Power Network Analysis

Dr. Abheejeet Mohapatra

Department of Electrical Engineering

IIT Kanpur

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Lecture-28

Hello everyone, welcome to Lecture Three of Week Six of the course Power Network Analysis. In which we continue our discussion on transmission line models and the use of these models for the performance assessment of transmission lines under different operating conditions. In today's discussion, we will specifically look at or talk about the long line model, which is the most precise, perfect, and accurate model compared to the short line or medium line models that we have discussed in the previous few sets of lectures. So, compared to the short line and medium line models, the long line model, as I mentioned, would be the most accurate representation of the actual transmission line. So, let us see what this long line model looks like. In fact, the short line model and the medium line model could also be replaced by the long line model.

The only word of caution is that the numbers or analysis we would expect from a short line model, the same set of analysis or numbers would also be obtained if a short line were to be modeled as a long line, and hence there would be hardly any difference in terms of accuracy or the numbers we would get. On the other hand, for analyzing the long-line model, the time required may be higher, or the complexity involved may be greater. So a trade-off is always desired between the computational complexity and the accuracy involved. So that's the reason we are segregating short-line, medium-line, and long-line models.

So, in a long line model, if we talk about it, it is pretty obvious that as the line length increases, the effect of total line capacitance and probably conductance, if it can be modeled, no longer becomes negligible; the quantum of line capacitance or conductance becomes appreciable. And obviously, as line length increases, the line resistance and inductance aspects would also increase. And the ratio of capacitance, resistance, and inductance would depend on what the per unit line capacitance, resistance, and inductance are, respectively. So, as a result of higher line capacitance, the associated line charging current, or the capacitive current, becomes significant or appreciable.

Specifically in the long line model or in the medium line model, which regulates the flow of reactive power in the transmission line or transmission network, thereby affecting the related terminal voltages.

And in the long line model, the lumped representation would no longer be applicable; actual network parameters are physically distributed in nature, so we would use the distributed network representation or distributed parameter representation in the long line. So typically, the long line model would be applicable for lines of length beyond 250 kilometers and an operating voltage above 400 kV line to line. Line capacitance, as I mentioned, is no longer negligible; it has to be considered. Distributed line parameter representation is a must, no matter how small or large the line section is, if the line long end model is to be considered. And in this small section or last section, whatever section we take for the long line model, every section would have a series impedance thanks to line resistance and inductance, and the same section would also have a shunt admittance, which would represent the line capacitance and conductance, provided conductance is modeled properly.

So shunt admittance would definitely have a line capacitance aspect. If conductance can be modeled, then conductance can also be considered in the long line model. And as a result, the resulting equations that we would observe would become no more algebraic; they would become differential equations. So let's understand how we get these differential equations instead of the beautiful, simple algebraic equations in short-line or medium-line models. So what I have considered here is one such long line whose length is ll kilometers.

And since we are developing these models from a single-phase or per-phase equivalent basis, I can always correlate or define that the potentials at the sending end V_S and the receiving end V_R are all being measured with respect to a reference. That reference could be the neutral or the ground point. So in this two-port network representation, the terminals marked as negative are actually fictitious negatives; they could be referred to as the fictitious neutral or ground from which the receiving end voltage V_R and sending end voltage V_S are measured. Similarly, I_S and I_R are the respective per-phase sending end currents and receiving end currents, respectively, and each section of this line, no matter how small or large, has a series impedance, denoted by small z, in units of Ω/km. The units we defined for r, l, small r, small l, and small c in the previous module were evaluated on a per-unit length basis, which could be meters or kilometers.

That's the reason why z, the per-unit line series impedance, is measured in Ω/km. Similarly, small y represents the shunt admittance on a per-kilometer basis, where it incorporates the effect of line capacitance, and if conductance can be modeled, then g would become nonzero in that case. And over here, ω, which is common in both small z and small y, is equal to $2\pi f$, where f is the source frequency at which the voltage sources V_S and V_R are operating. And in this line's length of capital, small l length, we are choosing or measuring the distance or length of this line from the sending end perspective. So x is the line length along which the measurements are being made or the analysis is to be made.

So for x equal to 0 , we would refer to our voltage and current as pertaining to the sending end quantities. And when x becomes equal to ll, which is the actual line length, then we would get the currents and voltages that would refer to the receiving end quantity. Any value of x, again remaining positive between 0 and ll, would refer to the voltage and current at a distance x from the sending end. That is the reason why the line length, which is marked as x, is being measured from the sending end. And if we look at the boundary conditions, which I have already explained, for x equal to $0, V(x)$, which is the voltage at a distance x from the sending end, and the per-phase current as $I(x)$, which is the current at a distance x from the sending end.

As per the boundary conditions, when x becomes $0, V(x)$ equal to 0 is nothing but the sending end voltage, and I at x equal to 0 is nothing but the sending end current.

$$V(0) = V_S$$
$$I(0) = I_S$$

Similarly, we also have the boundary conditions of the receiving end, which I mentioned: if x becomes equal to ll, then this voltage should refer to the receiving end voltage, and the corresponding current should also refer to the receiving end current.

$$V(ll) = V_R$$
$$I(ll) = I_R$$

So we have two sets of boundary conditions, one for the sending end and the other for the receiving end. Now, if we apply or take a section of width Δx at a distance x from the sending end and mark two nodes, which are nodes a and b, then if we apply KVL across nodes a and b, by KVL we have the voltage at node a as $V(x)$, which is written or marked over here, so $V(x)$ should be equal to the voltage at node b, which is $V(x + \Delta x)$, and since we are flowing along the direction of the flow of current. So the corresponding voltage drop that is observed is going to be equal to $I(x)$ multiplied by the impedance, which is $z(\Delta x)$.

So we have $I(x)$, the current through the series impedance, multiplied by the actual impedance of that section. Remember, the unit of z is Ω/km; Δx is also a distance. So when you multiply z by Δx, that results in the overall series impedance of section Δx. Similarly, $Y(\Delta x)$ multiplied by Δx would refer to the actual or total shunt impedance for the section of width Δx. So this is our KVL equation.

One could also counter or say, "Oh, why did I choose $I(x)$?" because at the initial or starting point of node a, the current is $I(x)$, whereas at node b, the current has now become $I(x + \Delta x)$. So yes. Instead of the current $I(x)$, this term can also be written as $I(x + \Delta x)$. The overall resulting equation won't change drastically. Please remember that we are assuming Δx to be a very small section of the actual line length.

So if we apply KVL, we have this approximate equation where $I(x)$ could also be $I(x + \Delta x)$.

$$V(x + \Delta x) - V(x) = -I(x)z\Delta x$$

The overall resulting equation won't change much, and if we now rearrange terms, we have $V(x+\Delta x)-V(x)$ equal to $-z\Delta x$ divided by $I(x)$,

$$\frac{V(x+\Delta x)-V(x)}{\Delta x}=-zI(x)$$

which we again, when we divide and take the limit of Δx tending to 0 , because remember we want to. Accurately model each minuscule line section, so that's why we are taking the limit of Δx running to 0 .

$$\lim_{\Delta x \to 0}\frac{V(x+\Delta x)-V(x)}{\Delta x}=-zI(x)$$
$$\frac{dV(x)}{dx}=-zI(x)$$

In that case, even if we had, let's say, instead of $I(x)$, we had $I(x+\Delta x)$, then when we apply the limit of Δx tending to $0, I(x)\Delta x$, if it would have been placed over here, then for limit, the i of delta $I(x+\Delta x)$ tending to 0 , it would eventually become $I(x)$, and that was the reason why I mentioned that we could have $I(x)$ or $I(x+\Delta x)$ in the current KVL equation. So if we take this limit, we have our first differential equation, the first one of the differential equations, which says the first derivative of voltage from the sending end with respect to the length or distance x should be equal to $-z$ times $I(x)$.

Remember, here has a unit of Ω per kilometer. Similarly, if we make use of the KCL along node a, we also have to consider the importance of the shunt admittance through KVL. We are considering the importance of the presence of the series impedance; the shunt admittance aspect can be taken care of through the KCL philosophy. So, if we apply KCL again at node a, before node a we have current $I(x)$, and just after node a we have current $I(x+\Delta x)$. Along $y(\Delta x)$, there is a voltage appearing whose quantum is $V(x)$.

So, the way we applied KCL for the nominal pi model, T model, and medium line model, on similar lines if we apply KCL , then the equation that we are expected to get by KCL at node a would approximately be equal to $V(x)$ into $Y(\Delta x)$ plus $I(x+\Delta x)$. So now you may again question or think, "Oh, why did I choose $V(x)$ voltage?" The logic remains the same, even if you had chosen $V(x+\Delta x)$ instead of x, because Δx is tending to 0 ; the overall differential equation that we are likely to get won't change. So, if we apply KCL at node a, we get our usual equation, which I wrote in the previous slide. We rearrange terms and take the limit of Δx as it approaches zero because the line section has to be accurately modeled for every minuscule section.

$$I(x) = V(x)y\Delta x + I(x + \Delta x)$$
$$I(x + \Delta x) - I(x) = -V(x)y\Delta x$$
$$\frac{I(x + \Delta x) - I(x)}{\Delta x} = -yV(x)$$
$$\lim_{\Delta x \to 0} \frac{I(x + \Delta x) - I(x)}{\Delta x} = -yV(x)$$
$$\frac{dI(x)}{dx} = -yV(x)$$

So we get our second differential equation, which states that the first derivative of current measured from the sending end with respect to the distance itself is equal to $-y$ times $v(x)$.

$$\frac{dI(x)}{dx} = -yV(x)$$

Please note the unit of y here is Ω^{-1}/km. So if we look at our overall governing equations, we have two first-order equations, and these two first-order equations can be rewritten in terms of pure homogeneous forms. So, what we do is take the second derivative of the first equation, which results in d^2v/dx^2, which is nothing but $-z$, because z is a constant and doesn't depend on the line length. So, $I(x)$ derivative comes into play here, and again if we substitute di/dx, which is nothing but $-y$ times vx, we get zy times vx. On similar lines, we have our second differential equation, which is purely in terms of currents.

$$\frac{d^2V(x)}{dx^2} = zyV(x)$$
$$\frac{d^2I(x)}{dx^2} = zyI(x)$$

So, if we compare these two equations, they are pure homogeneous second-order differential equations: one is entirely in terms of voltage, and the other is entirely in terms of currents. So both the current and voltage equations that we saw over here, please remember that these voltages and currents are phasor quantities; they are in the frequency domain, and hence if we have to find the solution of $V(x)$ and $I(x)$ in terms of x, we would still be getting voltage and current at a distance x again in the frequency domain and not in the time domain. We'll see the impact of time-domain modeling of the long-line model in the next lecture. So please have patience until then. So now we are wondering what the solution form of these equations is.

Again, as I mentioned, these are beautiful second-order linear homogeneous equations. By linear, I mean that if I take a combination of signals or a combination of phasors individually, if they satisfy these given differential equations, then the combination of these solutions would also satisfy these individual second-order differential equations. So now, if we are guessing what the standard solution form of these equations could be. So let's say we focus on the first equation, which states that the second derivative of voltage with respect to distance is equal to zy times $V(x)$. So, through experience and mathematical analysis, it is imperative that if we have v of x as an exponential function, let me write that exponential function as A_1 into $e^{\gamma x}$, where I still don't know what A_1 and γ are.

I have just taken a generic form of the solution or the general form of $V(x)$, and I am trying to understand or guess whether an exponential function can be a solution to this equation. So let's try that. In order to satisfy the corresponding equation, I should first find what the first derivative of voltage is for the chosen analytical form. So, A_1 is a constant which is unknown. A_1 remains the same; $e^{\gamma x}$ derivative would be γ into $e^{\gamma x}$ again.

I take the second derivative d^2v/dx^2; it would come as A_1 is constant, γ is also a constant which is not known, and here I have $\gamma^2 e^{\gamma x}$. Now, if I put these numbers individually over here, I get that $A_1\gamma^2 e^{\gamma x}$ should be equal to $zyA_1e^{\gamma x}.A_1$ is a constant that depends on the boundary conditions, and assuming that A_1 is not 0 , essentially for the solution or for the equation to satisfy, we have the condition that γ^2 is equal to zy, which can again be written as γ times the square root of zy.

$$\frac{d^2v(x)}{dx^2} = zyV(x)$$
$$v(x) = A_1e^{\gamma x}$$
$$\frac{dv(x)}{dx} = A_1\gamma e^{\gamma x}$$
$$\frac{d^2v(x)}{dx^2} = A_1\gamma^2 e^{\gamma x}$$
$$A_1\gamma^2 e^{\gamma x} = zyA_1e^{\gamma x}$$
$$\gamma^2 = zy$$

Now, what do you expect the unit of γ to be? That is the reason why I mentioned that the unit of γ is Ω per kilometer and the unit of y is Ω inverse per kilometer.

So, if I multiply z by y, the overall unit of z by y is nothing but 1 per square kilometer. And that is the reason why γ is the square root of zy with a unit of only kilometers inverse, because γ is the square root of zy, so the unit of this is 1 per kilometer only. Γ is called the propagation constant. On the lines of $A_1e^{\gamma x}$ satisfying the voltage equation, we can also have another exponential form where $e^{-\gamma x}$ comes in. So in a way, the general solution of the first equation can be written in terms of two exponential sums.

And $A_1e^{\gamma x}$ itself satisfies the solution. $A_2e^{-\gamma x}$ also satisfies the first differential equation on a line similar to this. So their combination also satisfies the solution and that's where these equations are linear. Γ is the root over z of y, which is called a propagation constant. Since z and y themselves are complex numbers, the square root of the product of two complex numbers would also likely be a complex number, and that is the reason why γ has a real part known as the attenuation constant.

And $j\beta$, where j is the square root of $-1, \beta$ is called the phase constant, with units of radians per kilometer, and the attenuation constant is in nepers per kilometer. Nepers and radians, as such, are dimensionless quantities. So whether the units represent anything or not, it's not important. Just to segregate the real and imaginary parts of the propagation constant, these units of attenuation and phase constant come in. So, if this is a general form equation of $V(x)$, can we not also derive the general form of the equation or solution for $I(x)$? Yes.

What we would do is go back to our actual equation that if $V(x)$ satisfies this particular general form of solution, then we can actually make use of this equation where $A_1 e^{\gamma x}$ plus $e^{-\gamma x}$ is derived once, and that same function is plugged in over here where we would get $I(x)$. So if we do that, then as I mentioned, A_1 and A_2 are two conditions, and two constants will depend on the boundary conditions. Similarly, if we take the derivative, we can also get the closed-form solution of $I(x)$, where capital Z_c is nothing but z by y, z by γ actually, and Z is called the characteristic impedance. Since it is impedance, its unit is Ω. How come Z_c had a unit of ohm? The overall analysis could be obtained very well from this resulting expression.

So, you collect z units; a small z unit is ohm per kilometer, y units are ohm inverse per kilometer. So if I take z by y, its unit would be ohm per kilometer divided by ohm inverse per kilometer.

Per kilometer gets canceled, and I have ohm squared. The square root of z divided by y will also have a root of ohm squared, which essentially leads to ohm as a unit. And that's the reason Z_c is a characteristic impedance.

It's unique for a given long line. The importance of characteristic impedance becomes very obvious when we actually delve into the performance assessment of the line under different operating conditions, which still has time. We will probably discuss it in the next two lectures. And as I mentioned, A_1 and A_2 are constants, so they can be obtained using boundary conditions since we are measuring our voltage and currents from the sending-end perspective. So we use our first boundary condition where at $x = 0, V = V_S$, and at $X = 0, I = I_S$. So if we put $x = 0$ in both these equations, and substitute v of $x = 0$ as $V_S, I(x) = I_S$, what we get here is for $x = 0$, this term becomes 1 , this term becomes 1 , so essentially I have $A_1 + A_2 = V_S$.

Similarly, if I put $x = 0$, this term becomes 1 , this term becomes 1 , so I have $A_2 - A_1 = I_S Z_c$. There are two simultaneous equations with two unknowns, V_S and $I_S; Z_c$ is known, and V_S and I_S are known from the measurements perspective. Z_c is a characteristic impedance that depends on the per unit series impedance and per unit shunt admittance. So if I solve these equations, it's pretty obvious that from here I can get A_2 as $\frac{V_S + I_S Z_C}{2}$ and A_1 would be $\frac{V_S - I_S Z_C}{2}$.

So, since V_S and I_S are known. So, in a way, A_1 and A_2 are also known, and from these known A_1 and A_2 values, we can substitute those constants, which are now known in terms of $V(x)$ and $I(x)$, and eventually, we would get $V(x)$ as these expressions. These two expressions or terms are essentially the sum of the mean of $e^{\gamma x}$ and $e^{-\gamma x}$; mathematically, it also resembles the cos hyperbolic function. Just as we have sine and cosine functions like $\cos\ x$ and $\sin\ x$, on similar lines we have hyperbolic functions such as $\cosh\ x$ and $\sinh\ x$, and essentially, $\cosh\ x$ refers to a function of this order. So basically this term is nothing but the hyperbolic cosine of $e^{\gamma x}$, and the second term within the brackets is nothing but the hyperbolic sine of γx.

So that's where we get our hyperbolic functions. On similar lines, we can express our $I(x)$ in terms of hyperbolic sine and hyperbolic cosine functions. $V(x)$ and $I(x)$ are now known in terms of V_S, I_S, Z_c, and depending on the line length X, we can figure out the actual voltage and current at any distance X from the line. One would wonder what the complexity involved in this was. The complexity involved in finding the currents and voltages through the actual long name model expression is the effort needed to figure out the hyperbolic cosine functions or hyperbolic sine functions. Please remember that cosh and sinh functions can exist; then we can have other hyperbolic trigonometric functions possible.

So tanh, can all exist. The effort required is in finding out or evaluating those hyperbolic functions, whereas, unlike in the short-line model, in the medium-line model, we had just known constants.

So that's the major computational complexity involved. We know $V(x)$ and $I(x)$ at a distance x from the sending end. So at the receiving end, if we let $x = l$, we have V_R and I_R in terms of V_S, I_S, Z_c, and actual line length. Γ again is the square root of zy, where small z and small y refer to per unit line series impedance and certain admittance. In the next lecture, we will continue our discussion on the long line model, and we will see the beauty behind the lossless condition: if a transmission line is well designed, long enough, and operating at high voltages, then certain specific phenomena happen specifically for the lossless condition, which actually becomes very useful to understand the cases where lightning strikes and how the voltage waveforms and current waveforms appear in a transmission network. The same analysis can also be used to perform fault analysis in DC networks. Essentially, the lossless long-line model creates the premise for such discussions. Unfortunately, given the limited time that we have, we won't be able to discuss the lightning analysis and the fault analysis of DC networks, but they do arise from the discussion on the lossless long-line model. Thank you.