolute locations that we have proved. If we worked out the expression for output correlation or auto correlation you can say it was this input rxx is k plus L minus R all right. So, this finally, becomes a function of k because L you know you sum over L, So, L goes, you sum over R, So, R goes. It right hand side is a function of k only there is no n. As a result output auto correlation does not depend on the index n it only depends on the gap k. So, it was stationary in the second order also and already shown that stationary in the first order that is in mean and therefore, it is WSS.

Therefore, what is the output power spectral density? Because if it is WSS I can take the DTFT discrete time Fourier transform of output auto correlation function and call it output power spectral density phi yy e to the power j omega. There is nothing, but this all right and then R yy k you you substitute by this expression. So, you will have triple summation. Now you see what is the meaning of triple summation you take some value of k for that take one value of R move L over the entire range then another value of R again move L over the entire range and that way after you finish all values of R L then again move k to another value repeat the same thing.

So, basically you cover a grid three dimensional grid one axis a k one axis R one axis L and all points of k L R this triplet you will get the same thing if you interchange the order that is maybe you take R first fixed array value for that take L fixed array value and then move k over the entire range then take L another value again for that move k over the entire range and then once you have L fixed then again change R to another value and then again do the same thing and like that you will you will cover the same grid it is called interchanging the order of summations. So, that is something which is always useful in our you know signal processing in particular whenever you come across double or tripping summation you have to interchange the order of summation then only again new results come that is what has been found. So, I keep the this summation outside range R this as it is the k summation I you know send inside earlier k was fixed to a value one time and then R and L were valid over the entire range then again k another value again R and L were valid over the entire range. you will get the same thing if you first take R equal to something then vary then again take L something move k over the entire range then again L another value keeping the same R k over the entire range that way you cover all L R and then again take another value of R do the same thing. So, you will cover all the you know all the ranges of k L R all the possible values of this triplet.

So, that is what we do. So, inner summation is over k. So, only those who are depending on k are inside those who are depending on L they are inside those who are depending on R they are outside they can be pushed outside. So, k equal to 0 to k equal to minus infinity I am very sorry. e to the power minus j omega k and now some simple manipulations.

What I do? I write Ry k as it is, sorry it is not Ry k I am very sorry. It is Rxx k plus L minus R. So, e to the power minus j omega k. So, here again Rxx k plus L minus R now one thing you see you take particular value of power.



So, that is fixed now ok in the inner summations and then again take another value of L fix that.

So, one R for that one L for those chosen L and R you take k over the entire range. Then again you vary L and then vary k R like that, but once you choose one R and then one L they are constant within this summation over k. So, L and R are constant and k plus L minus R e to the power minus j omega k. So, what I can do? I can make it k plus here I have j omega k only. So, I can make it k plus L minus R.

To cancel out this extra terms I have brought in e to the power minus j omega L. So, I must have e to the power j omega L ok and that depends only on L not on k. So, it can come outside the summation as a common goes outside the summation here and I brought it e to the power minus j omega minus R. So, e to the power plus j omega R. I have to cancel it by a e to the power minus j omega R.

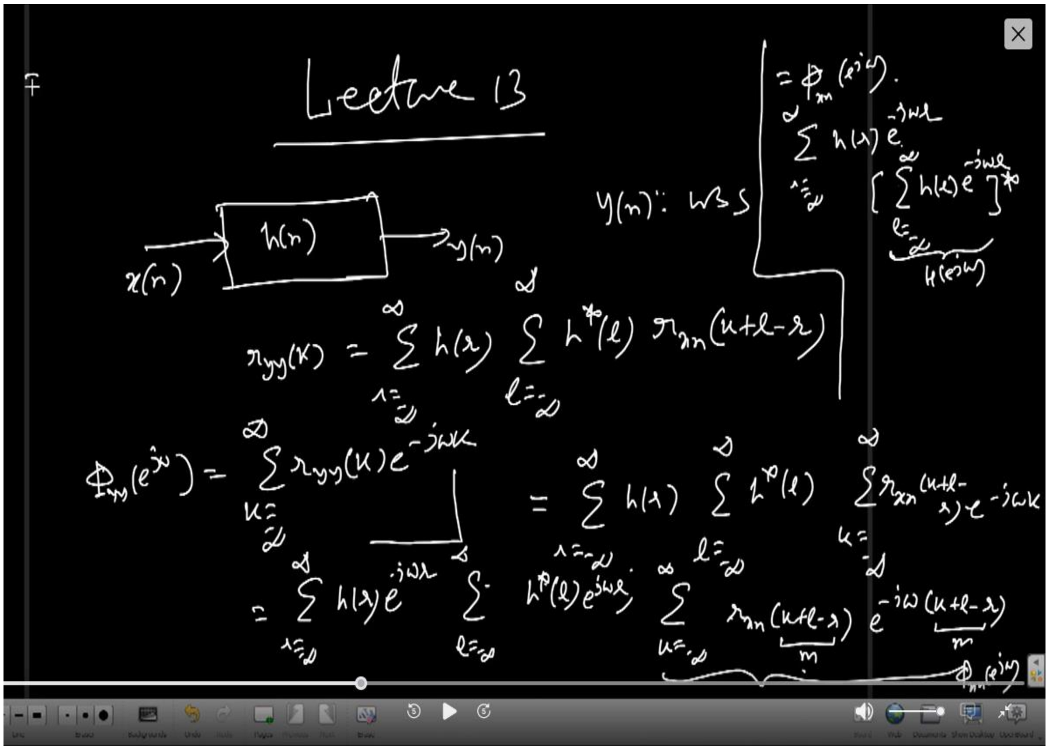
That will go out as a common again that depends only on R. So, that will again go out as a common of the summation, So, it will come here all right. you can call this as M. So, this index is M L and R are fixed. So, as k is minus infinity M goes to minus infinity as k goes to plus infinity M goes to plus infinity.

So, Rxx M e to the power minus j omega M summation over M. So, this is the discrete type Fourier transform of auto correlation function Rxx M which is the power spectral density of the input right phi xx e to the power j omega. So, this entire quantity is phi xx e to the power j omega this summation. And now this does not depend on L and R, So, this will go outside and this summation, So, what I am left with I am writing here this phi xx e to the power j omega it has already gone out, then you have got H R e to the power minus j omega R. For each R we put the value and they run L over the summation H star L e to the power j omega L, but that you can write as, so, this product H star L e to the power j omega L it is as good as a H L e to the power minus j omega L and then whole is co conjugated H L and then summation.

Let me explain again if I take the conjugate out. So, I should make it e to the power minus j omega L then again conjugate out. So, H L e to the power minus j omega L whole together conjugate and summation, but summation of conjugate is conjugate of the summation ok conjugate of the summation. So, this is this summation and then the conjugate ok. So, I take star out and from this I make it here I make it into your minus j omega L.

So, star goes out again ok. Because conjugate of e to the power j omega L is e to the power minus j omega L. So, e to the power minus j omega L then star make it e to the power plus j omega L. So, I have H L e to the power minus j omega L product star and then summation. So, summation of conjugated quantities is first you sum and then conjugate.

That is Z 1 star plus Z 2 star is Z 1 plus Z 2, then star Z 1 plus Z 2 here I have infinite sum. So, Z 1 plus Z 2 plus Z 3 dot dot dot plus that Z star. So, this is nothing, but the transfer function H e to the power j omega discrete time Fourier transform of H L or H N whatever we the index you choose H e to the power j omegas and then a star and that goes out as a common and this again you are left with another H e to the power j omega.

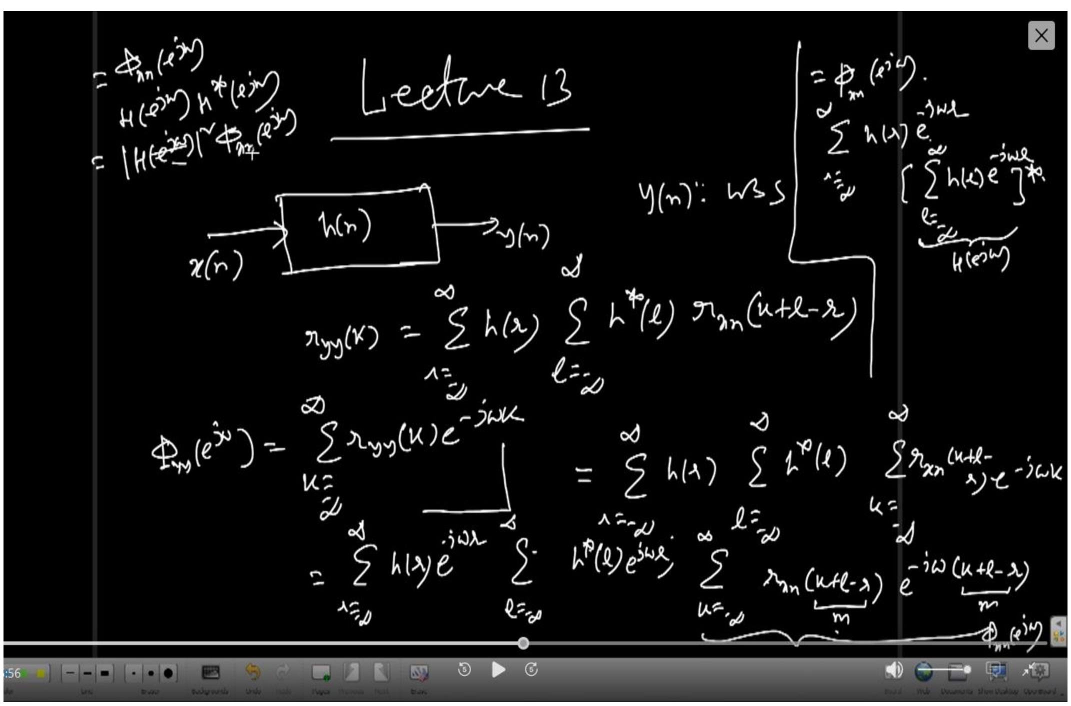


So, what you get is phi xx e to the power j omega then H e to the power j omega and then H star e to the power j omega ok H e to the power j omega star. So, H star e to the power j omega goes out as a common then H e to the power j omega which is mod.

So, this is mod even though H e to the power j omega is complex for every omega when you take mod and square this is real and this is already real and that shows output versus spectral density satisfies the condition that it should be real. This is a very important result important result because you can design a filter with a specific transfer function frequency response. So, that even if input power spectral density has a particular shape like high here low here for certain frequency and all that you can alter that by designing the filter. So, you can give good gain high gain to some zone in frequency and low gain in some zone other zone of frequency that way we can band pass band digit kind of a filter and output power spectral density can be shaped. Its shape can be determined by your design of a filter ok.

So, because remember only the magnitude response comes. So, you can design a filter with the low pass characteristics that is only up to some frequency its value is high otherwise 0 close to 0. So, input power spectral density only in the pass band will be amplified otherwise gone. So, it will be low pass random process ok. Low pass random process means not its individual discrete time Fourier transform of a particular y n, but in a statistical sense you take the auto correlation then dT fT that is power spectral density ok and that will be like this low pass signal because if you integrate the power spectral density from minus pi to pi we have seen already it will give you the average power.

Now, if it is low pass signal it will mean that average power is located only certain low pass pass band at nowhere else and likewise so and so forth ok. This is a product relation that is why even if this power spectral density is actually multiplied by this mod square of transfer function of frequency response because in this product if it is high overall will be high if it is low overall will be low.

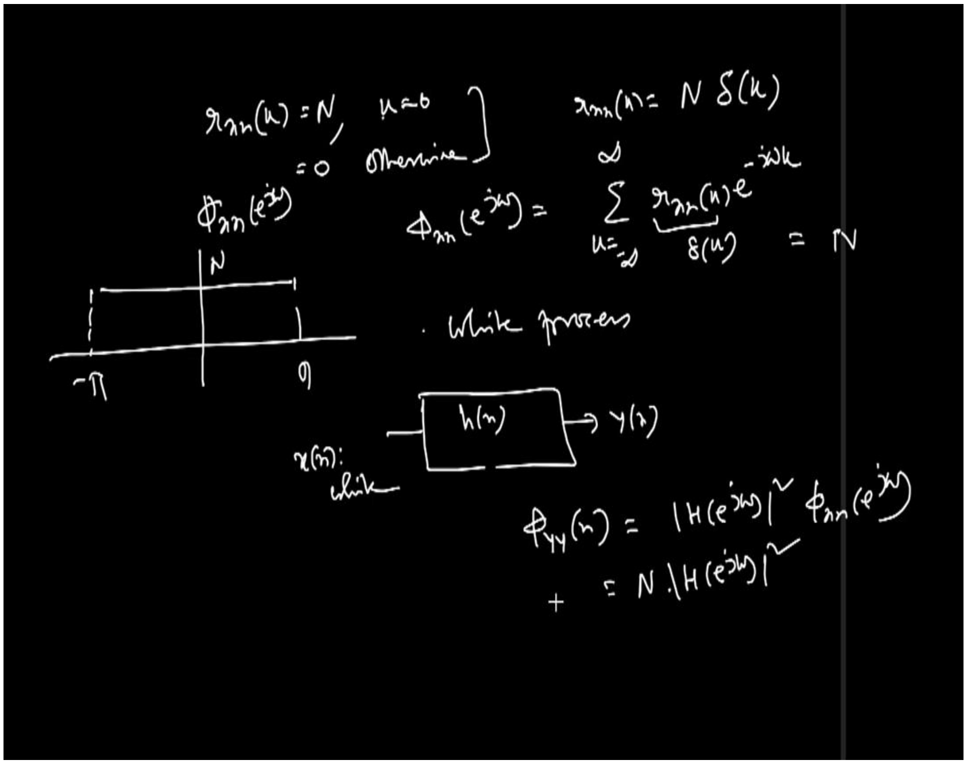


That way you can give a shape you can adjust the shape of the output power spectral density alright ok. A particular case suppose what is it suppose input is such input is such rxxk it is 1 if k equal to 0 and 0 otherwise alright that means, rxxk is delta k we know delta k is 1 when k equal to 0 0 otherwise. What does it mean it means that when k is 0 then only there is correlation that is variance.

Variance is this much, but for any k, gap of 1 k is 1, gap of 2 k is 2, gap of minus 1 for any gap the correlation is 0, which means every 2 different samples, So, that they are immediate neighbors or far away neighbors they are uncorrelated. Ok in that case rxxk is delta k and then your for this processes is, but rxxk is delta k input delta k only k equal to 0 case is to be taken otherwise 0 and e to the power 0 is 1. So, you get 1 ok if rxxk is not 1, but maybe some constant N then it will be and here it will be N. That is power spectral density it is constant N from minus pi to plus pi. This is called white process this random process is called white process.

Because from minus pi to pi are equally present like white light white process. So, if that LTI system makes it which is white then YN its power spectral density will be as it is mod h e to the power j omega whole square input power spectral density right. That we have proved in the previous page which is, but this is equal to capital N. So, in this case it does not depend on input just by designing the filter you can give a shape. Because only mod h e to the power j omega whole square comes as a function of frequency other factor is a constant independent of frequency.

So, for white input you can just design the filter and thereby make the sequence as a random process that is low pass or high pass or band pass meaning power spectral density is low pass type or band pass type or high pass type like that all right. So, that much for power spectral density now, but now one thing I left out I wanted to I said that I will show that this function is actually non negative ok.



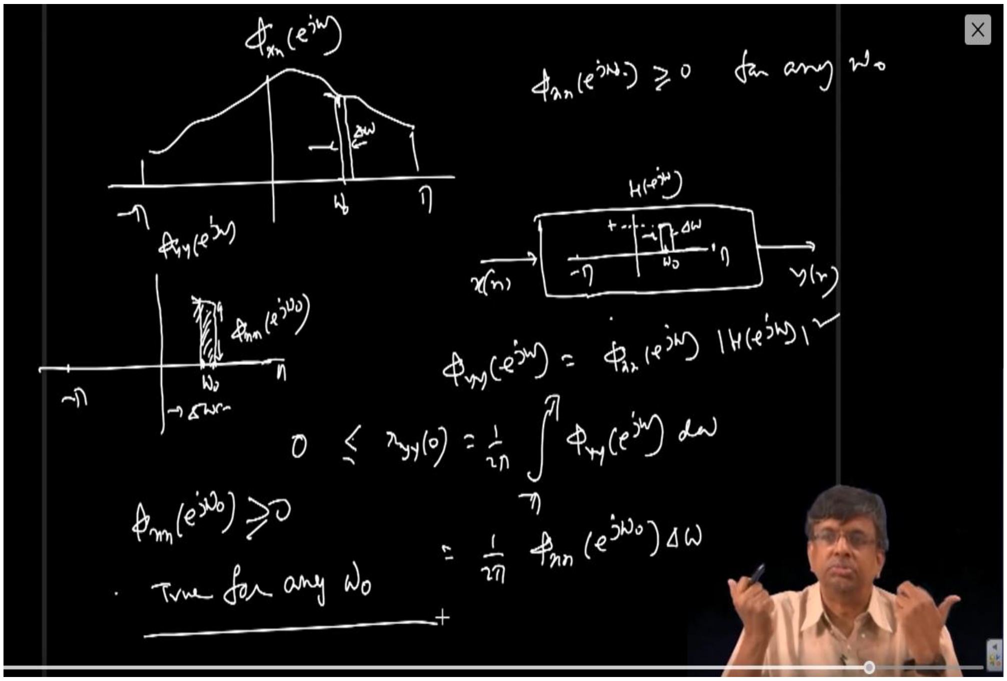
That is this power spectral density is non negative. So, suppose this is your power spectral density this is a point omega and you have to show that phi xx e to the power j omega you can call it omega naught, omega naught is greater than equal to 0 real and already same. So, it is greater than equal to 0 and for any omega naught.

That will mean that is always above 0 or equal to 0 this is non negative. So, omega 0 is given what I do I take a small segment of width delta omega. I have this random process xn whose power spectral density is this pass it through a band pass filter. h e to the power j omega which has just value 1 1 and center point is this omega naught and this width is delta omega this is pi minus pi and if I pass it through this output yn phi yi e to the power j omega will be input power spectral density times the mod h e to the power j omega square means It will be 1 square is 1 within this range otherwise 0. So, therefore, output power output power will be what is how will this look like phi yy e to the power j omega only at omega naught, it will be taking this kind of shape.

omega naught this kind otherwise 0 ok. You can for the time you can assume this to be a rectangle because delta omega is so small width delta omega is so small that the function hardly changes, So, it is approximately a rectangle. So, you have to just take the integral of these what is r yy 0 it is 1 by 2 pi integral this you have seen earlier phi yy e to the power j omega d omega right, but phi yy e to the power j omega d omega integral means the area under the curve. Now area under this curve is only this much ok and if it is a rectangle that area will be 1 by 2 pi height is because it is coming from phi xx ok. you are multiplying phi xx e to the power j omega at the point omega naught and at omega naught its value is phi e to the power j omega naught and you assume this value does not change because this delta omega is so small the function hardly changes.

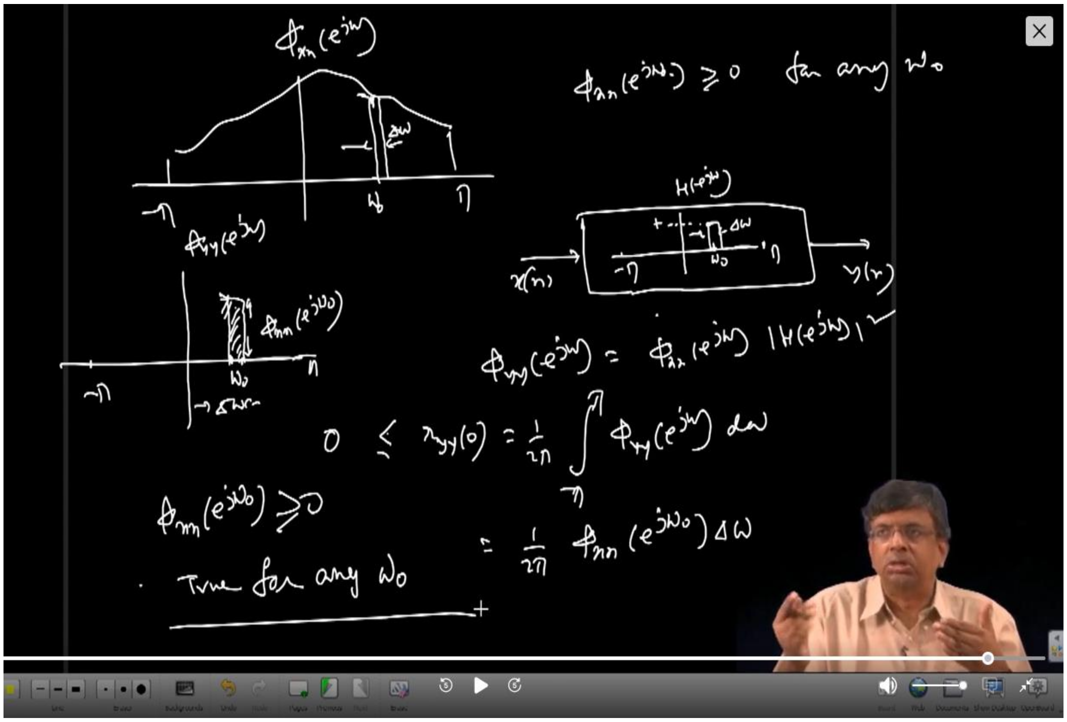
here phi e to the power j omega naught this height this is phi xx e to the power j omega naught. Because you are multiplying phi xx e to the power j omega by this mod square of the transfer function. So, elsewhere it is 0 only within this band its value will be high, I mean mod square of this is 1 because height is 1 1 ok. So, you are multiplying this by 1 at frequency omega naught and around that for a strip for a very small slot of delta omega infinitely small. So, value of this function at omega naught is phi xx e to the power j omega naught and it remains constant I am assuming it is a rectangle.

So, area will be this value phi xx e to the power j omega naught times this width. So, it will be phi xx e to the power j omega naught into the width only that much is area, but this is RYY0 we know RYY0 is a power average power it can never be negative. So, this now here right hand side 2 pi is positive delta omega positive means this is non negative is it not. So, this shows phi xx e to the power j omega naught greater than equal to 0 and omega naught which was arbitrarily.



So, this is true for all omega ok. So, this is for power spectral density estimation of power spectral density from random process data is a very big topic is called spectral estimation power spectral estimation. There are various approaches you know there is a order spectral estimation and many important signals are kind of analyzed spectrally most important of them is speech. A lot of research has gone into speech spectral analysis over decades and still going on. But I will not get into this because this is not a course on spectral estimation, But this is just come along with this adaptive signal processing sometimes we need.

So, that is why we built it up all right. So, with this background of random I mean probability density, joint probability density, conditional probability density, statistical independence, multiple random variables, correlation matrices, covariance matrices both for real and complex random variables. And then some matrix operations like Hermitian transposition and properties of matrices called Hermitian matrices all important properties including a special case of Hermitian matrix called positive definite matrices and its properties and then these random processes you know in particular white sense stationary random processes action of linear time invariant system on a WSS input what you get at the output that is also WSS what is the output power spectral density in terms of input power spectral density all these things we covered.



With this background we will now go to the next topic which is basically one step towards adaptive filter and that is called optimal filter all right. So, that I will do in the next class optimal filter. Optimal filter is half way through by this actually we try to estimate some unknown sequence from a given sequence just by some filtering of that if I have filtering of that.

So, we derive kind of optimal filter which will give a filter output. the best estimate for that unknown sequence, and then that will require knowledge of some statistical properties of or data on the input and that unknown thing which if not known will have to be learned will have to be adjusted you know from data only that brings us to training or adaptation where the filter will be learning itself and adjusting itself by you know learning from the incoming data and that will make it an adaptive system and in our case adaptive filter. So, that I do not want to you know start now because not much time is left. So, we will begin that we will start with that in the next class. Thank you very much.