Dynamics of Physical Systems
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Lecture - 5
Using the Lagrangian Equation to Obtain Differential Equations
(Part - II)

In the last class we have in the main seen that the differential equations of one dimensional systems. We could see how these are obtained from the Lagrangian equation.

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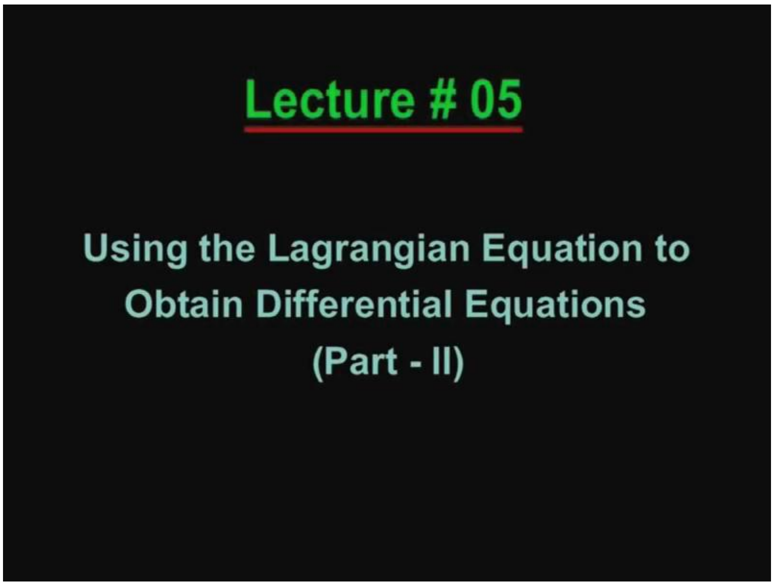

Let us now go to one stage higher to the two dimensional systems. Two dimensional means two spatial dimensions.

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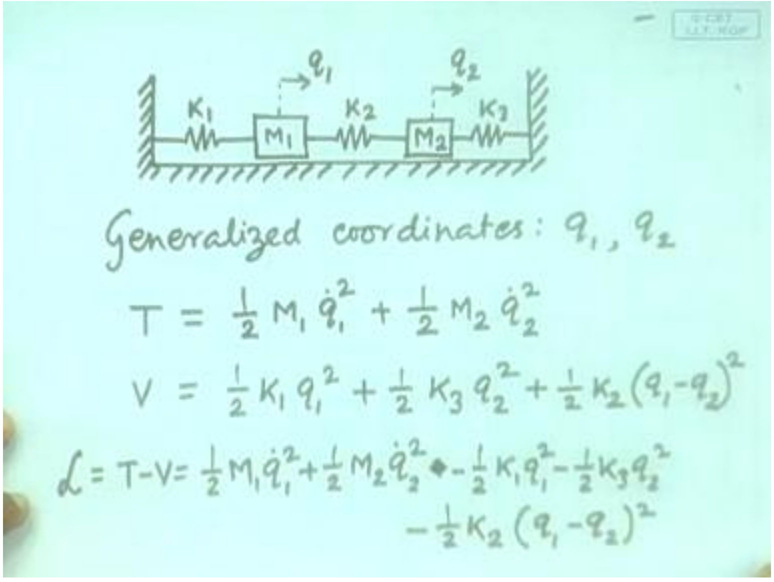

Now, let us take as an example a simple 2D system. Here is the constant a flat plane with two walls and there are two masses, they are linked with a spring and these are also connected to the walls by means of springs. And for the present moment we consider these as frictionless surface. In that case how we will define the generalized coordinates, can you suggest.

Simply the positions of this, so this is a q 1 and this is a q 2 measure from some kind of a data, see here. The moment we have written down these two as the generalized coordinates. So, let us write now in terms of these two generalized coordinates we need to write down the kinetic energy and the potential energy. So, let us first write down the T pretty simple, because this has say a mass M 1 and this has mass M 2. Let this be K 1, K 2 and K 3. Then the kinetic energy is half M 1 q 1 dot square for the first one plus half M 2 q 2 dot square, simple. The potential energy V, the potential energy in this spring will be half K 1 q 1 square, this one is ((Refer Time: 03:59)) q 1 minus q 2, any question?

Student: ((Refer Time: 04:29))

Here there is a question. Let us for the sake of simplicity assume, that for the this spring q 1 is measured from it is unstressed position. And for this spring q 2 is measured form it is unstressed position. You could assume anything otherwise what will happen is that, this will have some constant terms which on differentiation will vanish. But, this is easy way to write that is why we assume that. So, her question was where to assume the data, that we could assume anywhere, but that would be a convenient way of doing it. So, the Lagrangian function is T minus V, that would be half M 1 q 1 dot square plus half M 2 q 2 dot square minus half K 1 q 1 square minus half K 3 q 2 square minus half, is it visible K 2 q 1 minus q 2.

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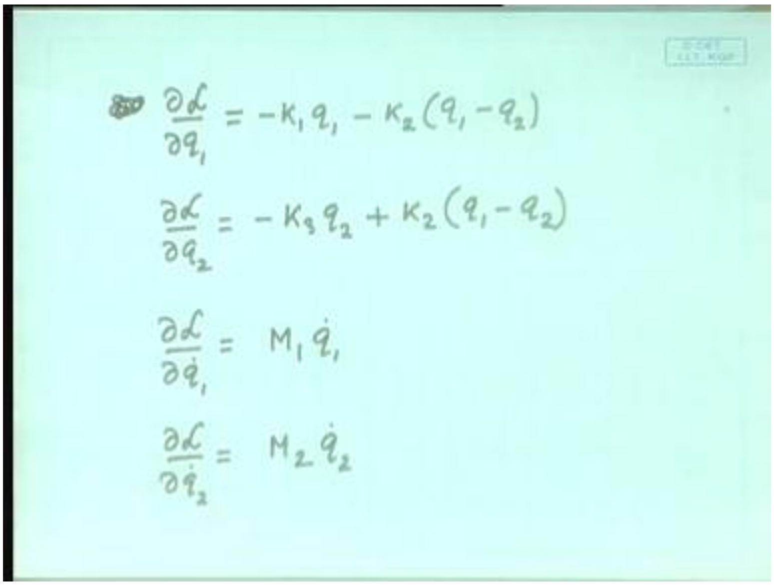

Now, take the derivatives, the Lagrangian which is split to say q 1, we will have to write the Lagrangian which is split to q 2. We will have to write the Lagrangian which is split to q 1 dot and we will have to write it in terms of do this. From here it is pretty simple and fair, you can see that the first one is, ((Refer Time: 06:57)) this one would be minus K 1 ((Refer Time: 06:58)) and...

Now, here it would be minus K 3 q 2 plus here K 2 q 1 minus q 2, the which respect to the generalized velocity, it is simple M 1 q 1 dot and M 2, these are the momentum can you see that q 2 dot. The moment you have written down this, to remember there would be two differential equations coming out of the Lagrangian formulism, one along q 1 another along q 2.

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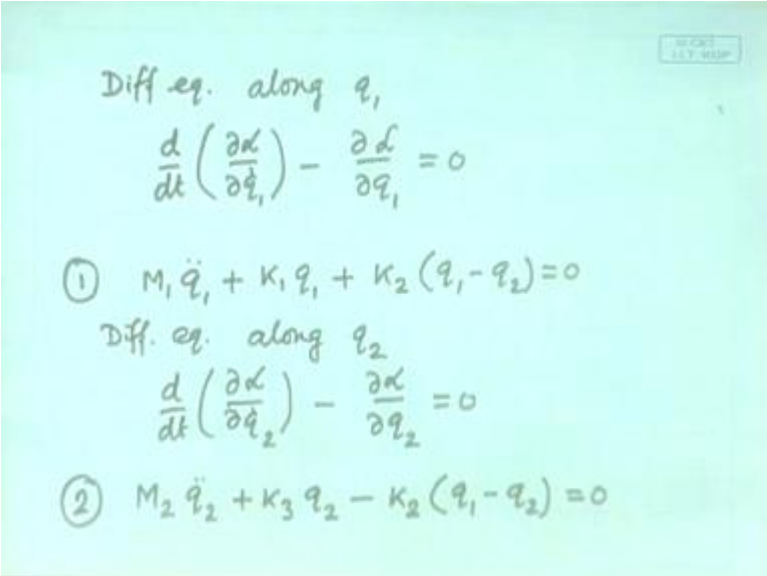

So, the differential equation along q 1 would be d d t of dot minus fine. Now, if you write down it will be substitute M 1, this fellow would be q 1 double dot, because you are taking derivative with respect to time once again. And then this would be minus let me copy from here this term plus K 2 q 1 plus K 2 q 1 minus q 2. Differential equation along q 2 will be notice the advantage of this formalism, that simply by substituting q 1 q 2 you get the accounting differential equation.

It will be just substitute this with change this to the equation with respect to q 2 dot minus Lagrangian with respect to q 2 is equal to 0. Substitute you get M 2 q 2 double dot plus K 3 q 2 minus K 2 q 1 minus q 2, so this is the first equation and this is the second equation. Along two different detections simple, we did not have to bother about individual bodies. And you can easily see that, if there where n number of bodies still we could do this same procedure pretty simply. Let us take another, we have already started doing 1, 2, 3 problems the last day, where if let us see do we have I lost the paper.

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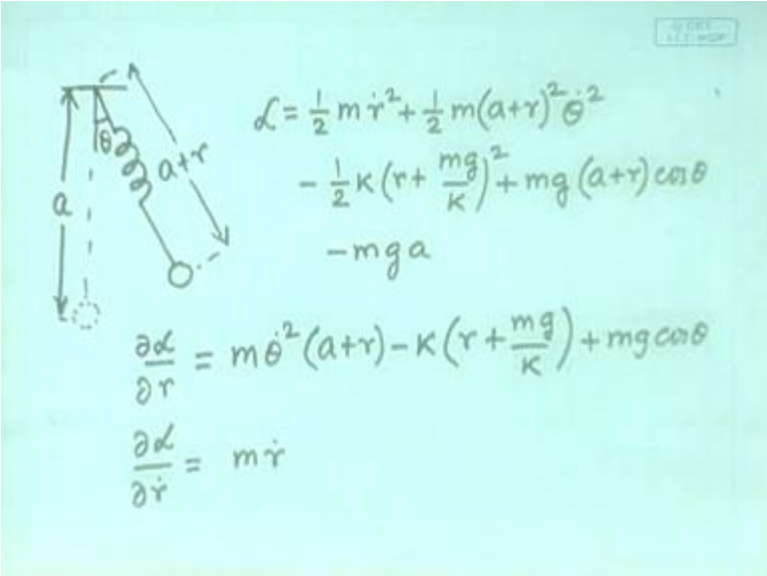

So, that day the problem was we took a spring pendulum. And our variables were theta and if the total when it hangs vertically, if the position is like this, then this distance was a. And here this distance was a plus r, that is how we have defined r and theta are the coordinates. And then we have already written down the Lagrangian function. Just recall the Lagrangian function in this case was half m r dot square plus half m a plus r whole square theta dot square minus half K r plus m g by K

We have all done these earlier, so I am not explaining, I am just writing down, so that we can start up from there. Plus I just illustrate something that students often make mistake, that is why I am doing this problem to it is completion. Cos theta minus m g a, that is what just check.

Student: ((Refer Time: 12:39))

Third term.

Student: R plus m g the whole square.

This is a square, that would be a square. So, now you start writing the derivatives with respect to the two coordinates. The first derivative that would write is Lagrangian with respect to r. When you write in terms of r, this has to be differentiated, this has to be differentiated, this has to be differentiated. So, how do you do it, you write it carefully 2 comes forward, so this m theta dot square a plus r differentiated, so it should be a plus r.

Then, because this also this one comes to this. Then this one it will be minus K, this thing will be different r plus m g by K and then from here plus m g cos theta. M g a basically ((Refer Time: 14:25)), so this is with respect to r. Then with respect to r dot, what do you have these remains, so m r dot enough. So, you can see from these two I can write down the first equation.

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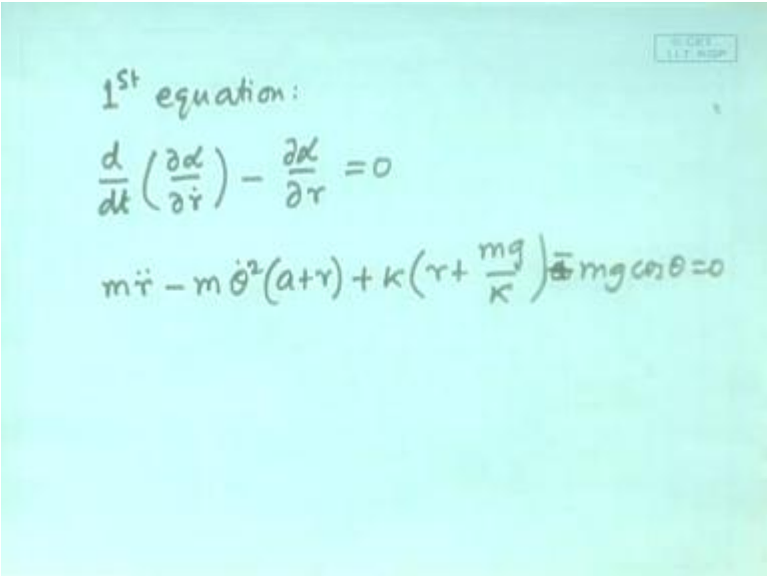

And that would be d d t of dou Lagrangian dou r dot minus dou Lagrangian it would be r equal to 0, so substitute the first term will come to m r double dot, no problem about it. Then the next term simply substitute it, minus m theta dot square a plus r plus K r plus m g by K minus cos theta equal to 0 done, first problem completed.

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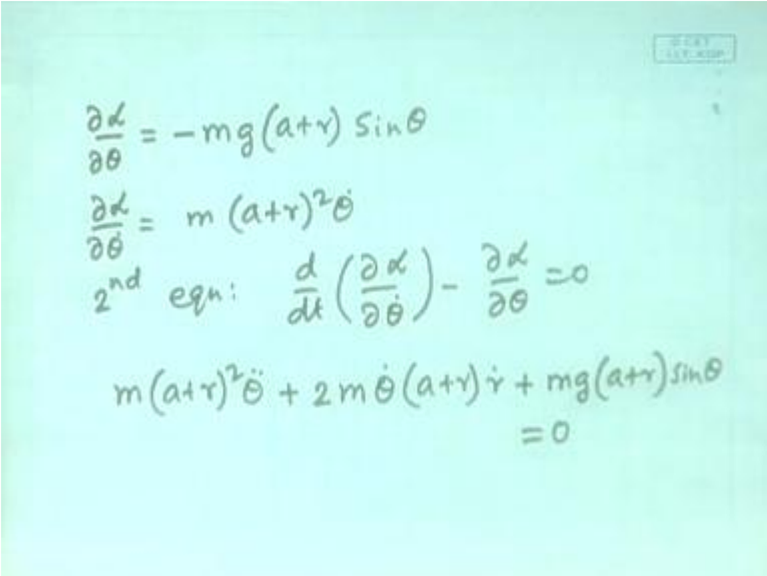

Second we write the same things with respect to theta, so Lagrangian doe theta will be well this was the form. ((Refer Time: 16:05)) This vanishes, this vanishes, this vanishes this is only here, so it is pretty simple. What remains only this term, it would be minus because cos to sin m g a plus r sin theta that does it. Then this term is put a dot [FL] simple, so now we have to write the second equation will be d d t of dot minus doe theta equal to 0. Now, this has to be differentiated, if you differentiate what do you have?

Student: ((Refer Time: 17:23))

So, differentiate this with respect to d d t, you take d d t, since these are both variables, so you have to do it carefully, that is what I want you to point out. So, you have by differentiating this, you have

Student: ((Refer Time: 17:52))

M a plus r whole square theta double dot plus, then you differentiate this one, so you have twice m theta dot a plus r a plus r into r dot into dot r, minus then becomes plus m g a plus r sin theta equal to 0. Now, this is what students often I have found in exams miss, here that both these are time variables. And therefore, when you differentiate with respect to time, both have to be differentiated, that is what often people miss. And I have seen hundreds of answer scripts saying that this times theta double dot.

No it is not, that is one point that is why I did this problem to its completion. So, these two are the differential equations of this system. So, second equation is this first equation was this, notice that these equations are hopelessly non-linear. Can you see these are not linear equations really. So, in such simple systems also you are having non-linear equation that we will deal with later.

Now, we need to talk about electrical circuits because, so far we are dealing with mechanical systems. But, we started with saying in our earlier class, that the electrical systems are equivalent to mechanical systems, you can have almost exact equivalence. So, you might ask, then it should be possible to do solve write equations for electrical circuits exactly the same way, yes that is true.

But, in that case we what the steps were, step number 1 was that define the generalized coordinates that are consistent with the constants. In a mechanical system we easy to see the constants a hanging ball with the wire I can easily see the constant. In the electrical circuit do you see the constant, where is the constant, no does not supply voltage is not the constant. Supply voltage is similar to a force, that is not a constant.

The constant is essentially the where the circuit is wired up, circuit is connected. Some things are connecting series means the two branch currents are forced to be the same that is the constant. Two lines are connected in parallel the voltages are forced to be the same, that is the constant. So, the point is that in electrical circuits the constants are the way the circuit is connected, that impose the constants. But, now our job is to define the minimum number of position coordinates.

That is what we did in mechanical system and what is the equivalent of position in electrical circuits charge, so we have to deal with charge, charge is equivalent to the position. So, charge becomes the configuration coordinate in electrical circuit, but charge wire. Again the rule was that I have to define the position coordinates, the minimum number of position coordinates that uniquely define the position less status of the system. That was what we did, in this case also we have to think in terms of the circuit is connected this way. What are the different charges that I need to consider as the minimum number of coordinates.

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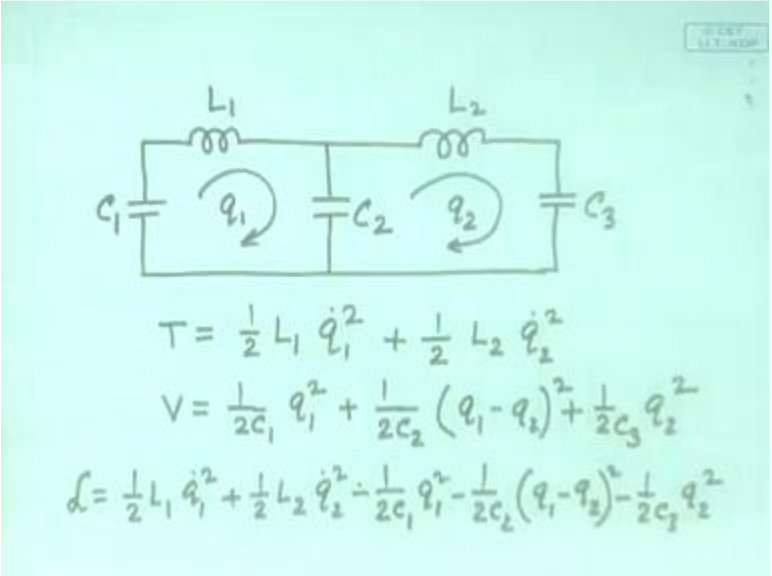

Say suppose the circuit is like this, there is an inductor, there is another inductor, there is a capacitor, there is a capacitor and there is another capacitor. Then there is a charge flowing here, there is a charge flowing here, there is a charge flowing here and what are the minimum number. Now, there anything could be taken as the minimum number, but from your 1st year knowledge of electrical circuits course you probably have learnt, that one simple possible solution to this problem is simply to consider the charges flowing in the loops like so.

At least it is guaranteed that they would be the minimum number of independent coordinates, that is what actually matters for us. So, let us say this is q 1 is here and q 2 is here simple. Q 1 is here q 2 is here, say this is L 1, this is L 2, this is exactly why we did not write the Lagrangian as L as in written in many of the physics text books. We wrote it with a script L, because we wanted to distribute only inductance, so that students do not get mixed up. C 1, C 2, C 3, in this case also we will have to write down the kinetic energy and the potential energy. What is the equivalent of the kinetic energy in a electrical circuit?

Student: ((Refer Time: 24:04))

No not the charge, the energy stored in the inductor that is the equivalent to the kinetic energy and the energy stored in the capacitor is the equivalent to potential energy, because the capacitor has the character of being a complaint element equivalent to a spring, that is why we have to do it that way. So, if that is so then you have your T in this case half L 1 q 1 dot square, q 1 dot is nothing but, the means current plus half L 2 q 2 dot square

V the potential energy, notice since we are writing the potential energy as V all the potentials are applied electromotive forces we always we will write as E not V. Because, we do not want to have duplicity of notations V, V is the charge energy stored here plus energy stored here plus the energy stored. If my energy is here, how much is the energy here, 1 by 2 C 1 notice, here the C is downstairs in the denominator. 1 by twice C 1 into q 1 square plus 1 by twice C 2, how much is the charge flowing to this q 1 minus q 2.

Q 1 minus q 2 square, how much is the energy stored here, 1 by 2 C 3 q 2 square. When we have written the T and V this way, we can write the Lagrangian as half L 1 q 1 dot square plus half L 2 q 2 dot square minus T minus V 1 by 2 C 1 q 1 square minus 1 by 2 C 2 q 1 minus q 2 square minus 1 by 2 C 3 q 2 square. Now, differentiate same procedure, strangely you will notice that it is yielding exactly the same equation as this, see here my can you see all of them ((Refer Time: 27:00)).

Here my T is half L 1 q 1 dot square half M 1 q 1 dot square, half L 2 q 2 dot square half M 2 q 2 dot square exactly the same, only M and L are interchanged. Therefore, the kinetic energy in this case is not only are conceptually the same, even in magnitude are the same, if provided that you were expressing M and L in the same quantities, whatever the units are. V half q 1 square 1 by 2 C 1 q 1 square half K 3 q 2 square 1 by 2 C 3 q 2 square exactly the same, only K 1 is equivalent to 1 by C 1, K 2 is equivalent to 1 by C 2 and all that.

And therefore, even without writing the differential equations here I can tell you pretty blindly, that it will be the exactly the same differential equations as this ((Refer Time: 28:10)) that we have already obtained. So, I do not really need to write, because these two systems are equivalent of each other.

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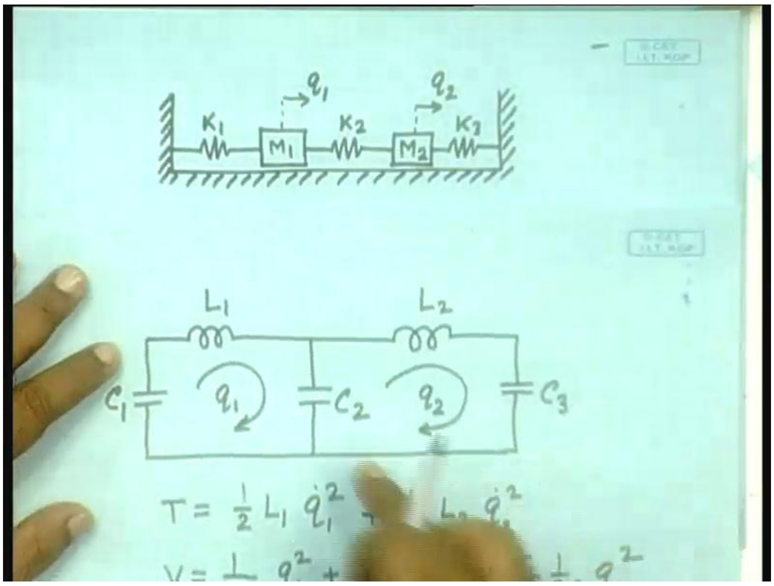

These two systems are equivalent of each other. Equivalent is what sense, equivalent in the following sense, that here what will happen if you say move this one and release, it will oscillate, it will transfer the oscillation here, if this will also oscillate, so there will be some dynamics. And the dynamics you will learn how to actually obtain the dynamics and plot it, but it is not difficult to see that there will be dynamics q 1 and q 2 will vary and you can draw the waveforms q 1 and q 2.

In this case you are not really moving something and releasing, what would be the equivalent of moving M 1 and releasing. What is the equivalent here?

Student: ((Refer Time: 29:04))

Yes, position is changed which means the K 1 is moved, K 1 is moved means some energy is stored initially in q 1, which is equivalent to some charge being stored initially in C 1. So, store some charge and switch it on you will see that this system will also oscillate. The currents and voltages will also oscillate, exactly the same way as this one and then if the L 1, L 2, L 3 and C 1, C 2, C 3 are numerically the same as these, they will be the exactly the same dynamics.

When it is excursion q as excursion reaches the maximum point, this current through the inductor it is or charge through to the inductor. It is excursion also reaches the same point, exactly the same dynamics. That is why the electrical and mechanical system would be same to be system wise not only element wise, system wise equivalent. So, far we were considering systems that had only one type of potential, either it was a we have considered the system with gravitational and potential. We have considered systems where the potential is represented by a spring and the energies in spring. What about applied force, if there is an applied force can it still be is still a conservative system, what was the definition of conservative system we said that…

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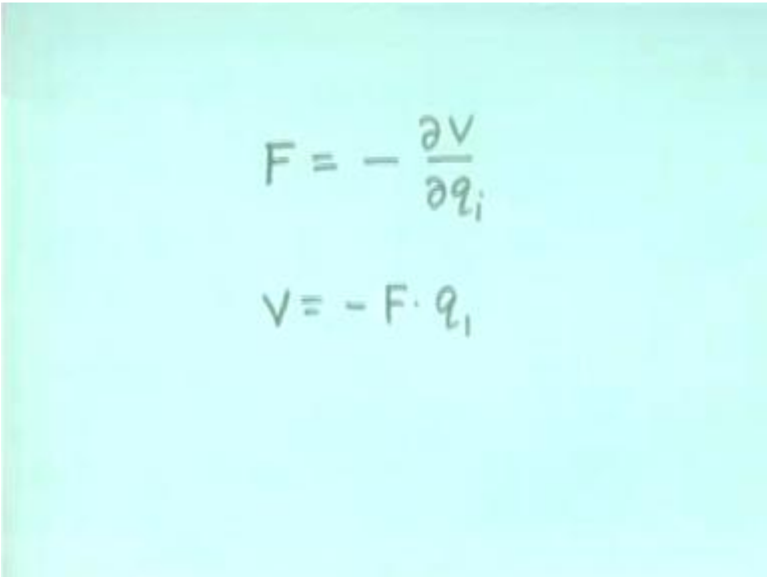

If you can write the force as minus the derivative of potential function with respect to, if you can do that then we say that it is a conservative system.

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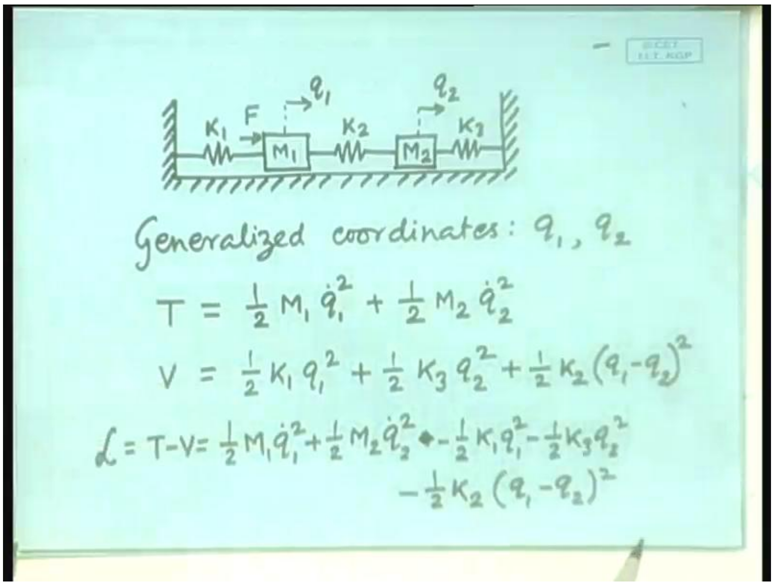

Now, if you imagine here on this, suppose I am applying some kind of a force here F, that force could be a sinusoidal forcing function or something like that. But, some would be applying a force here. Can that force be somehow included in this potential, so that ((Refer Time: 31:32)) this is valid, that is true. That means even if there is a externally impressed force, then also the system is conservative.

In the sense that it can be expressed in this form but how, in that case the rule is that you have to add to the potential function. Force times the direction of the generalized coordinate, that particular generalized coordinates along with it is applied. So, in this case we will write V is equal to F times q 1 with a negative sign, why you will see that it is necessary in order to set the sign, if you write it this way the V due to the applied force is this much. There we have already written the rest of the V function, we will only add if this is applied, let us do it separately.

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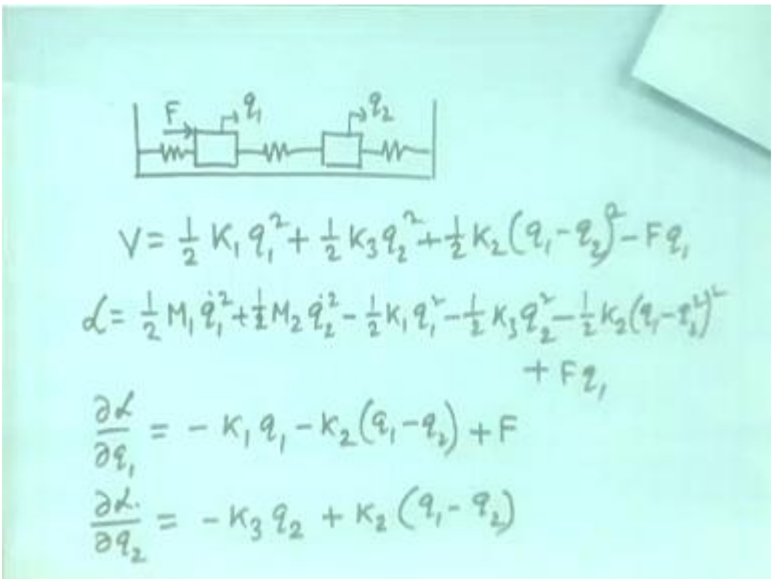

Now, we are talking about the system where and there is a force applied here, here this q 1 here is q 2. Then what, we have already written the rest of it, so let me write V is equal to K 1 q 1 square plus half K 3 q 2 square plus half K 2 q 1 minus q 2, this is that and this remains square, this remains. What gets additionally added to this potential function, so that the total force is still obtainable from that potential function by simple derivative.

What I said is that here you have to add minus F times q 1 that is it, if you do so let us see what happens to the Lagrangian equation. The Lagrangian would be the kinetic energy remains the same, so I will directly write the Lagrangian. It was half M 1 q 1 dot square plus half M 2 q 2 dot square minus half K 1 q 1 square minus half K 3 q 2 square minus half K 2 q 1 minus q 2 square plus F q 1. Because, this is minus it becomes plus.

Now, when we write the Lagrangian with respect to q 1, what you had now, here minus K 1 q 1 here, what was it earlier, let me just refer to that. Here earlier it was this minus K 2 q 1 minus q 2. Now, this will be added to that, so it would be plus F, but that will not happen if you differentiate with respect to q 2, so that remains the same minus K 3 q 2 plus K 2 q 1 minus q 2. Now, you write the first equation first differential equation. These two remain the same as earlier.

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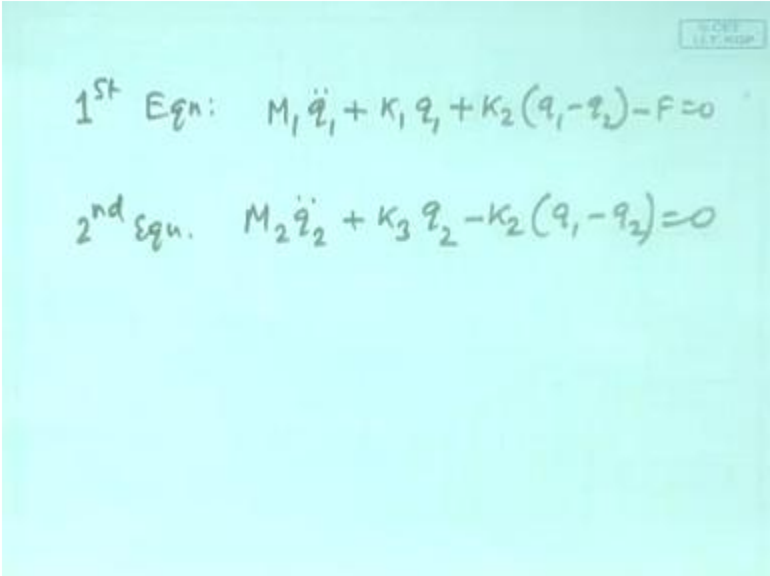

It will be M 1 q 1 double dot as usual, but then this one has to be written, it will be a plus K 1 q 1 plus K 2 q 1 minus q 2 minus F equal to 0. See that the signs have been right, because this is mass into acceleration, this is the force applied on it by the springs and this is the force applied externally. So, that is how the equation is written which is right, the second equation remains exactly the same. It would be M 2 q 2 double dot plus K 3 q 2 minus K 2 q 1 minus q 2 without any difference.

So, it is trivial to take into account the external forces. What will happen if we want to do the same thing in an electrical circuit, external force is equivalent to a better externally applied EMF. And what would be the equivalent of this ((Refer Time: 38:03)), a force being applied in the q 1 direction.

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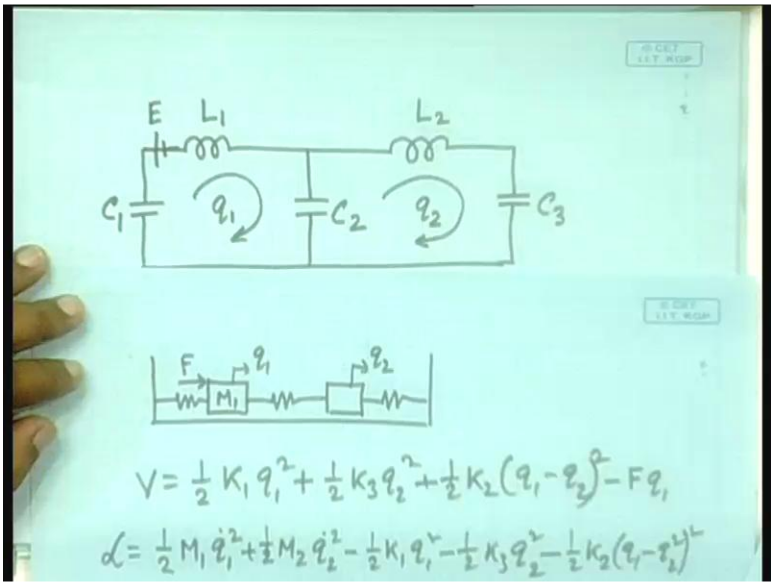

What is the equivalent to that?

Student: ((Refer Time: 38:09))

A source notice here, here is the M 1 which is equivalent to the L 1 and what is the character of this force, this force share the same velocity with M 1. And in order for that to happen in the electrical circuit you have to apply a battery here, that is it. So, if we really have a battery here and it is applied voltage is E, in that case the equation and the process of derivation of the equation will exactly would be the same. So, in that case what we will do, we will argue you that.

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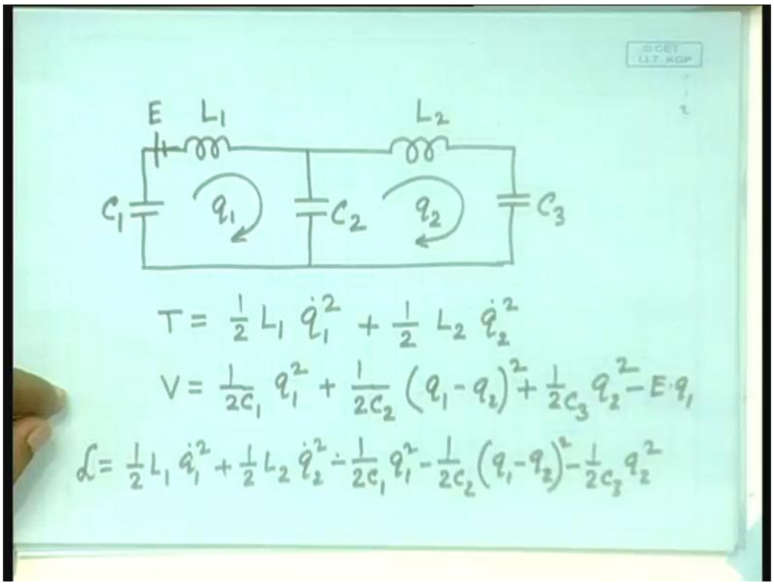

Here E is applied in the direction of the generalized coordinate q 1. And therefore, with the V we will have minus E times q 1 that is it, so that is how we will write the differential equation. Let us solve a relatively difficult problem which has an applied force on it, and which is the very practically relevant problem. Slowly we are going into engineering problems, have you seen ever launching of a spacecraft, a rocket for example.

You must have seen in the in the TV, so how does it happen, the spacecraft is launched alright, but initially it has to be held there. And it is held there with the help of braces, if the braces are there and it fires the whole thing will break off. So, that cannot be allowed, so the braces have to come off, if it comes off then the rocket will fall. So, how to prevent rocket from falling, what happens is that down there is actually a vehicle like thing.

There is actually a vehicle like thing which is allowed to move and that moves and there by keeps it vertical, it is like a inverted pendulum. The pendulum is vertical and here this point is allowed to move and it is actually a control system enabling which you keep it vertically and then you fire. That why it remains vertical and then it goes off that is how the rocket launching is normally done. And to model such a thing let us model the inverted pendulum problem.

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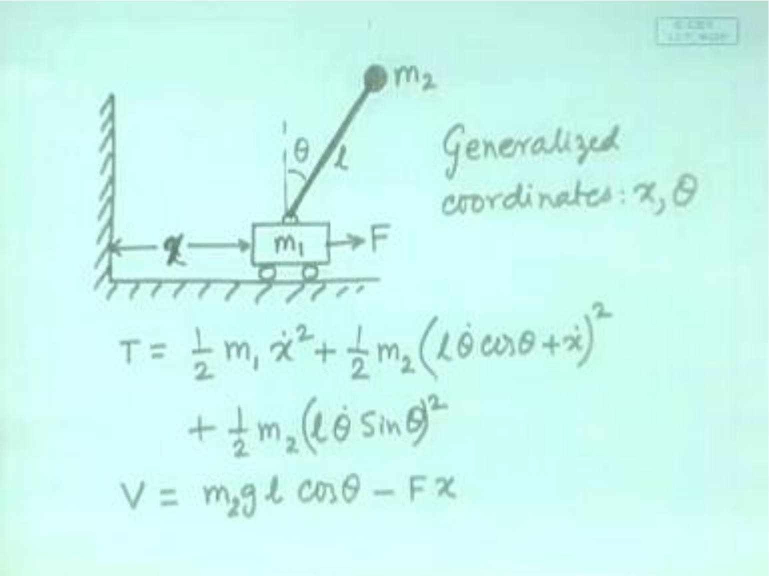

So, inverted pendulum problem is where you have the base along with the wall, and there is a car with wheels. And the pendulum is like this, say normally this could be a you know mass could be uniformly distributed. But, let us assume for our simplicity, that this is a mass concentrated at the top and here is another mass, this fellow also has mass. So, let us this be M 1 and this fellow be M 2. And then you have to apply force here, in order for this to move and balance it.

So, that force is F, well in this case how would you define the generalized coordinates, there are two generalized coordinates. One position of this fellow which has to be measured simply from the wall, it can be measured from the wall. And this fellow position can be measured in terms of the angle. Either you call this as q 1 and this as q 2, or if you want to retain the physical things theta, then let us call it x, then it would be is x and theta.

But, in our case it is essentially the q 1 and q 2 the 2 positional coordinates. Now, we have all set to write down the differential equations, so the generalized coordinates in this case x and theta. Now, the kinetic energy T, kinetic energy would be a combination of the two kinetic energies, kinetic energy of this one and the kinetic energy of this one. For this one it is simple, half m 1, then x 1 dot square x dot square. For this one it is a bit complicated, because it has a circumferential motion, there is no radial motion, but that circumferential motion has to be broken into two components x and y. And x has this additional x dot adequate, there are two components really. So, the horizontal component of this motion, if this is L, then L theta dot is the circumferential motion. L theta dot cos theta would be

Student: ((Refer Time: 44:19))

No, horizontal component and sin theta would be the vertical component. The vertical component is unaffected by this motion, the horizontal component has this motion erect to it, so that way we can write down the T. So, how what will it be a horizontal component would be plus half m 2, it would be l theta dot cos theta, l theta dot cos theta plus x dot. L theta dot cos theta is plus x we add up square plus half m 2 l theta dot sin theta l theta dot sin theta square.

This is the kinetic energy completed, potential energy potential energy there are because of two components one, because of gravitational energy in m 2. This does not have gravitational energies moves horizontally, but also there will be another component of the potential due to F. So, what will be this quantity, it would be m g l cos theta, now F is being applied in the direction of x. X increases in this direction, F is also applied in this direction, so it will be minus F x. What?

Student: ((Refer Time: 46:42))

M 2, now there will be a term here that will vanish, there will be one term coming out of this that we have cos square theta and here sin square theta and they will add up to one. So, this can be simplified, just simplify that, so let us write down the Lagrangian.

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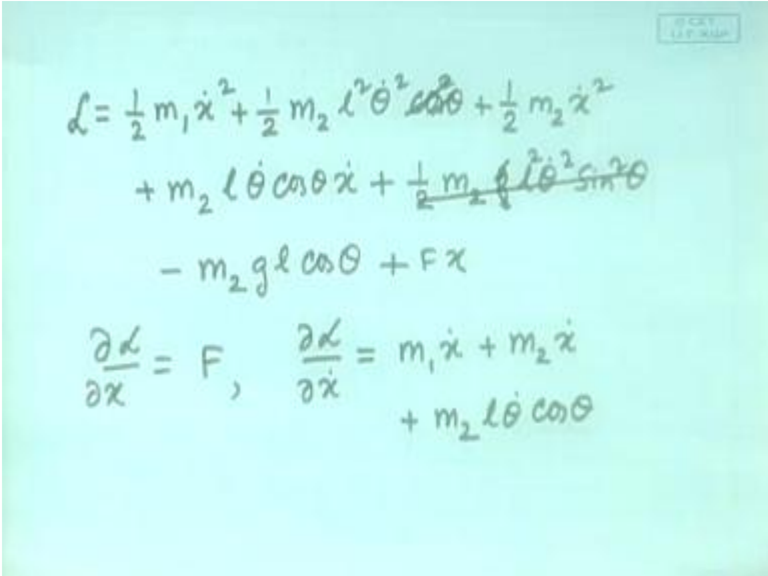

Lagrangian is half m 1 x dot square, let us breaks that that up half m 2 l square theta dot square cos square theta plus half m 2 x dot square. Plus half goes off m 2 l theta dot cos theta x dot. I have broken up just check it if I have written correctly, I have just expanded this square. Plus half m 2 l theta dot, I will not write it l square theta square sin square theta minus m 2 g l cos theta plus F x, that is the total Lagrangian.

Now, you will notice that this fellow and this fellow, this term gets common cos square plus sin square, so I will just drop this term and drop this term, I can do that. So, it becomes relatively simplified expression for the Lagrangian. Now, we will lead to take the derivative of the Lagrangian with respect to x, it will be this goes off, this goes off, this goes off. Now, you have m 1 x dot from here, this goes off here this remains, so plus m 2 x dot. This one we remains, but this is a complicated stuff differentiate it properly plus m 2 l theta dot cos theta. So, let us write the other one before proceeding, no let me write the equation from here.

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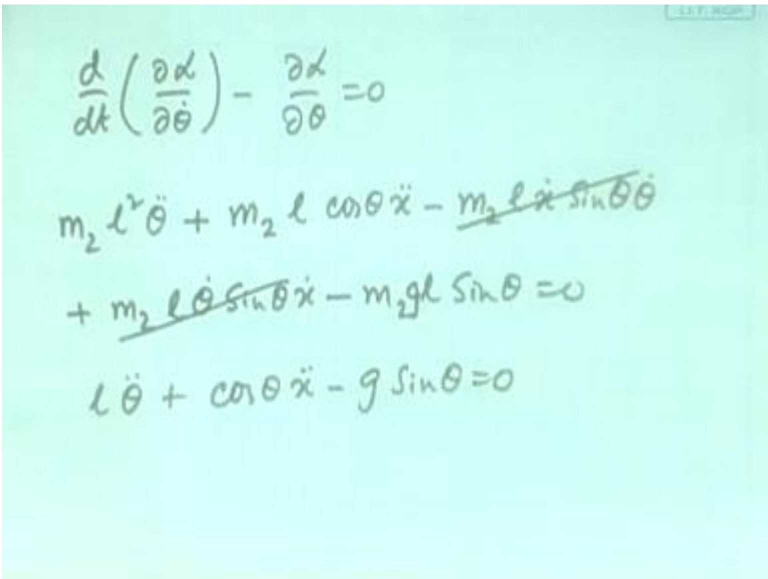

The equation would be then d d t of Lagrangian with respect to x dot minus Lagrangian with respect to x equal to 0. Now, when we expand it when we expand it would be write the d d t of this correctly. Now, I writing same thing, because most students making mistake here, write the d d t of this correctly. It will be m 1 plus m 2 x double dot plus m 2 l cos theta theta double dot theta double dot minus. Now, we have to differentiate this m 2 l theta dot...

Student: Sin theta.

Sine theta theta dot square

Student: ((Refer Time: 51:38))

So, this is the derivative of this. Now, we have to write minus this, so that would be minus F minus F equal to 0 done. Next one derivative of Lagrangian with respect to theta is if you do it with respect to theta, can you see the expression, yes. (Refer Time: 52:15) this goes off, this goes off, this goes off, this remains this remains, so take the derivative correctly, it will be minus m 2 l theta dot sin theta x dot plus m 2 g l sin theta. And with respect to theta dot is with respect to theta dot is this term remains m 2 l square theta dot plus m 2 l cos theta x dot.

Now, again take the derivatives properly when you write this, write it properly here we have this. The first term is simple m 2 l square theta double dot, no problem here. Next term will have problem plus m 2 l cos theta x double dot minus m 2 l dot sin theta dot. Then plus this one, so plus m 2 l theta dot sin theta x dot minus m 2 g l sin theta equal to 0, does anything cancel off.

Student: ((Refer Time: 54:22))

No third and the fourth, they cancel, m 2 l cancels off, so it is the actually simpler equation theta double dot l theta double dot plus cos theta x double dot minus g sin theta equal to 0, does the other equation simplify, no so that is the equation then good. So, this is how the equations are to be written. Remember this if you try with any other method would be enormously complicated, this particular problem. So, that is why the strength of the Lagrangian approach becomes salient in this kind of problems.

Thank you.