Introduction To Adaptive Signal Processing

Prof. Mrityunjoy Chakraborty
Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture No # 11

Introduction to Discrete Time Random Processes

Ok, so we considered complex random variables and their dis-absorbing complex covariance matrix and the Hermitian matrices their properties and special kind of Hermitian matrix called positive definite matrix. For the eigenvalues are not only real they are positive and it is always invertible that is showed that it is reasonable to assume that correlation or covariance matrices in most cases are actually just not Hermitian they are also positive definite that we explained in detail. Now, I will go to a topic which I think you may be aware of that is called random process, discrete time random process, that is discrete time random process. Suppose as before I am carrying out an experiment and I mean recording the observation, but experiment is like this you know every time I conduct an experiment a waveform results. A waveform can be continuous analog I can sample it. So, it will be a discrete time one or it can be discrete time itself.

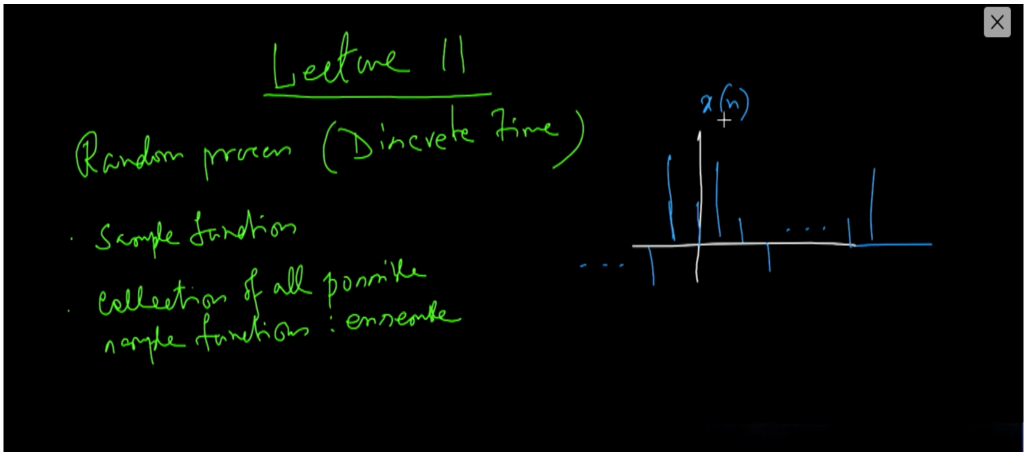
Like suppose I say iit and I record the waveform in this here and then sample it. So, I get I mean a block of samples there is a sequence. Next time I see the same make the same experiment say iit you are sure you will not see the same waveform ok. It is random you will say different waveform and so on and so forth.

But interesting thing is any of the waveform if you play into a loudspeaker it will produce the same sound iit. The waveforms are different isn't it very interesting. This because these waveforms they all belong to one particular class of random process or random phenomenon which is associated with the word iit. So, every time I make the sound iit I get different waveforms, but they all belong to the same category. So, this I mean phenomenon of generating I mean this you know random waveforms ok.

They basically I mean they are kind of you know I mean described as they are basically termed as random process. Like if I say x of n is a random process means x of n every time I observe there will be a discrete I mean I am talking of sequences. So, x of n is a sequence every time I make an observation there will be one sequence xn. Every such observed sequence is called a sample function ok and sample functions vary from experiment to experiment. So, xn as such is a random process and any observation of it will be a sequence which is called a particular sample function sample observation sample function.

xn as a whole will be called a random phenomenon and it is more popularly called as random process. I would rather love to call it random phenomenon. So, x of n is a random phenomenon like speaking the word iit and whenever I measure it observe it I see a waveform discrete time waveform which is a particular sample function. And if I collect all possible sample functions and in this case you know there will be infinite such sample functions then that collection this entire set is called ensemble. So, one is sample function and collection of all possible sample functions called an ensemble alright.

Sequences like this. It is just a sample function, it can be finite in duration, can be infinite in duration does not matter. Next time I observe I will take another I will find another discrete time waveform another sample function and so on and so forth, but each such sequence when played into a loudspeaker in that experiment in that example will produce the same word ok same sound iit. So, they all belong to one particular class of random phenomenon or random process ok. Now, here every sample is a random variable because if you consider this to be nth index this point.



Next time when I observe another sample function I am sure you will not see the same value this value will change it can be complex or real, but its value will change because every sample at every point is a random variable its random randomly fluctuating ok. So, that means, every sample itself is a random variable because if I make another observation of xn next time I will say totally different waveform. So, at nth point I will see another value of the sample ok not this. That means, if I restrict myself to a particular index n nth point remain there and keep observing the sample value I mean as I go on doing the experiment there is observe various sample functions one after another I will see different values scalar values for this sample. So, this sample will be sample that will located at nth point will itself be a random variable right.

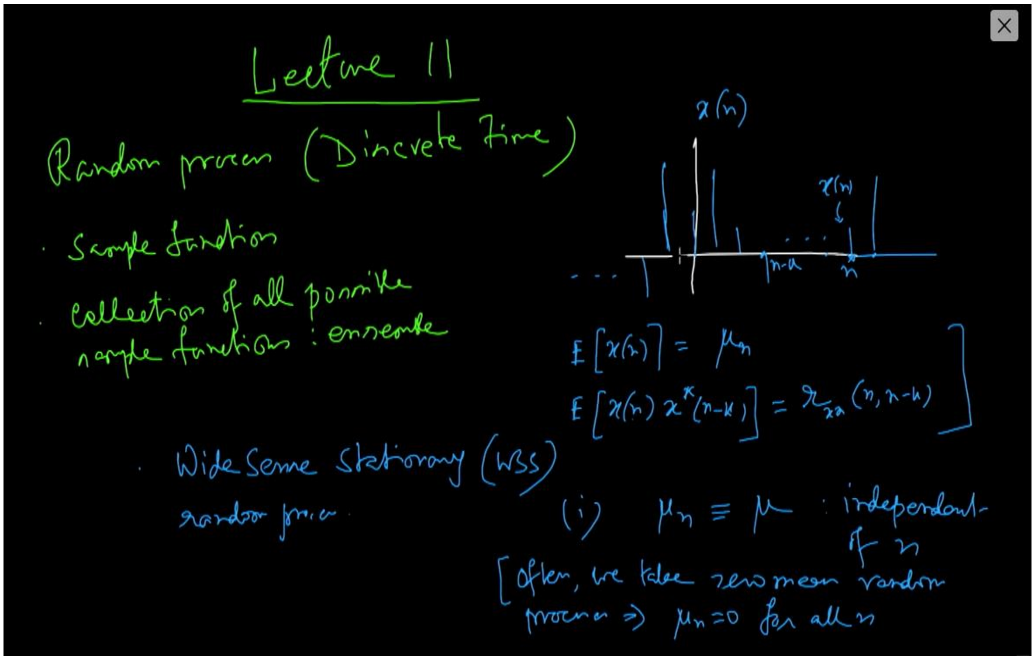
Now, suppose I take the random variable xn this is my xn suppose here it is a particular sample here there is a difference between these two xn this is the entire class random process and this is particular waveform particular sample ok. Those same notations are used when I say E of xn in a sample its random variable I take its expected value. So, expected value will be mu in general it will depend on where the sample is located because I I may not get the same mean at n plus 1 or at n minus 1 or n minus 2. That is why I should put an index n all right. And there is another thing correlation, suppose xn and I am talking of complex case to be more general.

So, in the case of complex case if one random variable is x another is y then we take x y star as the correlation right. So, x x is xn here or y may be another sample k k point away from nth point may be here n minus k if k is positive it could be right hand side also if k is negative ok. And this product x into y star expected value there is a correlation. Correlation also should in general depend on what are the locations of these two samples because I can pick up any other two samples take their correlation that will not be in general same as this. So, it will be dependent on r x x if I call x x it will be a function of both n and n minus k this point this point in general.

There is a class of random process called wide sense stationary WSS random process. Here first we assume mu n does not depend on n it is fixed at all point. It means the waveforms are fluctuating one after another as I do experiment after experiment or experiment. But the randomness the fluctuating property randomness ok is not having any bias on a particular zone of this axis. I will see the same degree of randomness at say this point nth point to same degree of fluctuation or randomness at n minus 1th point and so and so.

And therefore, the mean of that is this random variable here at nth point or this random variable here at n minus kth point that all that will be same because there is no bias on a particular time point or on a particular zone in this axis. You will find the it is the under this assumption that the randomness the property this property this is a fluctuation fluctuating nature randomly fluctuating nature. This is uniformly distributed all over this axis you will find the same you know behavior at nth point same behavior of fluctuation random fluctuation as at this point as at this point. Therefore, the mean if you make this assumption of course, this is an assumption we make ok. Therefore, mean should not depend on whether I am at nth point or n minus kth point or 0th point or minus 1th point.

So, it should become independent of n independent of often we take 0 mean random processes. So, this mean meaning mu n 0 for all n.

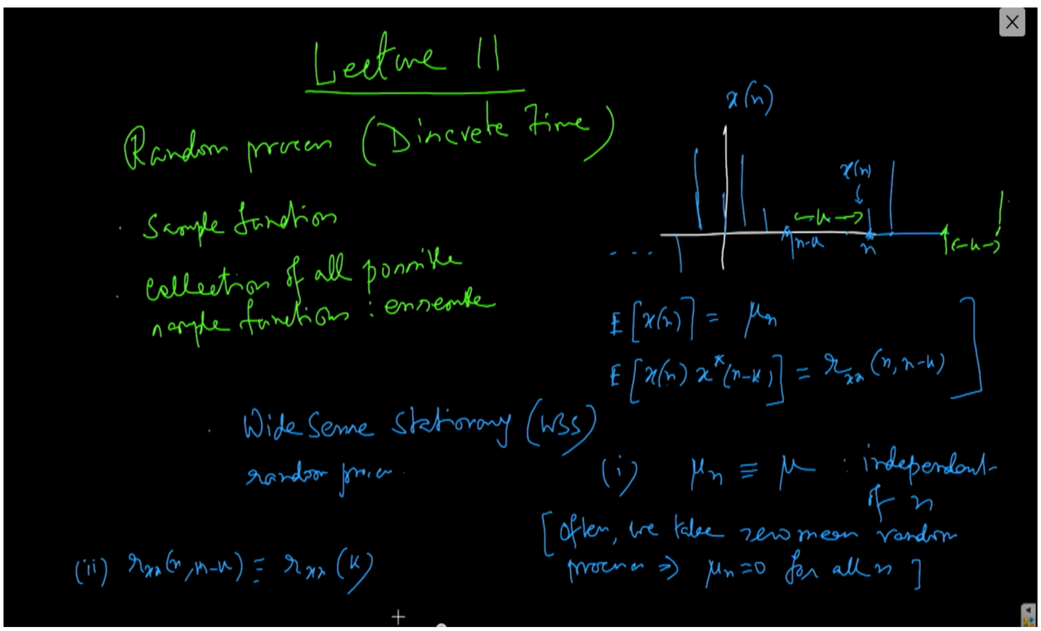


So, if mu n is 0 for all n that is that means, again mu n is independent of n because you may whatever n you choose whatever point you choose is a mean is same. So, that becomes a special case of this. So, often actually in practice we take 0 mean random processes which satisfy of course, qualify for this all right.

And making 0 mean is not a problem if it is really satisfying this kind of thing that mean is same at all points of time then it is just a question of subtracting mean from every sample. So, it becomes 0 mean every sample is 0 mean. So, it is not a big deal. So, this should be for this assumption to hold wide sense stationary random process mean should be independent of n mean of any point of time should be independent of n. And secondly, correlation correlation should depend on only how close or how far these two samples are ok.

So, what is the gap? If they are in general if they are close by correlation is naturally you know expected to be higher if they are far away correlation is supposed to be less. So, it should depend on just the gap. So, whether I take these two samples or whether I take two samples here if the gap is k like gap is k here between these two samples or between these two samples correlation should not change because correlation should depend only on the gap not on the absolute locations. Because again the correlation property is not localized on this axis ok. It is uniformly given uniformly distributed over this axis.

So, correlation between two samples at k point apart whether I see them here or elsewhere like between them I should be see the same thing. It is the gap that should matter not on the absolute locations ok. So, that means, in this case n minus k should depend only on k alright only on k. So, if these two assumptions are satisfied or you make these two assumptions then it is called wide sense stationary WSS random process. So, correlation depends only on the gap between the two sampling points not on the absolute values absolute locations and mean is independent of the sampling point chosen n ok.



It is same for all indices alright. One thing so, I come to the next page. So, now then I rewrite rxxk is nothing, but as we have seen E of xn x star n minus k it depends only on the gap. So, one thing is what is rxx0 rxx0 is E of xn x star n minus 0 that is x star n.

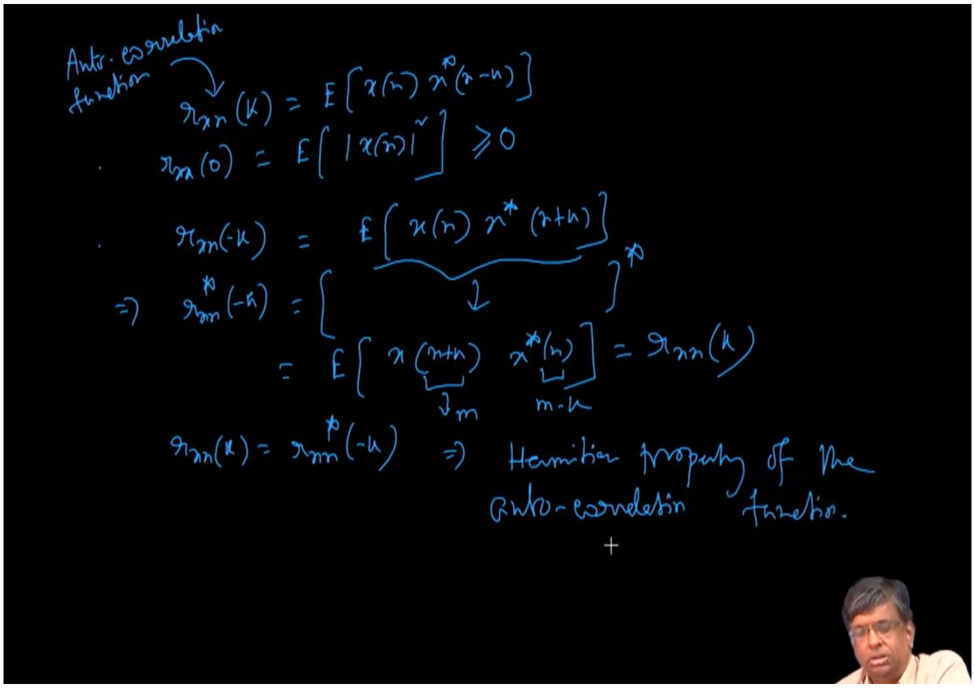
So, xn x star n ok. So, of course, mod xn makes it real square makes it and mod itself makes it non negative it cannot be negative square ok expected value. So, it is it is a power, total power, total expected power ok. Total sample is taken its mod value squared I am not subtracting mean here, but if the mean is 0 then it is also xn also is a increment around the mean. So, it is the incremental power expected which is the variance. So, for 0 is a total power and expected power for the nth sample ok and this should not be under WSS assumption this should not depend on n.

So, it is just for any sample total first this is power total power because I am taking a whole of xn I am not subtracting mean from it. So, mod of that square total power expected value that is rxx0 and obviously, that is greater than equal to 0 usually it is greater than 0, but equal to 0 means expected value of mod xn square 0 means in all trials xn takes 0 value then only if xn is 0 then only mod of xn is 0 mod square is 0 and if in all trial it takes 0 value then only expected value of that mod xn square can be 0 which means xn is a 0 random variable its value is always fixed to be 0 in all experiments that normally does not happen in our cases ok and that is why it is usually greater than 0. Anyway this is one property of the this called auto correlation function this is called auto correlation. And another thing what is rxx minus k it is E of xn x star n minus k then what is rxx conjugate minus k. So, you have to take conjugate this entire thing here conjugate outside, but and this is a complex number right xn is complex this complex complex number if you call it z.

So, E of z star we have seen earlier is E of z star. So, it is nothing, but E of if you call this product as z, E of star of that and star of that is means star on this star on this. So, x star n x star n I write on this side and star and star cancels. So, x n minus n sorry this is plus k n minus k.

So, it was minus k is minus k. So, if it is minus k it becomes plus k all right if n plus k is a point you call it m. So, this is n is n plus k is m n plus k is m. So, this is m minus k. So, it is E of x of m x star m minus k. So, again because it is WSS it does not depend on the absolute location m and m minus k it only depends on the gap between the two points.

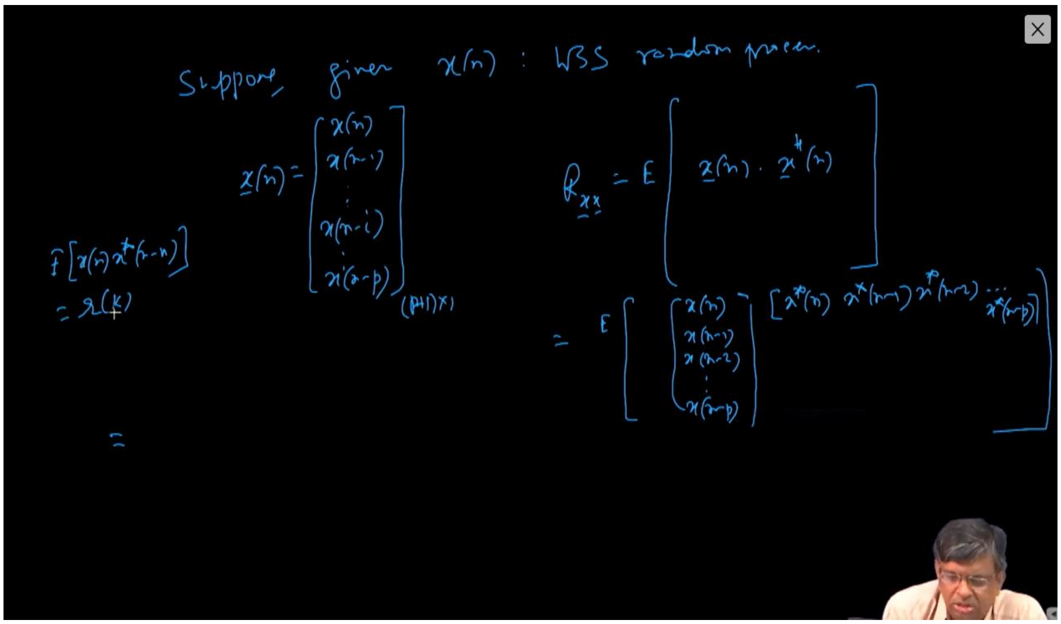
The one without star take the point to be m the one with x star take the point to be say m minus k and then subtract this from this that is the gap. So, m minus bracket m minus k which is k. So, again it is nothing, but the autocorrelation at k. So, I write this way rxxk this I write on the left hand side is same as rxx star minus k. This is called the Hermitian property of the auto correlation function.



What is the consequence? Suppose given xn to be wide sense stationary random process. Let xn be a vector for system of this random variables x of n n minus 1 dot dot dot maybe n minus i dot dot dot maybe n minus p. So, n minus 0 n minus 1 n minus i n minus p. So, total p plus 1 that is p plus 1 cross 1 is a random vector. So, corresponding correlation or auto correlation matrix rxx will be as we have seen already and again after e I am not putting anything under subscript it is up to you to understand.

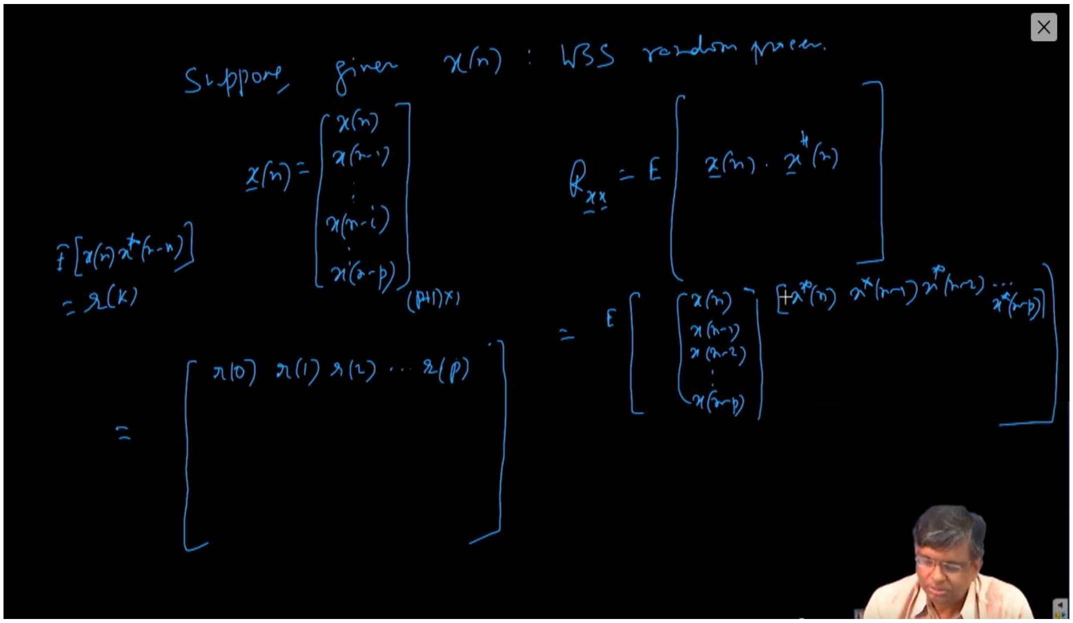
So, it is xn that is this vector into its Hermitian. So, row column vector then it becomes row after transposition and all elements conjugated. This is the meaning of rxx. What is the rxx shape of rxx now? So, start with x of n I am writing it here x of n minus p and then x of n star x of n star n minus 1 maybe I write n minus 2 also x of x star n minus 2 dot dot x star n minus p. Therefore, first see xn this will be matrix.

So, 1 comma 1 is element will be xn into x star n expected value of that xn into x star n is we have already seen this is a variance e of what is your notation axn x star n minus k this was r I use that subscript xx for the time being to avoid you know clumsiness and all that let me drop that subscript just rxxk rk that xx I am dropping because otherwise this place will be filled with rxx and you know it will be clumsy.



So, just rk I am using as a notation rk. So, if k is 0 then xn into x star n e mod xn square r which is r0 and r0 will be the total power which is real non-negative and usually positive. So, that will be xn into x star n there is mod xn square e of that. So, this will be r of 0 then xn x star n minus 1 expected value that is r1 then xn x star n minus 2 ok.

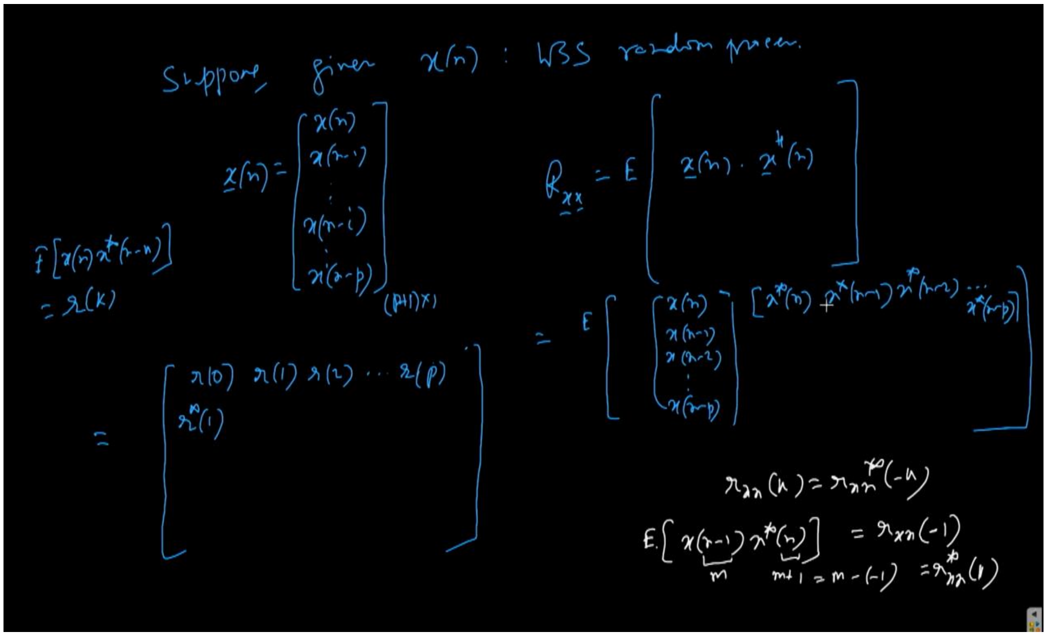
So, that will be r2 dot dot dot dot see n is disappearing from here because of wide state stationary assumption because it depends correlation depends only on correlation or auto correlation depends only on the gap k. So, gap is 0 gap is 1 gap is 2 n n n n minus 1 n n minus 2 like that. So, lastly xn into x star n minus p it will be rp. Now xn minus 1 x star n xn minus 1 x star n what it will be? It will be this is if you call it mth index. So, this is this will be what m plus 1 or if you call it mth m plus 1 and then star ok.



Let us see here x we have already seen in the previous page the Hermitian property sorry the Hermitian rx k and rx star minus k they are same. We have seen. So, x now we have got this x n minus 1 x star n E of that alright. So, if you call the index m n is m plus 1. So, this m plus 1 ok m plus 1 meaning m minus minus 1.

So, x of n x star m minus k k is minus 1 here alright x of n x star m minus k and k is minus 1. So, this is your rxx minus 1. So, rxx minus k if I make it, it becomes rxx star k. So, this is rxx star k 1. So, I am not using the xx subscript here for avoiding clumsiness you know because too congested.

So, x n minus 1 x star n it will be nothing, but r same thing this has to be because after all this correlation matrix is Hermitian we have seen. So, i comma j then 1 comma 2 and 2 comma 1 those elements must be you know conjugate of each other conjugate symmetry and that is what is happening r star 1. Then x minus x n minus 1, x star n minus 1 So, index is n minus 1 if you call it m x m x star m. So, mod of x m square expected value which will be again r of 0 whether m m here or n here does not matter x m x star m k is 0 or there is x n minus 1 x star n minus 1 or x n x star n k is 0. So, it does not change because gap is 0 what matters is the gap.

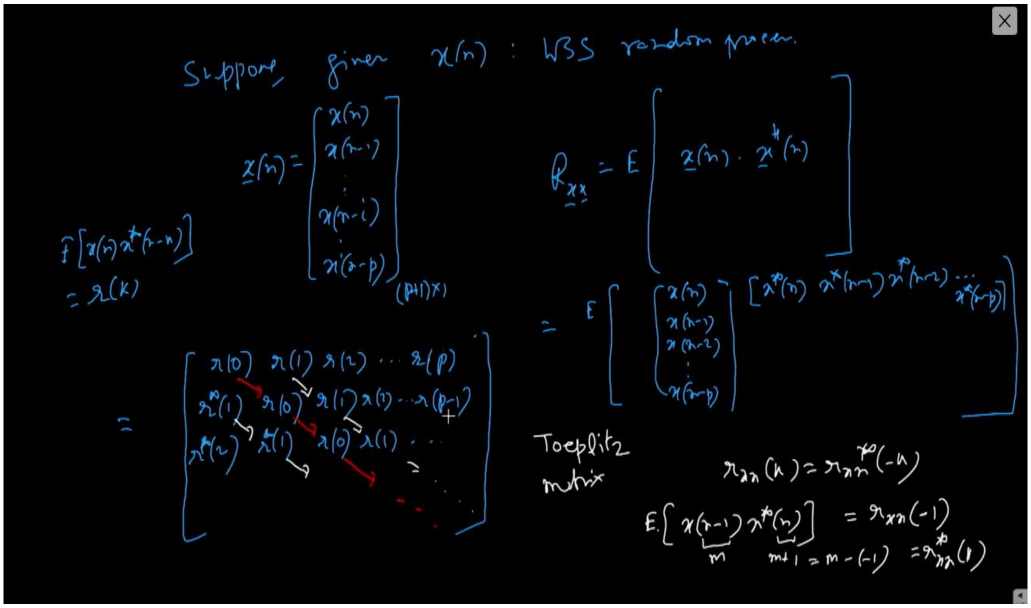


So, it will be rxx it will be r 0 because n minus 1 n minus 1 no gap. So, gap is 0. So, there will be again r 0 same element then n minus 1 x star n minus 2. So, n minus 1 n minus 2 now gap is plus 1 all right. So, it will be r 1 next element would be you know if it is x star n minus 3 n minus 1 n minus 3 gap is 2.

So, it will be r 2 dot dot dot dot. So, n minus 1 and n minus p. So, gap is p minus 1 n minus 1 minus this this index n minus 1 minus this index n minus p. So, if you subtract it because p minus 1. So, r p minus 1 then again x n minus 2 x star n and by the same logic here n minus 2 if it is n minus 2 and x star n ok. If you call it m it is m plus 2 which is m minus minus 2 ok.

So, it becomes rxx minus 2, but rxx minus 2 is rx star plus 2 if it is minus k because plus k rx star plus 2. So, again from the Hermitian property also you know if it is 1 comma 3th element then 3 comma 1th element will be conjugate of that. Then n minus 2 n minus 1 in the same way n minus 2 n minus 1 if you call n minus 2 as m. So, n minus 1 will be m plus 1 and same thing will follow. So, it will be this r star 1 then n minus 2 n minus 2 no gap.

So, it will be again r0 then will be n minus 2 n minus 3 now it will be r1 dot dot dot dot. So, you can see one thing along the diagonal same element is propagating then along this diagonal sub diagonal r star 1 r star 1 r star 1 is propagating. Similarly, along these r1 r1 r1 is propagating. So, if you know one row entire matrix can be you know cos factored. Any matrix where you know of this form that elements you know just go on elements just repeat along a diagonal and again along each sub diagonal there is called toe-plitz matrix.



So, along the diagonal r0 propagates along next sub diagonal r1 propagates here r2 propagates like that here again r star 1 in a general top of this matrix it may not be r star 1 it may be any same thing like you can be a b c dot dot dot. So, a a a a b b b b b if it is c c c c c c if it is d d d d d d and like that, but here along with toe-plitz this hermitian also say r1 if it is r1 it is conjugate only coming here and then propagating. So, it is an additional property on this. So, these are very important property of the correlation structure correlation matrices of discrete time random sequences.

So, we will continue from here in the next class. Thank you very much.