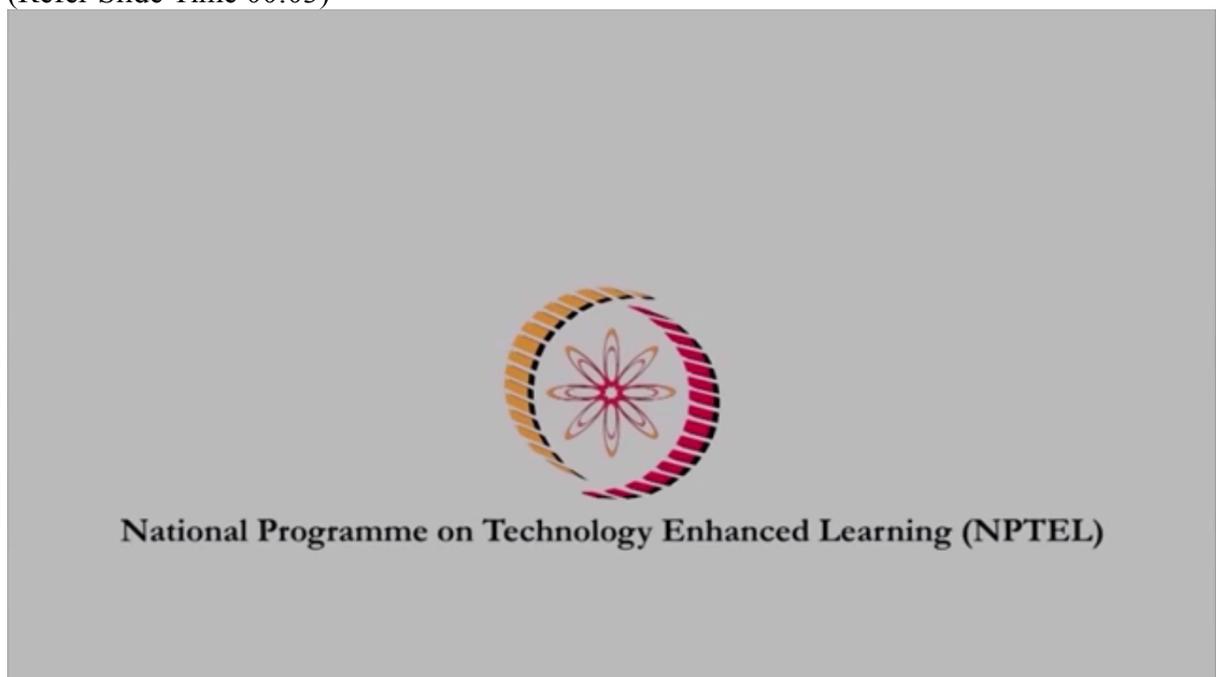


(Refer Slide Time 00:00)



Indian Institute of Technology Kanpur

(Refer Slide Time 00:03)



National Programme on Technology Enhanced Learning (NPTEL)

(Refer Slide Time 00:06)

Course Title
Electromagnetic Waves in Guided and Wireless

Course Title
Electromagnetic Waves in Guided and Wireless

(Refer Slide Time 00:08)

Lecture - 28
Solution to Electric Scalar Potential and Magnetic Vector Potential Equations

Lecture - 28
Solution to Electric Scalar Potential and Magnetic Vector Potential Equations

(Refer Slide Time 00:11)

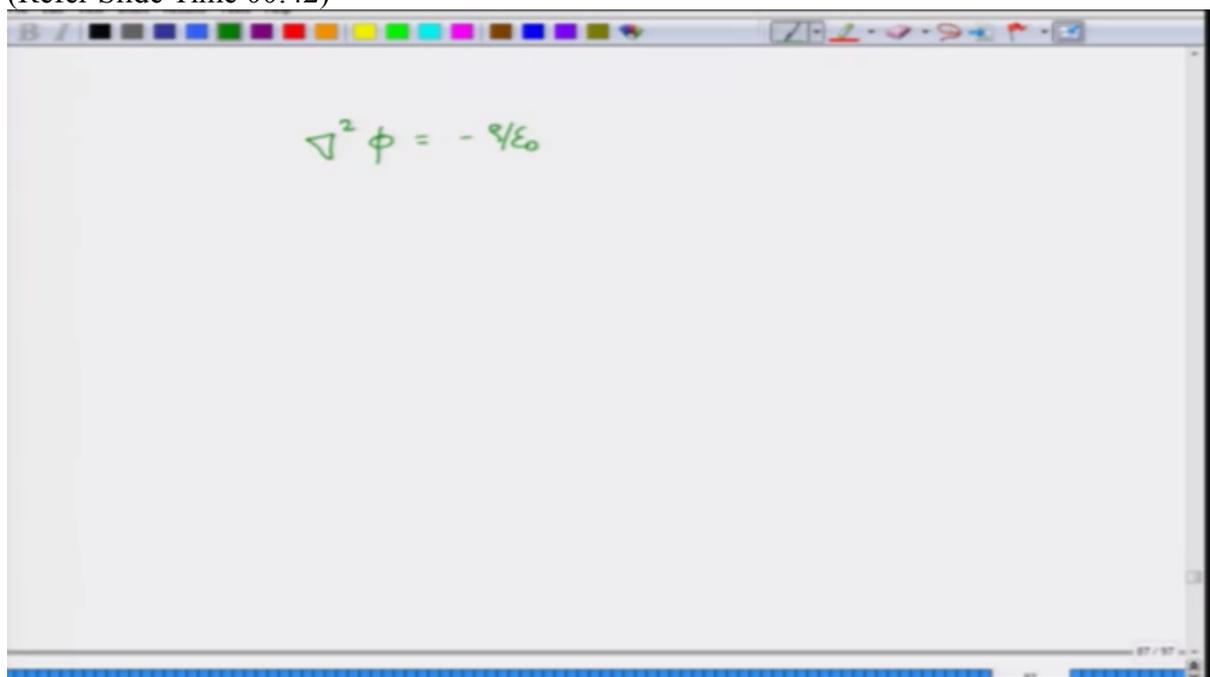
by
Dr. K Pradeep Kumar
Department Of Electrical Engineering
IIT Kanpur

by

Dr. K Pradeep Kumar
Department Of Electrical Engineering
IIT Kanpur

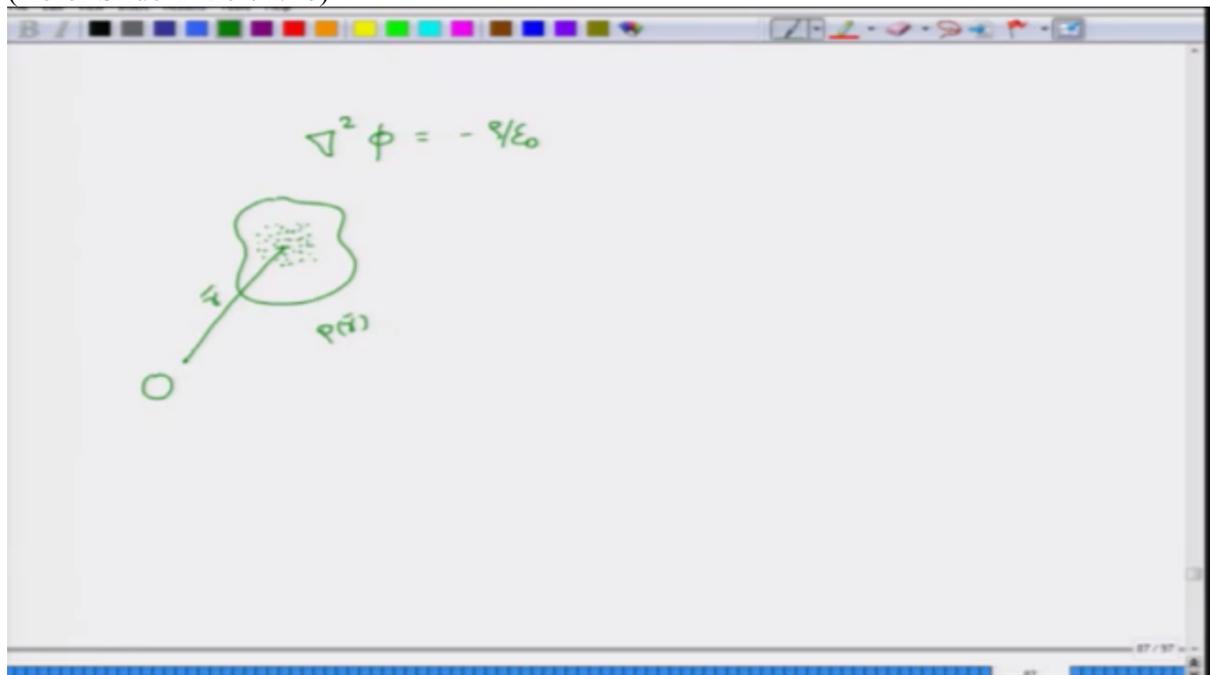
Hello and welcome to NPTEL MOOC on Electromagnetic Waves in Guided and Wireless Media. We continue the development of radiation topic by starting with ϵ_0 equation that we wrote in the last model, which was $\nabla^2\phi = -\rho/\epsilon_0$ where ϕ is the electric scalar potential, which, of course, is related to the electric field. Okay. So please note that there is no time here.

(Refer Slide Time 00:42)



Now in general these charges would be centred in space at some particular, you know, they can be clustered around. So I'm assuming that there is a certain volume over which the charges are present and this would be characterised by the charge density, which we will describe as having, I mean, as a scalar field meaning that the amount of ρ at different points in the space can actually keep changing. So, of course, I'm assuming that there is some origin O and then I'm measuring the distance at any point onto this volume by the position vector r . Okay.

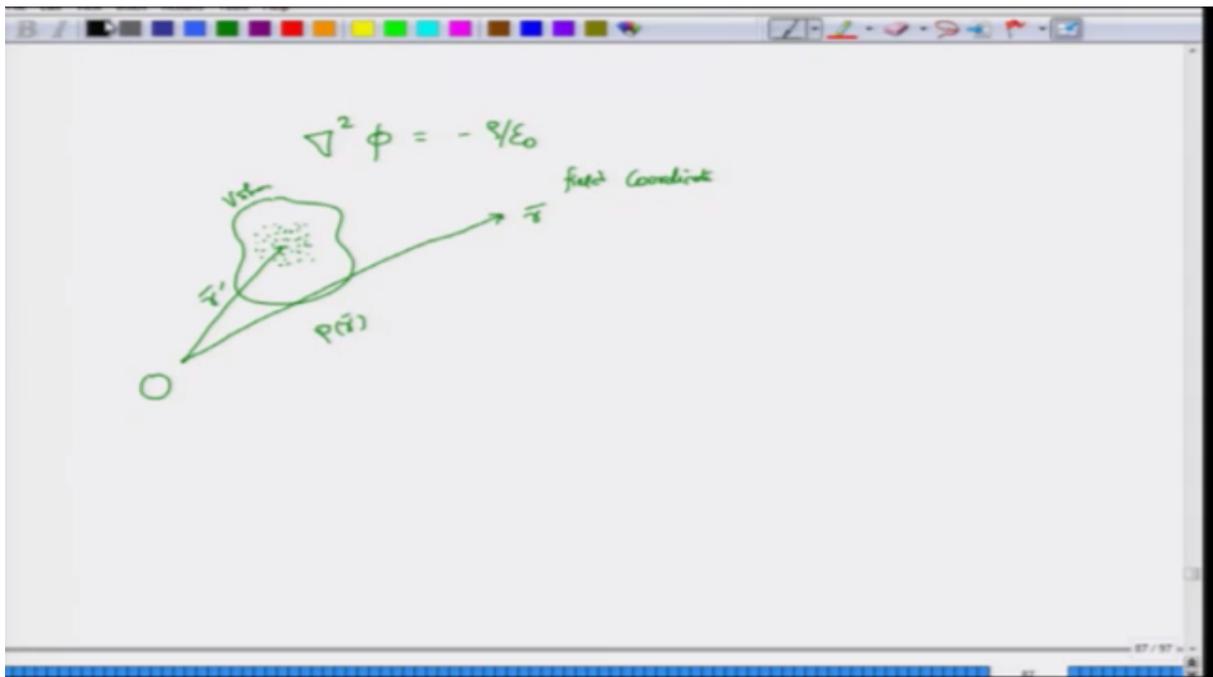
(Refer Slide Time 01:16)



Now because I want to look at what is the potential at another point, which could also have a potential, so which would also have a, you know, what is called the position vector r , I can't use the same r for both, right? So I am going to call this, you know, volume where the charges are located, the position vector to the volume or the points within that volume where the point charges are located as r' . Okay.

The charge distribution will be referred to the charge distribution in the source this r' would essentially be the source coordinates and this r we would call it as the field coordinate meaning that this is the point where we seek the field.

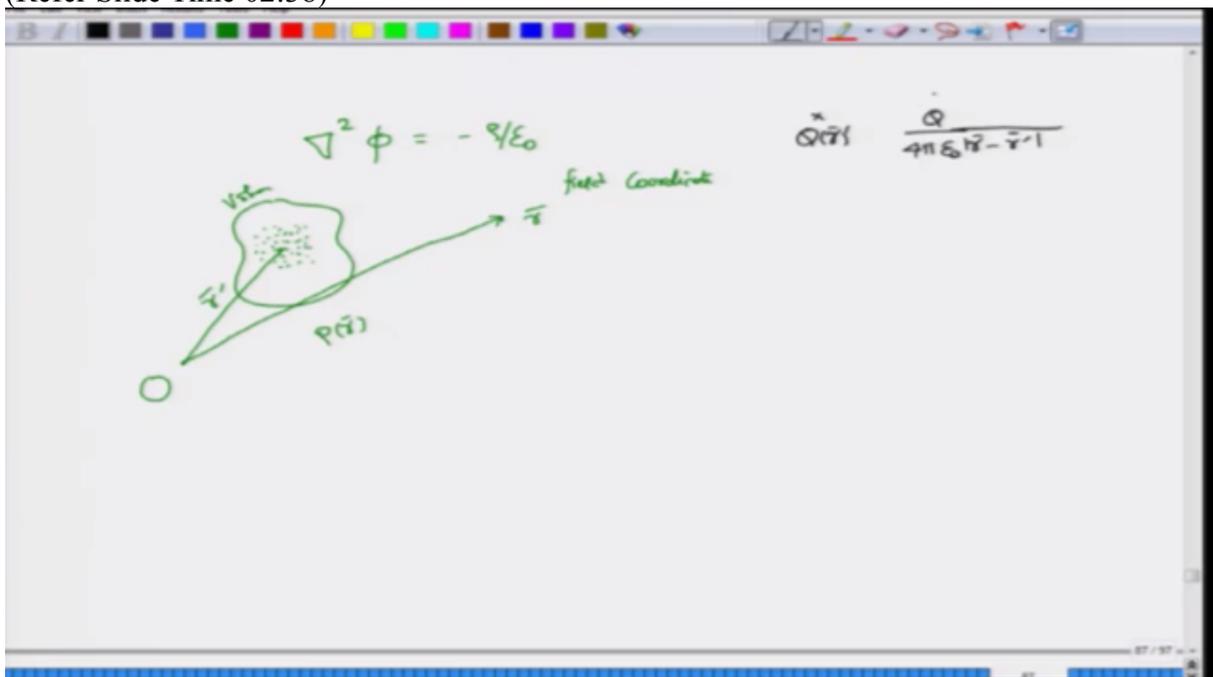
(Refer Slide Time 01:58)



In this case, of course, we are seeking a scalar field, which is potential from the knowledge of this charge distribution that have been given to us and the material medium is assumed to be of permittivity ϵ_0 . Okay.

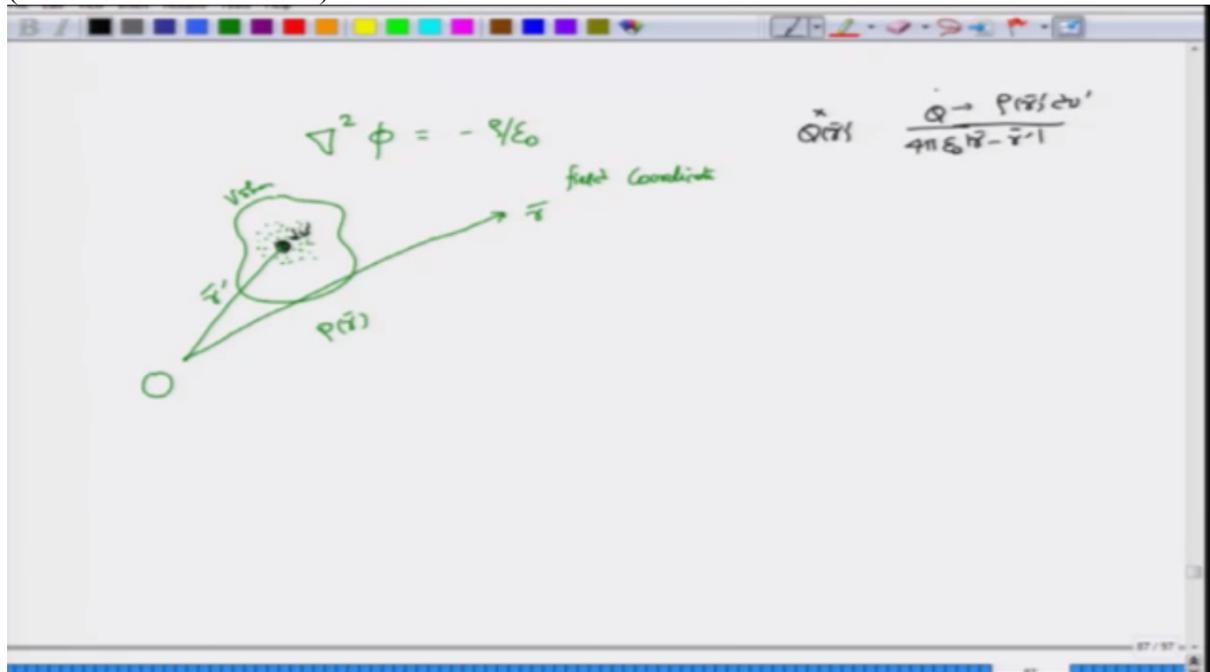
Now for the single charge, which is kept at the origin let's say Q , the corresponding potential at any other point, you know, away from the charge would be something like $Q/4\pi\epsilon r$. Correct? If instead the charge is kept at r' , then the potential would be actually $r - r'$, the vector distance between the two, right, the magnitude of the distance, which we are calling it as $r - r'$. So this is for a single charge.

(Refer Slide Time 02:38)



Now when you have multiple charges, then you can imagine that this charge can be written as $\rho(r')$, which is to say that you consider a small volume here, which is at a distance of r' or the position vector of r' times the small volume around this fellow, that is dv' , that would give you the overall charge located in this small infinitesimal volume. Okay. And then the potential because of that would be $\rho(r') dv'$ divided by the same thing. I am assuming that this dv' is very, very small. Okay. Point like volume is what I'm considering.

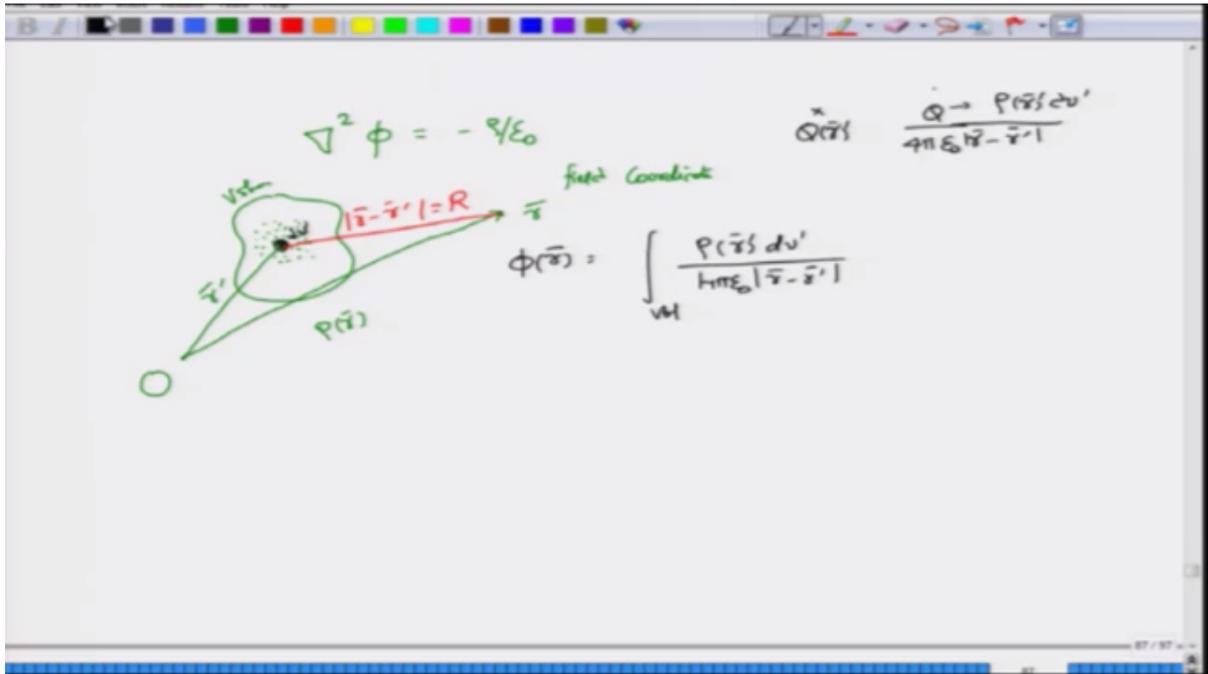
(Refer Slide Time 03:15)



So if I want to obtain the full distribution, I have to basically integrate this expression. So I'm going to get $\rho(r') dv'$. This is the volume integral. So I am going to write this as volume divided by $4\pi\epsilon_0 r - r'$ distance, right?

So this is what you're going to get as the potential at this particular position vector r or at the field point r where the summation would actually come from many different charges that are located. So the red line that I just drew actually tells you what is this $r - r'$ magnitude. Okay. This actually has a very common letter that we use, which is given by R and R would be the radial distance from the source coordinate to the field coordinate. Okay. So this is the expression that you have.

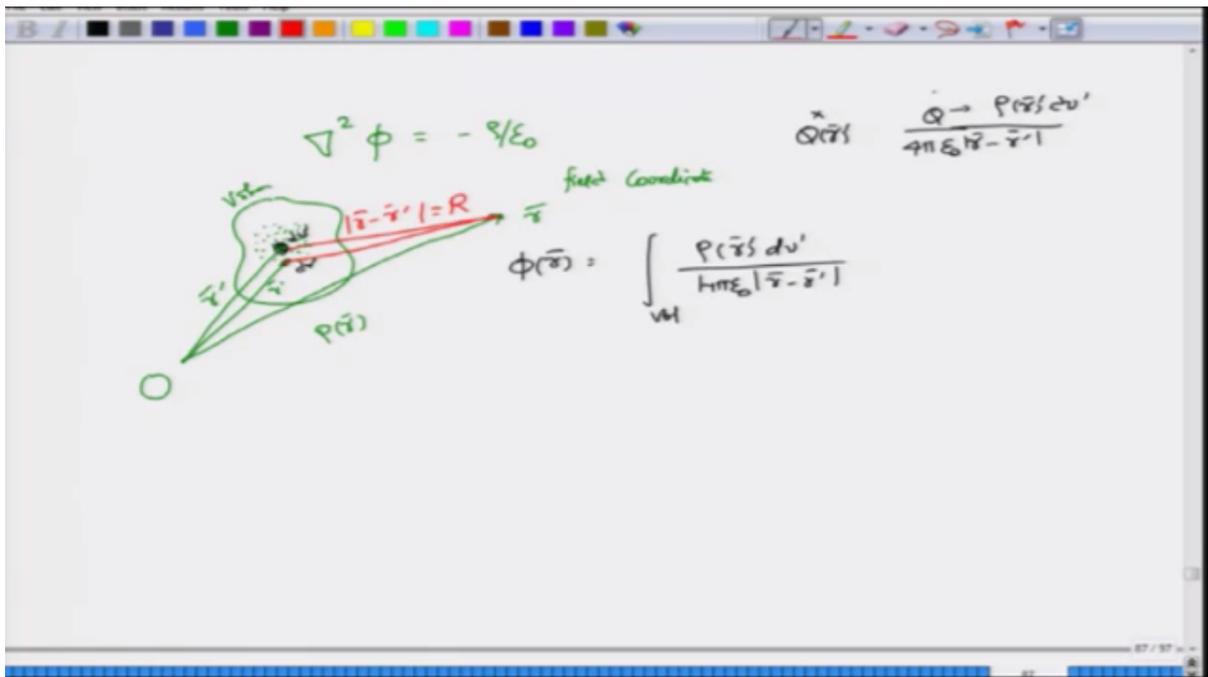
(Refer Slide Time 04:05)



So notice that this would be a volume integral in general and it can be reduced to a surface integral if the charge distribution is actually on the surface. It could reduce to a line distribution if the charges are piled up along a line or it could reduce to a point like distribution if the charge is an isolated charge or a bunch of charges located at a single point, right? So all these different cases from the point charge, line charge, surface charge, and in general the volumetric charge is all taken care by this expression. Okay.

I want you to notice two things. One is that there is an integration which is fine, but you notice that the integrated quantity inside is the source coordinate. While you keep the field coordinate are fixed. what you're moving or what you're doing is to kind of consider different r' . Okay. So at this point you have another r' and then you have, you know, the volume around that one, which we will call as dv' and then you look at the distance between that.

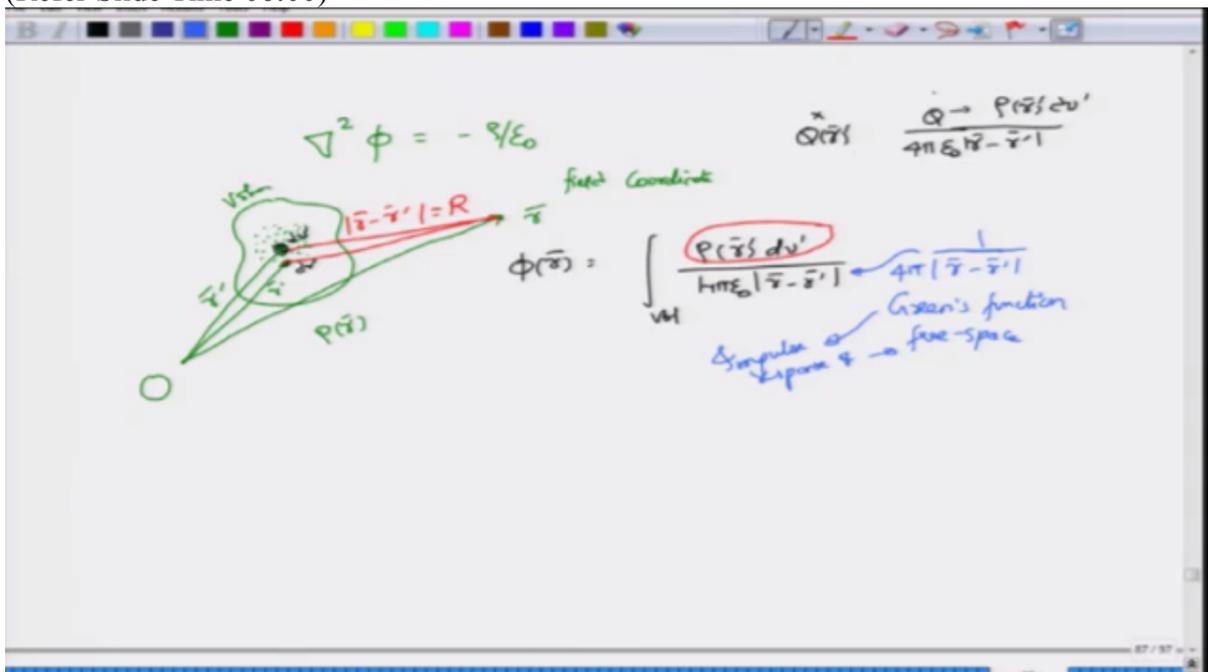
(Refer Slide Time 05:03)



So this particular one, the newly drawn charge distribution would be influencing slightly differently on to the overall field or it will be contributing slightly differently to the overall field at the field coordinates. Okay. So that I want you to notice. So there is an integration and the integration is with respect to the source coordinates. Okay.

The second thing what I want you to notice is the appearance of the $1/(r-r')$ term. Okay. So this is very important because as we will see, but not really derive, this quantity $1/r-r'$ or in general $1/4\pi(r-r')$ is what is called as Green's function for the free space. Okay. Green's function for the free space. If you are not happy with the terminology called Green function, you can simply think of this as the impulse response, okay, of the free space. Okay.

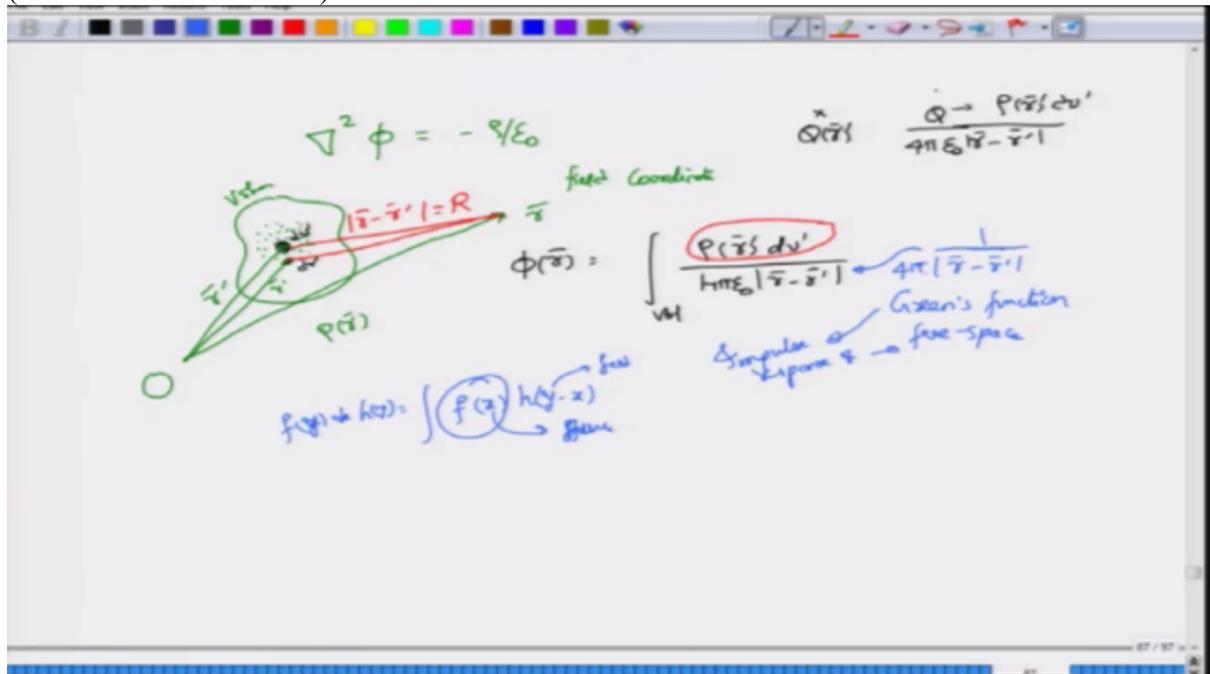
(Refer Slide Time 06:00)



And now if you, you know, kind of look at this, this is kind of a convolution integral that is going on, right? See what is your convolution integral? You would have some $f(x)$ and then you have g of or let's say $f(x)$ and then the impulse response is usually some $h(y-x)$, right? So this fellow would be the function or whatever the convolution $f(x)$ convolved with or $f(y)$ convolved with $h(y)$.

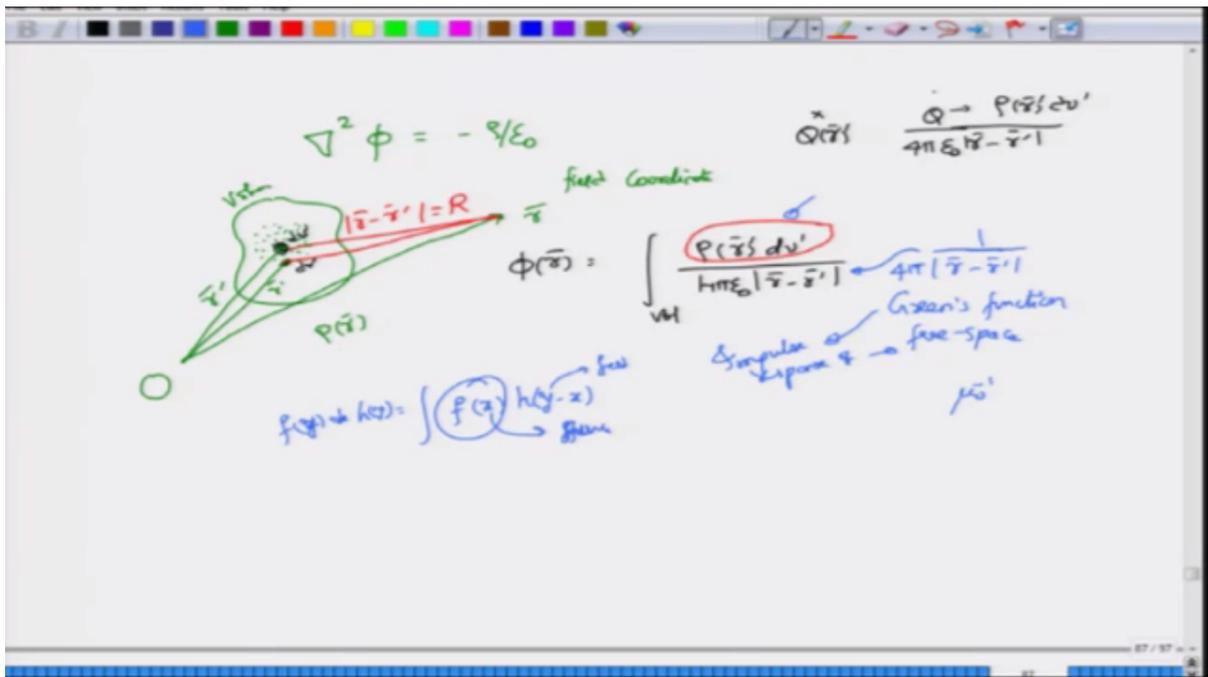
So this convolution integral would have the function which is in this case acting like the field, sorry, source coordinate and then this fellow is acting like the field coordinate. So this is exactly the same thing, right? Of course, I have not rigorously proven this, but the analogy should be essentially clear.

(Refer Slide Time 06:44)



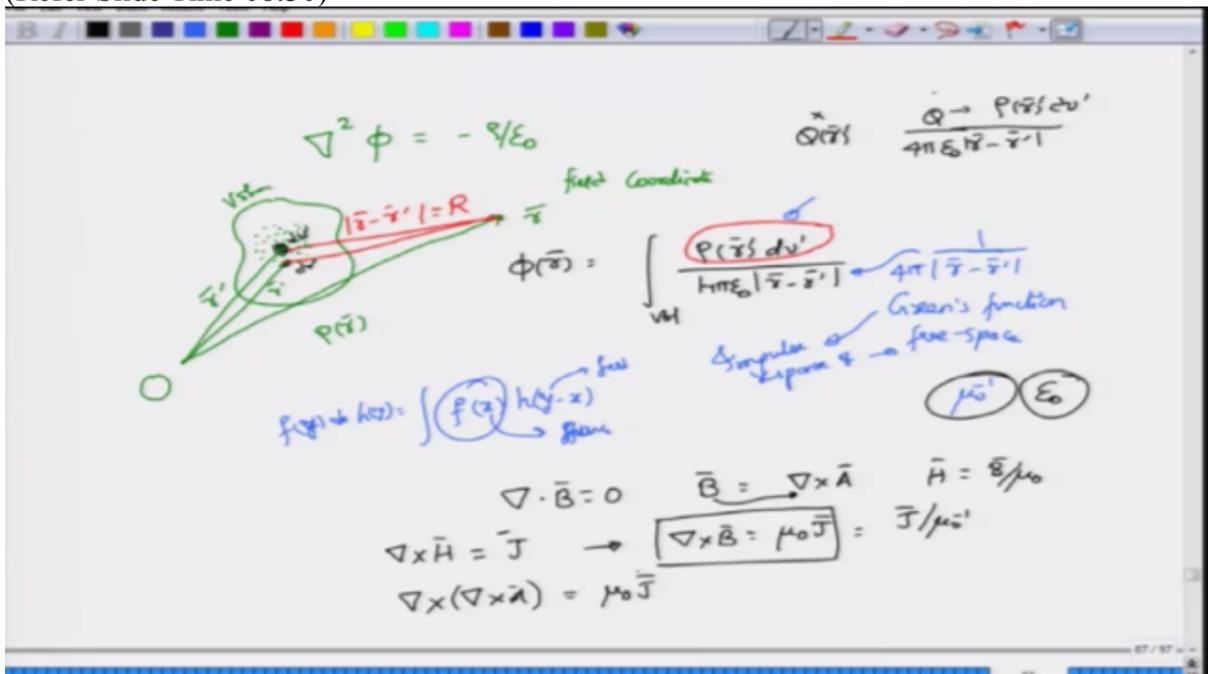
The factor of $1/\epsilon_0$ in the case of the scalar potential and the factor of μ_0 in the denominator or in the denominator of μ_0 inverse would be also multiplied to represent the Green's function for the scalar potential and for the vector potential. Okay. So this equation that we have written is essentially a convolution between the source coordinate or the source rather the source charges or the charge distribution, okay, and this particular Green's function, which is $1/4\pi\epsilon$ or general $1/4\pi(r-r')$. Okay. So please note these two points. Okay.

(Refer Slide Time 07:20)



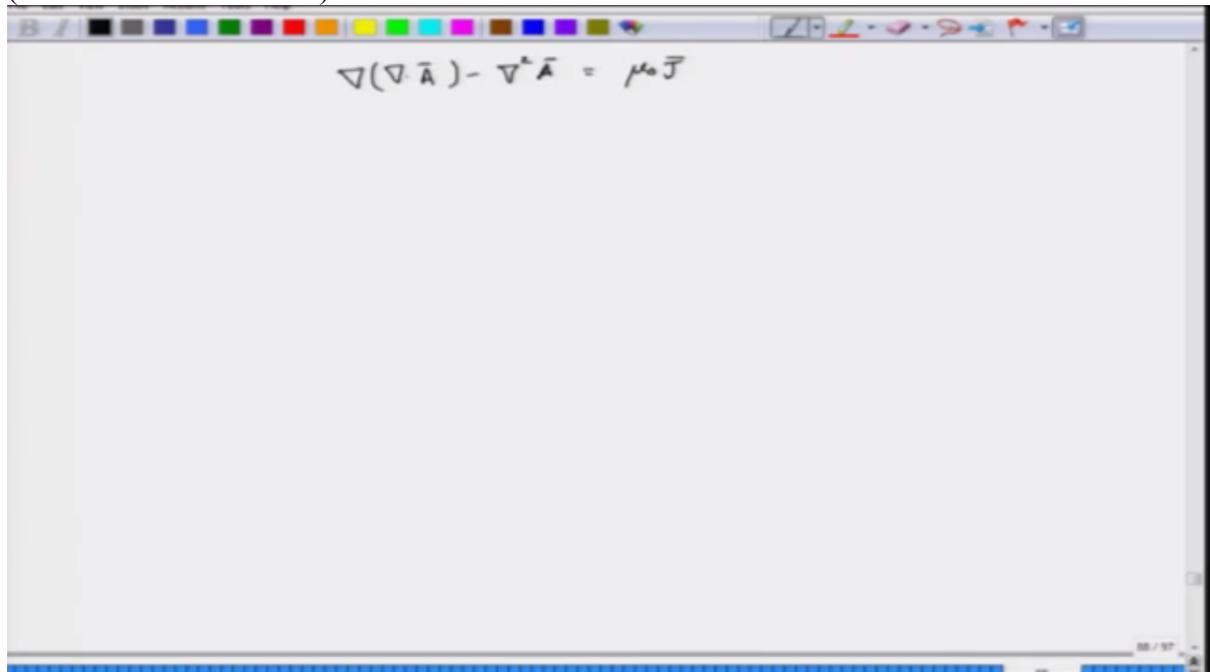
Why are they important? Well, we had this other equation, which said $\nabla \cdot \mathbf{B} = 0$, and therefore we wrote \mathbf{B} as $\nabla \times \mathbf{A}$. Correct? Now I know that $\nabla \times \mathbf{H} = \mathbf{J}$ and I know that \mathbf{H} is basically \mathbf{B}/μ_0 . Correct? So if I substitute for \mathbf{H} in terms of, you know, in this expression here, I can write $\mathbf{H} = \mathbf{B}/\mu_0$, so I will actually have an equation, which tells you $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ or you can write this as \mathbf{J}/μ_0^{-1} . Okay. You don't have to write it, but it is just another way of thinking about the denominator part being μ_0^{-1} for the Green's function and it would be just ϵ_0 for the electric field Green's function. Okay. So, in any case, you have this equation, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ and I know that \mathbf{B} is basically $\nabla \times \mathbf{A}$. Therefore, I can write down the equation, which tells you curl of curl of \mathbf{A} to be equal to $\mu_0 \mathbf{J}$.

(Refer Slide Time 08:30)



Something interesting is going on here because I can express this curl of curl of A as $\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, which, of course, would be equal to $\mu_0 \mathbf{J}$. Okay.

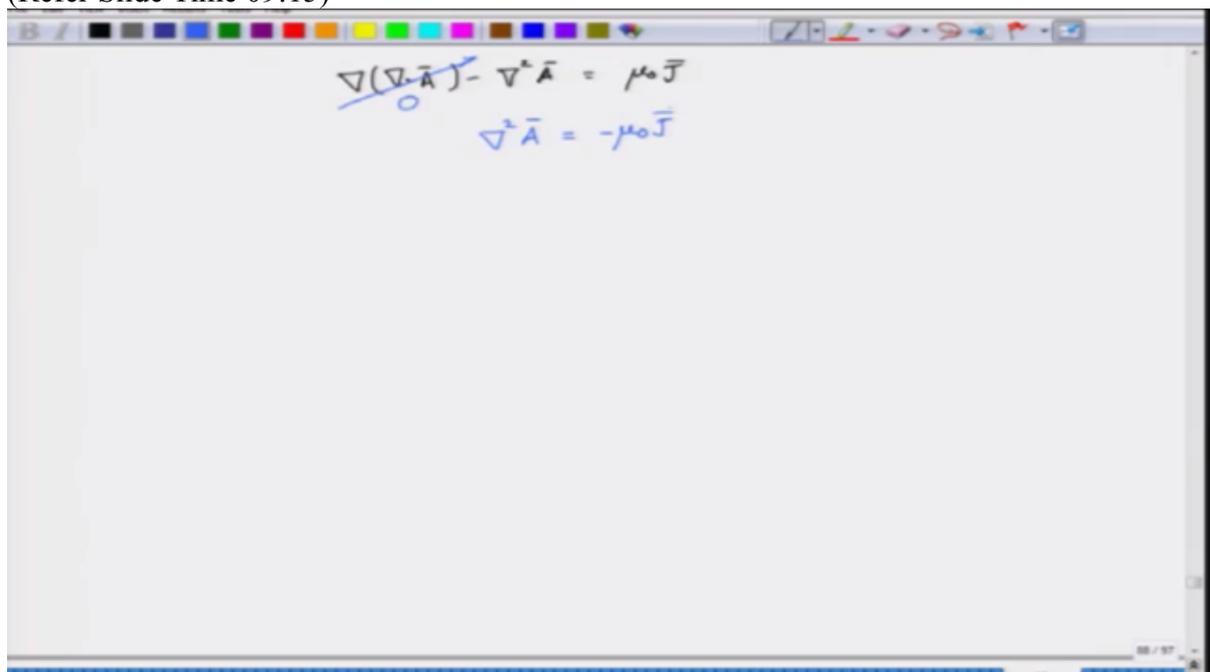
(Refer Slide Time 08:46)



A whiteboard with a toolbar at the top. The equation $\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$ is written in the center.

Now what we have seen from vector analysis is that B can be expressed as the curl of A, but we haven't said anything about what we should be choosing or what we should be writing for $\nabla \cdot \mathbf{A}$. It turns out that this is under our control and since this is under my control or our control, I am going to take $\nabla \cdot \mathbf{A} = 0$. Gradient of 0 is also 0. So I get a simplified equation, which says that $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$.

(Refer Slide Time 09:15)



A whiteboard with a toolbar at the top. The equation $\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$ is written in the center, with a blue circle around the $\nabla(\nabla \cdot \mathbf{A})$ term and a blue arrow pointing to it. Below it, the equation $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ is written in blue.

Isn't this equation very similar to $\nabla^2\phi = -\rho/\epsilon_0$? Yes, it is very similar except for two differences. One, you're dealing with a vector field quantity. Here you are dealing with a scalar field quantity. Here the source is a vector field. Here the source is a scalar field. Okay. However, in the Cartesian coordinate systems at least this ∇^2 operator can be applied separately to A_x , A_y , and A_z , okay, which would then give out $-\mu_0 J_x$, $-\mu_0 J_y$, and $-\mu_0 J_z$ being the, J_x , J_y , J_z being the three components of the current densities along the x, y, and z. So these are the fields. Okay.

(Refer Slide Time 09:56)

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\nabla^2 \phi = -\rho/\epsilon_0$$

$$\nabla^2 \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} -\mu_0 J_x \\ -\mu_0 J_y \\ -\mu_0 J_z \end{pmatrix}$$

In general, A_x will be a function of x, y, z. A_y will be a function of x, y, z. A_z will be a function of x, y, z as well as J_x , J_y , and J_z . Everything is a function of x, y, and z, but because a ∇^2 operator can be split in terms of operating on A_x alone, on A_y alone, on A_z alone in the case of Cartesian coordinate system, you have this equation.

And if you just take this part of the equation, the solution of this equation is very simple. It would simply be A_x at any field coordinate which is given by r will be given by integral of $J(r')$ or rather $J_x(r')$ dv' divided by 4π . Now I will have to have a μ_0 in the numerator, right? So I have $r - r'$. Okay. So this is the expression for A_x .

(Refer Slide Time 10:47)

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\nabla^2 \phi = -\rho/\epsilon_0$$

$$\begin{pmatrix} \nabla^2 A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} -\mu_0 J_x \\ -\mu_0 J_y \\ -\mu_0 J_z \end{pmatrix}$$

$$A_x(\vec{r}) = \int \frac{\mu_0 J_x(\vec{r}') dV'}{4\pi |\vec{r} - \vec{r}'|}$$

In general, the expression for A will be obtained by simply removing this particular letters and then making instead everything into a vector field. Okay. So if I take this entire thing as a vector field, then the vector field for the magnetic vector potential can be written in this particular manner, and this completes our solution for the magnetic vector potential.

What is important is the direction of the magnetic vector potential will be the same as the direction of the J field. However, the direction of the B field will be perpendicular to the direction of the J field, right, because there is a curling operation involved out there.

(Refer Slide Time 11:31)

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\nabla^2 \phi = -\rho/\epsilon_0$$

$$\begin{pmatrix} \nabla^2 A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} -\mu_0 J_x \\ -\mu_0 J_y \\ -\mu_0 J_z \end{pmatrix}$$

$$\vec{A}(\vec{r}) = \int \frac{\mu_0 \vec{J}(\vec{r}') dV'}{4\pi |\vec{r} - \vec{r}'|}$$

So these are the two equations and the two solutions that we were actually looking for and given that the current distribution is known, that you can find A and from finding A, you can find out B.

Yes, it is tedious in some sense. Perhaps there is nothing we have gained from introducing this vector potential is what we will think of, but not really. I mean, in many calculations, we can actually start with the vector potential which simplifies a lot of our calculations in calculating the fields of the radiation, right? The radiation fields can be somewhat easily calculated provided you start off with the current distributions, and calculate the vector potential and from there you go to the magnetic field.

The reason why it becomes simple is because J and A direction coincide with respect to each other. Contrast this with Biot-Savart law. Okay. Biot-Savart law now I, what I would like you to do is to go look up Biot-Savart law and then you will see that there will be a $\hat{n} \times J$ something sitting there in that Biot-Savart law meaning that we are actually going to look at this direction, right? So that would actually also -- that would not be in the same direction as the J field. B and J are not in the same direction of the J field, right? So because of that the calculations within Biot-Savart law are slightly more complicated, actually very complicated. Simply write this as A(r). Okay.

(Refer Slide Time 12:53)

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\nabla^2 \phi = -\rho/\epsilon_0$$

$\nabla^2(A_x)$	$-\mu_0 J_x$
A_y	$-\mu_0 J_y$
A_z	$-\mu_0 J_z$

Biot Savart

$$\vec{A}(\vec{r}) = \int \frac{\mu_0 \vec{J}(\vec{r}') dv'}{4\pi |\vec{r} - \vec{r}'|}$$

$$\hat{n} \times \vec{J}$$

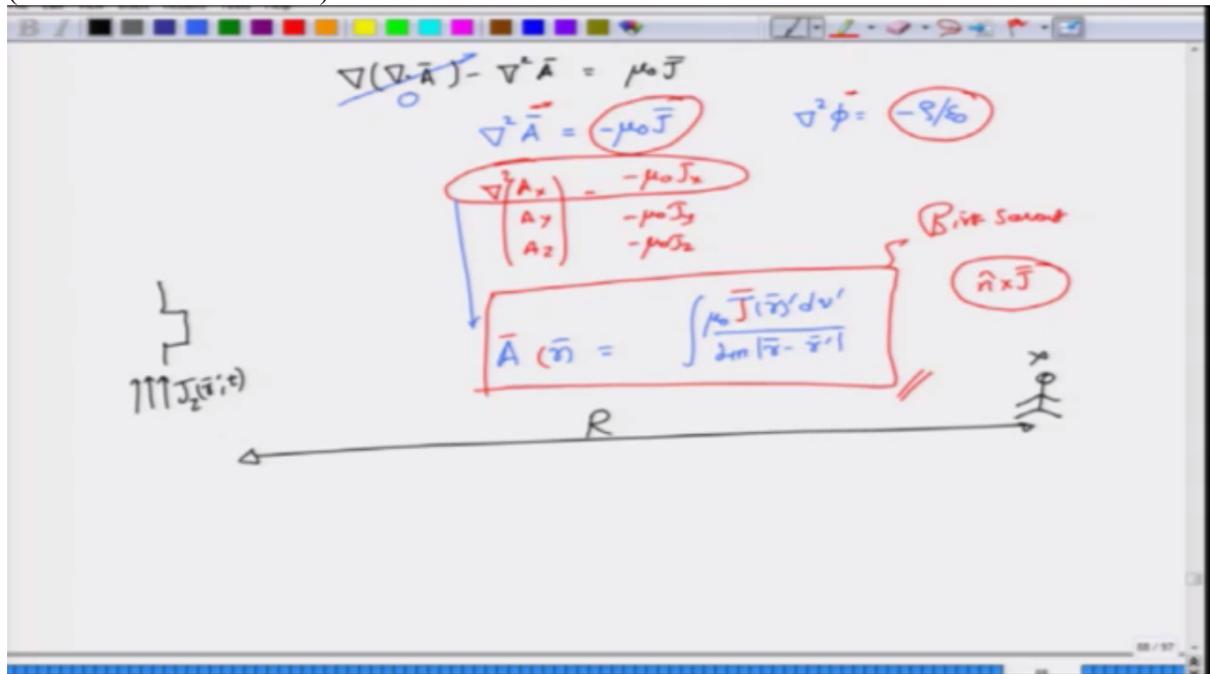
I mean, what we mean to say is that it's an easier method to calculate A and then calculate what is the magnetic field B. Okay.

Now all these things is actually okay for the static case, but I'm not really interested in the static case. See for one primary reason, let's assume that my current distribution is along this z direction, okay, and it is slowly now varying. Okay. So I will assume that this is only J_z component and it is slowly changing with respect to time Okay. And I am sitting very, very

far away here. So I am actually standing here or sitting here very, very far away from the source distribution, and I won't notice.

So let's say there is some source which has come, turned the current on and then turned the current off. Do you expect that this changing in the current, so if the current is actually turned to 0, and then suddenly turned on, and then suddenly turned off, will you be able to see this sudden change in the current or in general, even if it is not sudden, it's a slow variation of the change, will you be able to see directly sitting at a distance R away? That is the distance between these two is actually R no? R, which is the distance between the field and the -- source and the field coordinates.

(Refer Slide Time 14:08)



I won't be able to see this because clearly if anything is changing, right, that characteristic will be visible after a propagation delay, which is given by the distance between the source and the observer or rather the source and the field coordinates and divided that one by the velocity of propagation, right?

So taking a simple thing of the distance between them to be R and being in the free space, there has to be an at least delay of R/c seconds before this change in the current distribution is actually visible to me, right? But such a coding is not possible in this expression that we have written. I mean, where is the time dependence on this? There is no time dependence on this.

So, therefore, we have to go back and look for what kind of a equation we should write such that the equation is valid for the time varying case, that is for the dynamic scenario as well. Okay.

So, clearly, in the dynamic scenario, things are going to change slightly because $\nabla \times \vec{E}$ is now no longer equal to 0, but you have $-\text{del } B/\text{del } t$.

(Refer Slide Time 15:15)

$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$
 $\nabla^2 \vec{A} = -\mu_0 \vec{J}$
 $\nabla^2 \phi = -\rho/\epsilon_0$
 $\begin{pmatrix} \nabla^2 A_x \\ \nabla^2 A_y \\ \nabla^2 A_z \end{pmatrix} = \begin{pmatrix} -\mu_0 J_x \\ -\mu_0 J_y \\ -\mu_0 J_z \end{pmatrix}$
 $\vec{A}(\vec{r}) = \int \frac{\mu_0 \vec{J}(\vec{r}') dV'}{4\pi |\vec{r} - \vec{r}'|}$
 $t_d = R/c$
 $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Now right away when you see this equation, your hope should actually extinguish of finding a scalar potential ϕ because this scalar potential ϕ was valid only when the $\nabla \times E$ term was actually equal to 0.

Now because it is not zero, in fact, there is a term which is $-\text{del } B/\text{del } t$, clearly, I cannot take this electric, express electric field as a gradient of ϕ . So what am I going to do with this now?

(Refer Slide Time 15:51)

$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$
 $\nabla^2 \vec{A} = -\mu_0 \vec{J}$
 $\nabla^2 \phi = -\rho/\epsilon_0$
 $\begin{pmatrix} \nabla^2 A_x \\ \nabla^2 A_y \\ \nabla^2 A_z \end{pmatrix} = \begin{pmatrix} -\mu_0 J_x \\ -\mu_0 J_y \\ -\mu_0 J_z \end{pmatrix}$
 $\vec{A}(\vec{r}) = \int \frac{\mu_0 \vec{J}(\vec{r}') dV'}{4\pi |\vec{r} - \vec{r}'|}$
 $t_d = R/c$
 $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
 $\nabla \times \vec{E} = 0$
 ϕ

Now what we do is an interesting thing. I know $\nabla \times E = -\text{del}/\text{del } t$ and there is a B, right? Forgetting for the moment that I'm dealing with the dynamic scenario, I will write B as curl of A.

Now I am actually guaranteed that this is okay because the condition for defining B in terms of A was that $\nabla \cdot \mathbf{B}$ was equal to 0. That is the B field had no divergence, which is true whether you're dealing with static or in the dynamic case, right? So even when the fields are changing with respect to time, okay, when $\nabla \cdot \mathbf{B}$ as a function of both space and time is equal to 0, because it is equal to 0, the corresponding B can always be expressed as the scalar potential A except that the scalar potential A is now varying both with respect to r and t. Okay.

(Refer Slide Time 16:38)

Handwritten mathematical derivation on a whiteboard:

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\nabla^2 \phi = -\rho/\epsilon_0$$

$\nabla^2 A_x$	$= -\mu_0 J_x$
A_y	$= -\mu_0 J_y$
A_z	$= -\mu_0 J_z$

Retarded potential formula:

$$\vec{A}(\vec{r}, t) = \int \frac{\mu_0 \vec{J}(\vec{r}', t')}{4\pi \epsilon_0 |\vec{r} - \vec{r}'|} dV'$$

Time delay: $t' = R/c$

Maxwell's equations for B:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

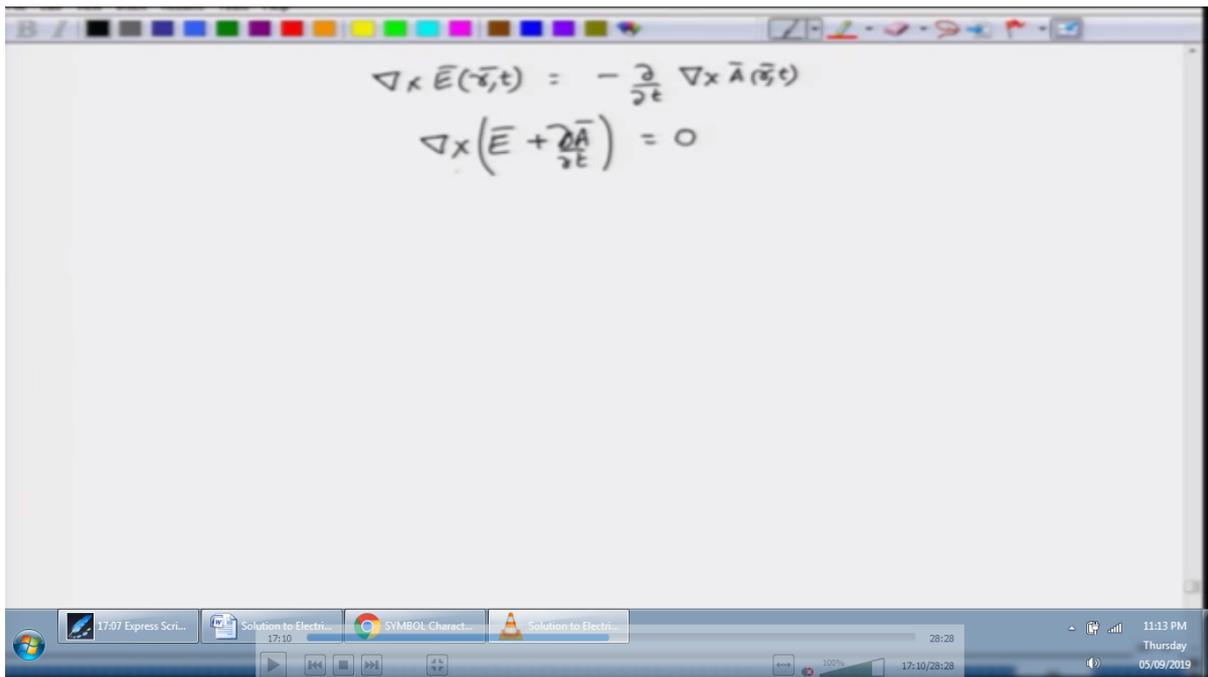
Intermediate equation for the curl of E:

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t} (\nabla \times \vec{A})(\vec{r}, t)$$

Other notes on the whiteboard include: $\vec{B} \text{ with source } \vec{n} \times \vec{J}$, $\vec{J}_z(\vec{r}, t)$, and ϕ .

So you get this sort of an intermediate equation, right, and what we do now is that we write $\nabla \times \vec{E}$, which is now a function of both r and t given by $-\partial/\partial t \nabla \times \vec{A}(\vec{r}, t)$. Correct? I'll put this on to the left-hand side, interchange the operations of the curl as well as the time derivative here. So I obtain $\nabla \times (\vec{E} + \partial \vec{A}/\partial t)$. This entire thing should be equal to 0. Okay.

(Refer Slide Time 17:10)



Now because I have a curl of some new field is equal to 0, this field which is $\vec{E} + \text{del } A/\text{del } t$, I can express this as negative gradient of ϕ . Okay. Of course, this ϕ is now not just constant. It is actually a function of, I mean, not constant in the sense not time independent. It is now function of time as well.

So earlier you had this ϕ , which was time independent, now the scalar potential is also time dependent. Okay. Of course, every quantity here in this expression is time dependent.

So I can rewrite this \vec{E} in terms of a time dependent scalar potential, okay, as well as time dependent vector potential. So now what you've actually seen here is a very interesting thing. See in the static case, this term would be equal to 0, and then we will not have this \vec{E} being, you know, a function of time and ϕ being function of time. They would simply be functions of r . But when time is not constant, you have this coupling term, right? And this coupling term that you have is actually the coupling term that is arising because of the current distribution.

(Refer Slide Time 18:16)

$$\nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \nabla \times \vec{A}(\vec{r}, t)$$

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi(\vec{r}, t)$$

$$\vec{E}(\vec{r}, t) = -\nabla \phi(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t}$$

Handwritten note: $\frac{\partial \vec{A}}{\partial t}$ is a Coupling term

So electric field is now being influenced by the magnetic field. Okay. Similarly, magnetic field will be influenced by the electric field because $\nabla \times \vec{H}$ is having a component of $\text{del } E / \text{del } t$. So, in this way, the magnetic field is influencing electric field. Electric field in turn will induce the magnetic field.

Of course, if you are interested only in writing what is \vec{B} , then \vec{B} can still be written as $\nabla \times \vec{A}$, right? And any variation of \vec{A} with respect to time will change the electric field and why I change the magnetic field by these expressions $\nabla \times \vec{E} = -\text{del } \vec{B} / \text{del } t$.

(Refer Slide Time 18:51)

$$\nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \nabla \times \vec{A}(\vec{r}, t)$$

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi(\vec{r}, t)$$

$$\vec{E}(\vec{r}, t) = -\nabla \phi(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t}$$

$$\vec{B}(\vec{r}, t) = \nabla \times \vec{A}(\vec{r}, t)$$

Handwritten note: $\frac{\partial \vec{A}}{\partial t}$ is a Coupling term

Handwritten note: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (circled)

So please don't lose of the fact that we have now graduated to the fields, but then we have this kind of a, you know, expressions out there.

Now there's an interesting thing that we want to find out. We already know how in the static case the general expressions for the scalar potential and vector potential are going to look like. Correct? Now what would be the corresponding equations for the scalar potential and the vector potential so that we can solve them and find out the solutions in the time varying case?

In order to answer that question, let us start with $\nabla \times \mathbf{H}$, which is given by $\mathbf{J} + \epsilon_0 \text{del E}/\text{del t}$. See the set of equations that we are about to write actually work better when we are dealing with the free space conditions. Okay. So because we are in the antenna problems or in the radiation, we are interested in this scenario; we will continue to assume that we are dealing with free space with an ϵ_0 out there. Okay.

(Refer Slide Time 19:51)

The image shows a whiteboard with the following handwritten equations:

$$\nabla \times \bar{E}(\bar{r}, t) = -\frac{\partial}{\partial t} \nabla \times \bar{A}(\bar{r}, t)$$

$$\nabla \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t} \right) = 0$$

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = -\nabla \phi(\bar{r}, t)$$

$$\bar{E}(\bar{r}, t) = -\nabla \phi(\bar{r}, t) - \frac{\partial \bar{A}(\bar{r}, t)}{\partial t}$$

There is a handwritten note in purple: Δ / *Coupling term* with an arrow pointing to the $-\frac{\partial \bar{A}(\bar{r}, t)}{\partial t}$ term.

$$\bar{B}(\bar{r}, t) = \nabla \times \bar{A}(\bar{r}, t)$$

The equation $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ is circled in purple.

$$\nabla \times \bar{H} = \bar{J} + \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

So I have this $\nabla \times \mathbf{H} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$. There is nothing I can do with \mathbf{J} , but for \mathbf{E} , I can use this expression and replace and for \mathbf{B} , now \mathbf{B} is basically $\mu_0 \mathbf{H}$. Correct? Therefore, \mathbf{H} is given by $\nabla \times \mathbf{A}/\mu_0$. Okay.

(Refer Slide Time 20:06)

$$\begin{aligned} \nabla \times \vec{E}(\vec{r}, t) &= -\frac{\partial}{\partial t} \nabla \times \vec{A}(\vec{r}, t) \\ \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) &= 0 \\ \vec{E} + \frac{\partial \vec{A}}{\partial t} &= -\nabla \phi(\vec{r}, t) \\ \rightarrow \vec{E}(\vec{r}, t) &= -\nabla \phi(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \quad \text{Curl term} \\ \mu_0 \vec{H} = \vec{B}(\vec{r}, t) &= \frac{\nabla \times \vec{A}(\vec{r}, t)}{\mu_0} \\ \nabla \times \vec{H} &= \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

So because I know that, I can rewrite the, I mean, I can include this H expression in here on the left-hand side and for E I can substitute here, right? So after doing these substitutions, I get $\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$. I hope we understand where this is coming from. This $\nabla \times \vec{A}$ is coming from $\nabla \times \vec{A} / \mu_0$.

Now I have multiplied that μ_0 on to this side. So that is coming from definition of H and then ∇ cross of that is already present. Okay. So $+\mu_0 \epsilon_0$, which, of course, you can recognize this as $1/c^2$ and then you have, so $\text{del } E / \text{del } t$. So E is what? E is basically $-\nabla \phi - \text{del } A / \text{del } t$. All these quantities are varying with time. Okay.

(Refer Slide Time 20:56)

$$\begin{aligned} \nabla \times \vec{E}(\vec{r}, t) &= -\frac{\partial}{\partial t} \nabla \times \vec{A}(\vec{r}, t) \\ \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) &= 0 \\ \vec{E} + \frac{\partial \vec{A}}{\partial t} &= -\nabla \phi(\vec{r}, t) \\ \rightarrow \vec{E}(\vec{r}, t) &= -\nabla \phi(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \quad \text{Curl term} \\ \rightarrow \vec{H}(\vec{r}, t) &= \frac{\nabla \times \vec{A}(\vec{r}, t)}{\mu_0} \\ \nabla \times \vec{H} &= \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \nabla \times (\nabla \times \vec{A}) &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{\partial \vec{A}}{\partial t} \right) \end{aligned}$$

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Now I can, you know, write down this expression as $1/c^2$ minus $\text{del}^2 A/\text{del } t^2$ minus interchanging the gradient and you know the derivative term, I can rewrite this $\text{del}/\text{del } t (-\nabla\phi)$ as this particular quantity. Okay. I still have $\mu_0 J$ and on to the left-hand side, I can write this as $\nabla \nabla \cdot \bar{A} - \nabla^2 \bar{A}$. Okay.

(Refer Slide Time 21:28)

Handwritten derivations on a whiteboard:

$$\nabla \times \bar{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \nabla \times \bar{A}(\vec{r}, t)$$

$$\nabla \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t} \right) = 0$$

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = -\nabla \phi(\vec{r}, t)$$

$$\rightarrow \bar{E}(\vec{r}, t) = -\nabla \phi(\vec{r}, t) - \frac{\partial \bar{A}(\vec{r}, t)}{\partial t}$$

Complicated term

$$\rightarrow \bar{H}(\vec{r}, t) = \frac{\nabla \times \bar{A}(\vec{r}, t)}{\mu_0} \quad \frac{1}{c^2} \quad \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J} + \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

$$\nabla \times (\nabla \times \bar{A}) = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{\partial \bar{A}}{\partial t} \right)$$

$$\nabla (\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu_0 \bar{J} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} - \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t}$$

Now what I like to do is to kind of get inspired from the free space propagation equation that we wrote. In the free space propagation equation that we wrote for the electric field, we came across what is called as a “Wave equation” where you had ∇^2 of electric field E, E being, you know, equal to some constant of proportionality. In fact, that constant of proportionality was $1/c^2 \text{del}^2 A$ by or $\text{del}^2 E/\text{del } t^2$.

So you had the second-order spatial derivative in terms of electric field being equated to the second order derivatives of the electric field with respect to time. Okay. You do have such a scenario in this set of equations. Okay. Observe carefully that you have this $-\nabla^2 \bar{A}$ term, which is the spatial derivative being related to $-1/c^2 \text{del}^2 A/\text{del } t^2$ in the time derivative case. Okay.

(Refer Slide Time 22:28)

$$\begin{aligned} \nabla \times \vec{E}(\vec{r}, t) &= -\frac{\partial}{\partial t} \nabla \times \vec{A}(\vec{r}, t) \\ \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) &= 0 \\ \vec{E} + \frac{\partial \vec{A}}{\partial t} &= -\nabla \phi(\vec{r}, t) \\ \rightarrow \vec{E}(\vec{r}, t) &= -\nabla \phi(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \quad \checkmark \text{ Coupling term} \\ \rightarrow \vec{H}(\vec{r}, t) &= \frac{\nabla \times \vec{A}(\vec{r}, t)}{\mu_0} \quad \frac{1}{c^2} \end{aligned}$$

$$\nabla \times \vec{H} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{\partial \vec{A}}{\partial t} \right)$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \left(\frac{-1}{c^2} \nabla \frac{\partial \phi}{\partial t} \right)$$

What we don't want are these two quantities. Okay. So I don't want this quantity that is messing up my equations and I would like to remove this one, but since this is the situation where we are actually dealing with the sources, we are actually trying to find out how radiation is happening, I cannot remove this. Okay. So always keep this term here because you're actually working with the case for finding the fields when there are current distributions and eventually charge distributions.

(Refer Slide Time 22:49)

$$\begin{aligned} \nabla \times \vec{E}(\vec{r}, t) &= -\frac{\partial}{\partial t} \nabla \times \vec{A}(\vec{r}, t) \\ \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) &= 0 \\ \vec{E} + \frac{\partial \vec{A}}{\partial t} &= -\nabla \phi(\vec{r}, t) \\ \rightarrow \vec{E}(\vec{r}, t) &= -\nabla \phi(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \quad \checkmark \text{ Coupling term} \\ \rightarrow \vec{H}(\vec{r}, t) &= \frac{\nabla \times \vec{A}(\vec{r}, t)}{\mu_0} \quad \frac{1}{c^2} \end{aligned}$$

$$\nabla \times \vec{H} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{\partial \vec{A}}{\partial t} \right)$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \left(\frac{-1}{c^2} \nabla \frac{\partial \phi}{\partial t} \right)$$

The way to remove these terms which I have identified in this dashed lines is kind of very simple. What I do is I make this left-hand side actually equal to the right-hand side. Okay. What that actually means is that, let me erase this here and/or write it here. Please note that

I'm writing this equality conditions here. Okay. That is gradient of $\nabla \cdot \mathbf{A}$ is equal to $-1/c^2 \nabla(\text{del } \phi/\text{del } t)$. Okay. Is it $\text{del } \phi/\text{del } t$? Yeah. Okay. This is $\text{del } \phi/\text{del } t$, right?

Now because gradient is a common factor, I can put that outside and then pull the right-hand side on to the left-hand side. I get $\nabla \cdot \mathbf{A} + 1/c^2 (\text{del } \phi/\text{del } t)$ is equal to 0. Okay.

(Refer Slide Time 23:39)

Handwritten mathematical derivation on a whiteboard:

$$\nabla \times \bar{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \nabla \times \bar{A}(\vec{r}, t)$$

$$\nabla \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t} \right) = 0$$

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = -\nabla \phi(\vec{r}, t)$$

✓✓✓ Ampere's law

$$\rightarrow \bar{E}(\vec{r}, t) = -\nabla \phi(\vec{r}, t) - \frac{\partial \bar{A}(\vec{r}, t)}{\partial t}$$

$$\rightarrow \bar{H}(\vec{r}, t) = \frac{\nabla \times \bar{A}(\vec{r}, t)}{\mu_0} \quad \frac{1}{c^2}$$

$$\nabla \times \bar{H} = \bar{J} + \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

$$\nabla \times (\nabla \times \bar{A}) = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{\partial \bar{A}}{\partial t} \right)$$

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu_0 \bar{J} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} - \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t}$$

$$\nabla(\nabla \cdot \bar{A}) = -\frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t}$$

$$\nabla(\nabla \cdot \bar{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t}) = 0$$

Now one way of making the gradient of some field quantity equal to 0 is to set this entire thing itself equal to 0. So I can write $\nabla \cdot \mathbf{A}$ as $-1/c^2 (\text{del } \phi/\text{del } t)$. So I actually have one equation here and when I substitute that, I can remove this equation or this term from this equation. I remove this entire term here. Okay.

(Refer Slide Time 24:05)

Handwritten mathematical derivation on a whiteboard:

$$\nabla \times \bar{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \nabla \times \bar{A}(\vec{r}, t)$$

$$\nabla \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t} \right) = 0$$

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = -\nabla \phi(\vec{r}, t)$$

✓✓✓ Ampere's law

$$\rightarrow \bar{E}(\vec{r}, t) = -\nabla \phi(\vec{r}, t) - \frac{\partial \bar{A}(\vec{r}, t)}{\partial t}$$

$$\rightarrow \bar{H}(\vec{r}, t) = \frac{\nabla \times \bar{A}(\vec{r}, t)}{\mu_0} \quad \frac{1}{c^2}$$

$$\nabla \times \bar{H} = \bar{J} + \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

$$\nabla \times (\nabla \times \bar{A}) = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{\partial \bar{A}}{\partial t} \right)$$

$$-\nabla^2 \bar{A} = \mu_0 \bar{J} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2}$$

$$\nabla \cdot \bar{A} = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$$

So what I now have is $\nabla^2 \vec{A}$. $\nabla^2 \vec{A}$ is present. So you can pull this $\nabla^2 \vec{A} / \nabla^2$ back into this. So this is $1/c^2$ and then rearrange the equation. So I'm going to erase this or I can first make this into plus here, plus here and a minus sign here and then pull this fellow onto the left-hand side. So I now have $\nabla^2 \vec{A} + 1/c^2 (\nabla^2 \vec{A} / \nabla^2)$, which must be equal to $-\mu_0 \vec{J}$. Okay. And then I have $\nabla \cdot \vec{A}$ is equal to this equation.

(Refer Slide Time 24:45)

Handwritten mathematical derivation on a whiteboard:

$$\nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \nabla \times \vec{A}(\vec{r}, t)$$

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi(\vec{r}, t)$$

Complex term

$$\rightarrow \vec{E}(\vec{r}, t) = -\nabla \phi(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t}$$

$$\rightarrow \vec{H}(\vec{r}, t) = \frac{\nabla \times \vec{A}(\vec{r}, t)}{\mu_0} + \frac{1}{c^2} \nabla \times \vec{E}$$

$$\nabla \times \vec{H} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{\partial \vec{A}}{\partial t} \right)$$

$$\nabla^2 \vec{A} + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\nabla \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$$

Side calculation:

$$\nabla (\nabla \cdot \vec{A}) = -\frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t}$$

$$\nabla \left(\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = 0$$

Finally, we can also show that there will be another equation for ϕ as well, which is basically $\nabla^2 \phi - 1/c^2 (\nabla^2 \phi / \nabla^2)$ that should be equal to $-\rho$ by, sorry, this is $+\rho/\epsilon_0$. Okay. So that would be equal to $+\rho/\epsilon_0$, right?

(Refer Slide Time 25:11)

Handwritten mathematical equation on a whiteboard:

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = +\rho/\epsilon_0$$

So this is another wave equation except that in this wave equation, the source term is the charge distribution and in the previous case, the source term for the magnetic vector potential is the current distribution, right?

So I will leave the next part as an exercise for you to find. I don't know whether there might be a + sign or a - sign. You can actually put what, you know, the correct sign here. So I will leave this also as an exercise for you to figure out.

(Refer Slide Time 25:37)

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{\rho}{\epsilon_0}$$

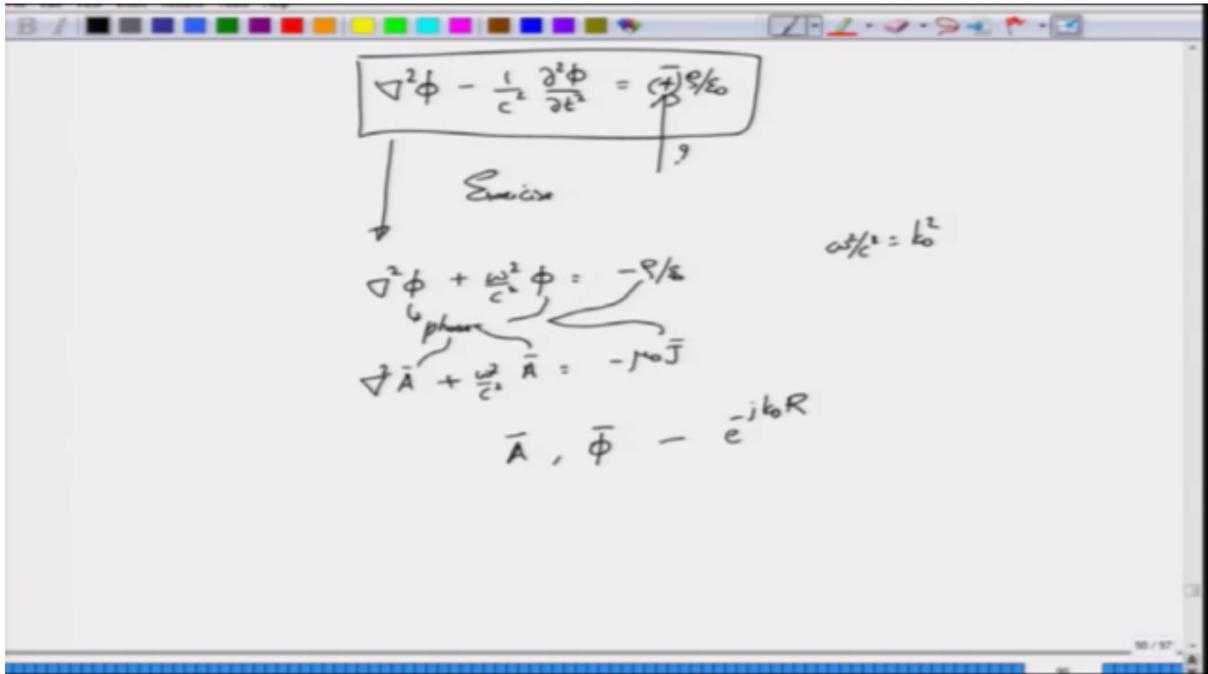
Exercise

But what I would like to stress is that if the right-hand side were actually equal to 0, both the electric scalar potential, which is now time dependent as well as the magnetic vector potential, which is, of course, also time-dependent, both will satisfy the same type of wave equation, which means that it is possible for us to think of a sinusoidal excitation of the current distribution as well as a charge distribution, therefore, leading to what is called as, I mean, what is the sinusoidal distribution for the magnetic field or magnetic vector potential as well as for the electric scalar potential meaning that I can, you know, use the phaser ideas and convert all these time quantities into the frequency quantities, right?

So if I do that, for example, into this expression, what do I get? I get $\nabla^2 \phi$, now ϕ is the phaser that we are dealing with plus ω^2/c^2 because remember $\text{del}/\text{del } t$ goes as $j\omega$; $\text{del}^2/\text{del } t^2$ goes as ω^2 times 5 must be equal to $-\rho/\epsilon^0$.

Similarly, you will have $\nabla^2 \mathbf{A} + (\omega^2/c^2) \mathbf{A}$ to be equal to $-\mu_0 \mathbf{J}$. So every quantity here is a phaser that I am looking at. So this is also a phaser. This is also a phaser. This is also a phaser. Of course, $\omega^2/c^2 = k_0^2$ and in the absence of the right-hand source terms, the plausible solutions for \mathbf{A} as well as for ϕ could have been e^{jk_0} depending on the direction of propagation.

(Refer Slide Time 27:13)



In general, the direction of propagation is along R. So this would be $e^{-jk_0 R}$. Okay. That is this is the exponential phase dependence, but we are going to show this correctly, right?

So the equations that we have written are what are called as inhomogeneous or the equations are called as inhomogeneous differential equations because the source terms are not actually set to 0. Okay. They will complicate the solutions, but that is what the whole point of radiation is. I mean, we want to understand what kind of current distribution and what kind of charge distribution is going to generate a given vector field, vector potential as well as the magnetic potentials.

Thank you very much.

an IIT Kanpur Production
(c) copyright reserved