Principles of Signals and Systems

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Lecture - 14
Example Problems LTI Systems-Convolution, Periodic Convolution, BIBO Stability

Hello welcome to another module in this massive open online course. So, we are looking at example problems in the analysis of continuous time and discrete time LTI systems right linear time invariant systems let us continue looking at these examples.

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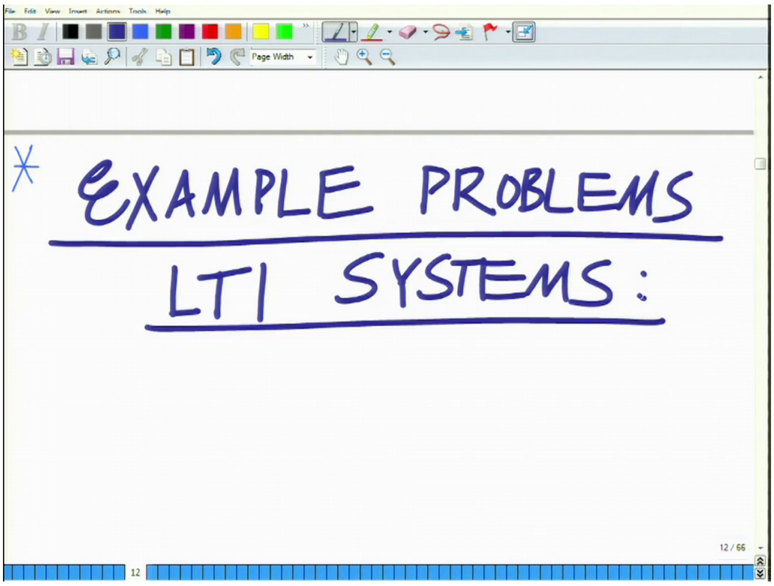

So, we are looking at example problems for the analysis of with the analysis of LTI systems.

And let us start looking at another problem which is basically we are given that x t equals u t minus u t minus 1.

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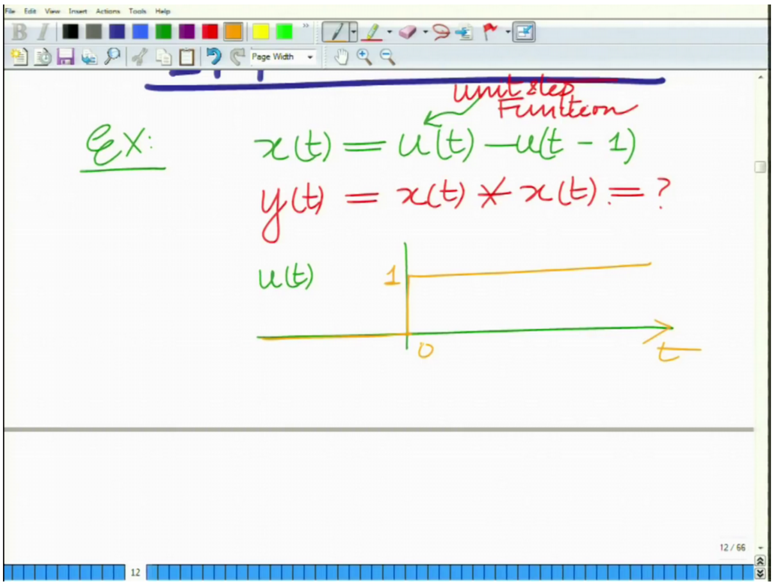

Where u t remember, this is the unit step function. U t is the unit step function. We want to find out what is y t equal to x t convolved with x t. What this? First let us try plotting u t x t now we have we know u t is the unit step function. This is given by 1 t is greater than or equal to 0 and is 0 otherwise this is the unit step function, which is equal to 1 if t is greater than equal to 0 and 0 otherwise.

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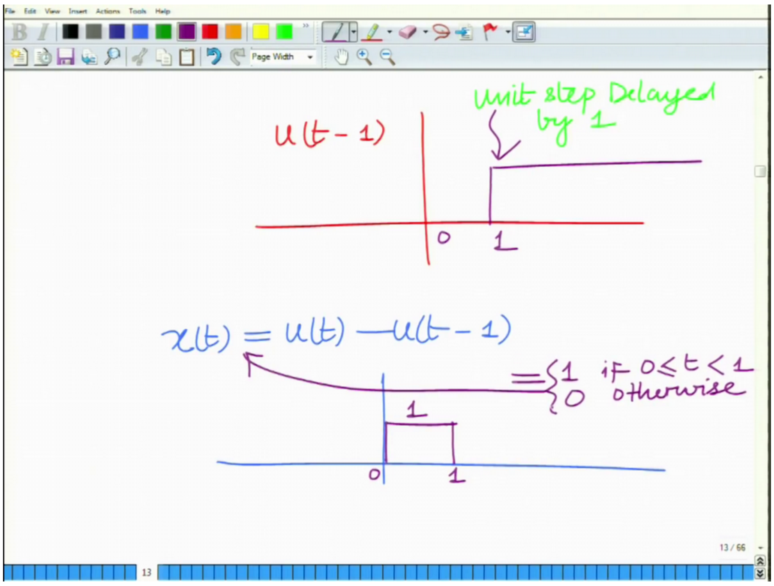

Now, u t minus 1 is the unit step function delayed by 1. So, this will be take the unit step function. And delay it by 1 unit. So, this is unit step delayed by unit step delayed by 1.

And therefore, x t you can see which is equal to u t minus u t minus 1. This will be given as this will be given by the function, which is equal to 1 when 0 or less than or equal to t less than 1. So, this is x t which is u t minus u t minus 1 equals 1 if 0 less than or equal to t less than 1, and this is 0 otherwise. So, this is your this is your x t.

Basically, it is a pulse of height one between 0 and 1, you can clearly see this is a pulse of height equal to 1 between 0 comma 1 that is from t equal to 0 to 1.

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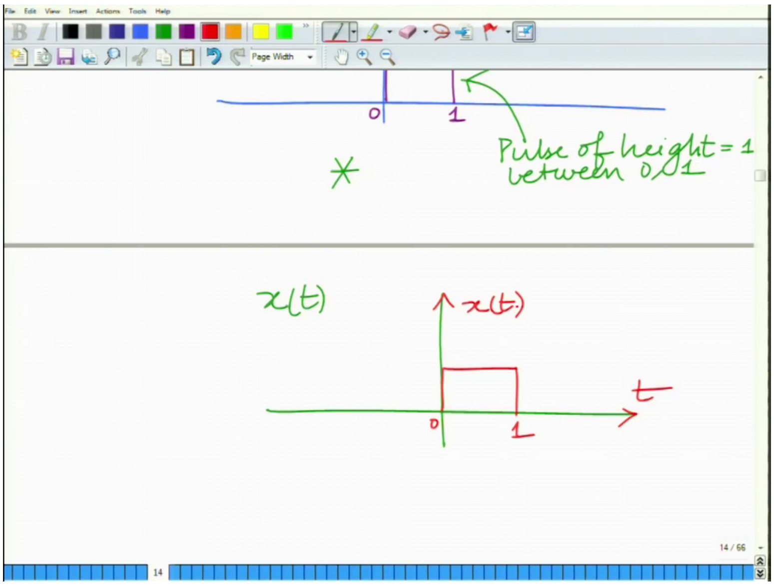

And now we want to convolve x t with itself convolve x t with another x t which is again; obviously, this is going to be a pulse of height one between 0 and between 0 and 1.

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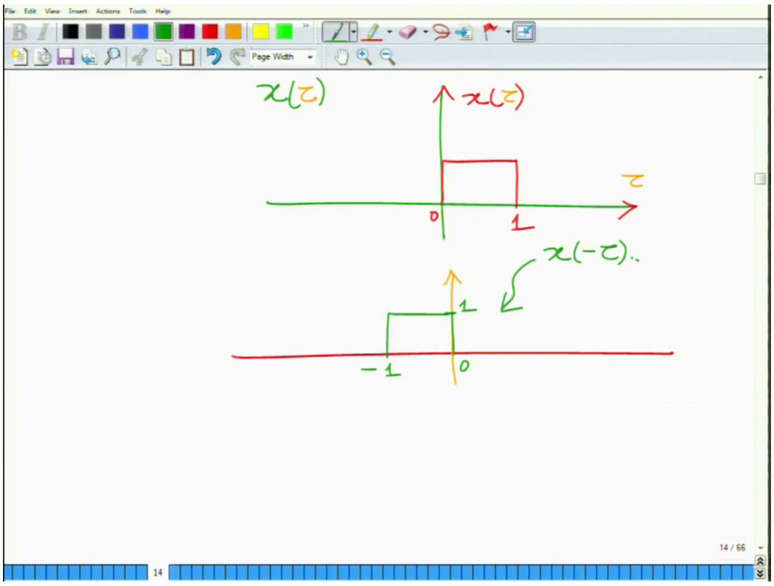

Now to convolve remember we need to flip first we are convolving x t with x t. So, we need to first flip it about 0.

So, let us say this we make it as the tau axis rather than the t axis. So, we make this as x of tau and this is becomes your x of tau. So, x of minus tau will be flip about the y axis. So, this will be pulse of height one this will be pulse of height one, and between minus 1 and 0. So, this is your x of minus tau.

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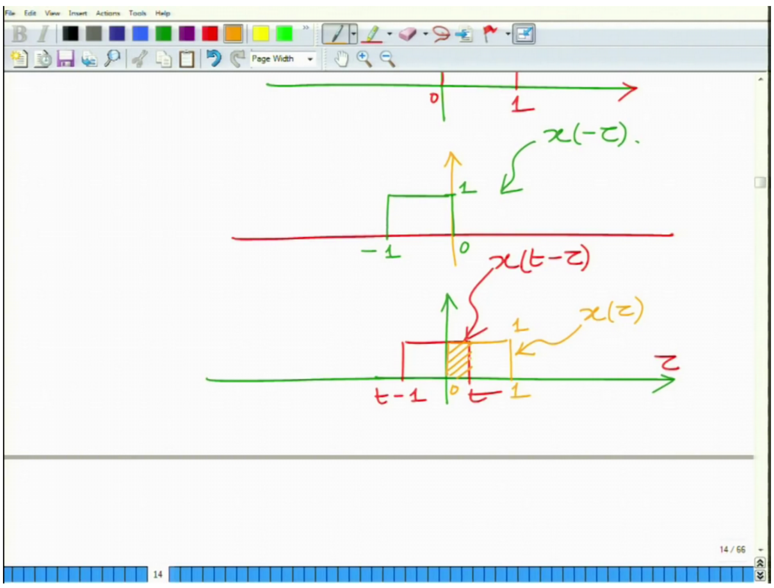

And x of t minus tau will be delayed version of this by t. So, x of t minus tau will be shifted to the right y t as you as we have. So, this is t this is t minus 1 and this is x of t minus tau. So, this is your tau axis, this is x of t minus tau. And now multiply this with x of tau remember x of tau is still a pulse of height one between 0 and 1.

So, this is a pulse of height one between 0 and 1, and this is your x of tau. So, multiply x of t minus tau by x of tau integrate from 0 to infinity. So, this is basically the area of overlap between these 2 pulses which can be seem to be; so, you can see integral minus infinity to infinity x of tau x of t minus tau d tau.

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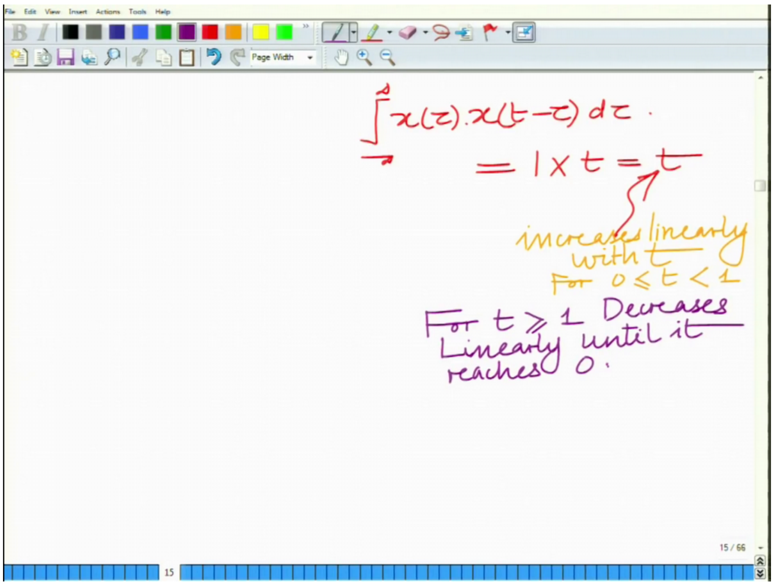

Remember for x of t minus tau we flip about the y axis; that is, look at x of minus tau and then delay it by t shifted to the right by t. And now you can see the overlap corresponding overlap is of t height 1.

So, this will be equal to 1 into t equals t. And therefore, this increases linearly with t increases linearly with t for 0 less than equal to t less than 1. Now once t greater than 1, from t grade for 3 greater than 1 it decreases again if you can see right for t greater than 1. You can see it again decreases linearly greater than 1 or greater than equal to 1 decreases linearly because overlap progressively decreases until it reaches 0.

So, which means this will look something like this will look something Like from 0 to one it increases.

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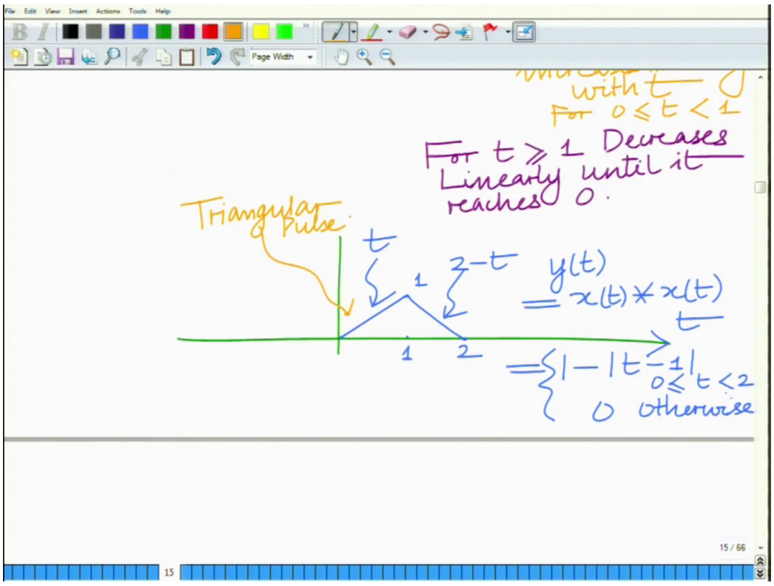

So, this is at one from one it decreases until it hits 2. So, this is t this height corresponds to the maximum overlap which is 1 and this is therefore, this portion will be 2 minus t. So, this is your y t equals x t convolved with x t. And you can see this can be described as this can be described as 1 minus mod magnitude t minus 1 or for 0 less than equal to t less than 2, and this is 0 and this is 0 otherwise.

So, this is basically your triangular waveform or not a wave form a triangular pulse basically. So, the convolution of 2 square pulses of equal width character the convolution of 2 square pulses of equal width, and height there is convolution of square pulse with itself yields the triangular pulse. This is 1 minus modulus of mod of t minus 1 for 0 less than equal to t less than equal to 2, and this is 0 otherwise. So, this is basically the result of the convolution.

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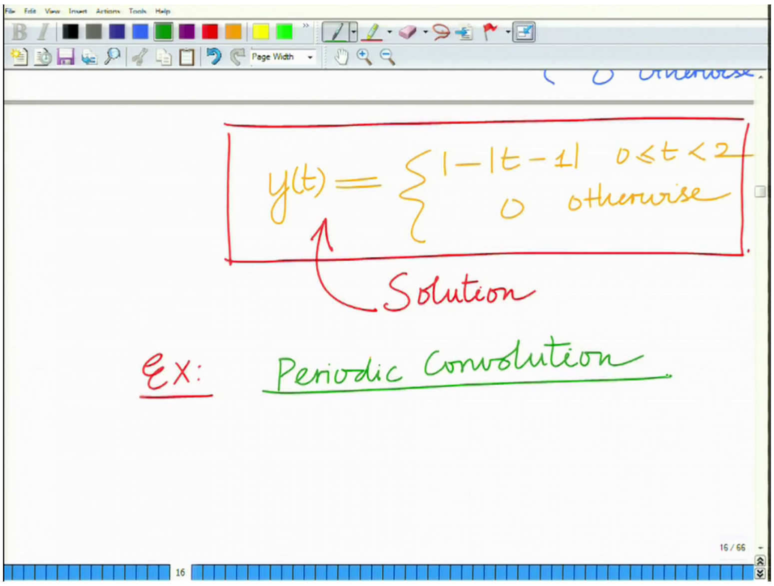

Let me just write it over here once again this is 1 minus magnitude t minus 1 0 less than equal to t less than 2 and this is 0 otherwise. So, this is the solution. This is the final solution.

Let us now look at the next problem which is that of a periodic convolution between 2 periodic signals. Periodic convolution is defined thus let x 1 x 2 be periodic with a common period T naught let there be let x 1 to x 2 b 2.

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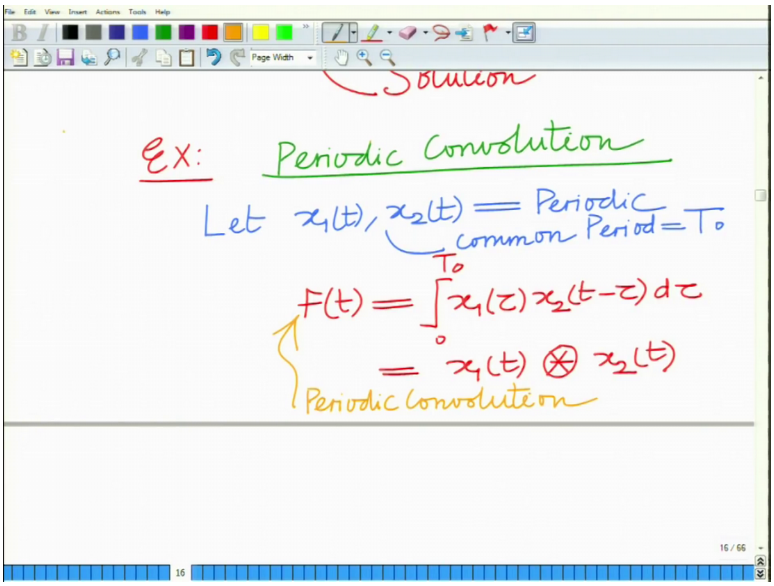

2 signals which are periodic with common period T naught capital T naught the periodic convolution f of t is defined as 0 to T naught x 1 tau that is convolution over one period x 1 tau x 2 t minus tau d tau and it is represented as x 1 t with this kind of a symbol.

And this is termed as a periodic convolution.

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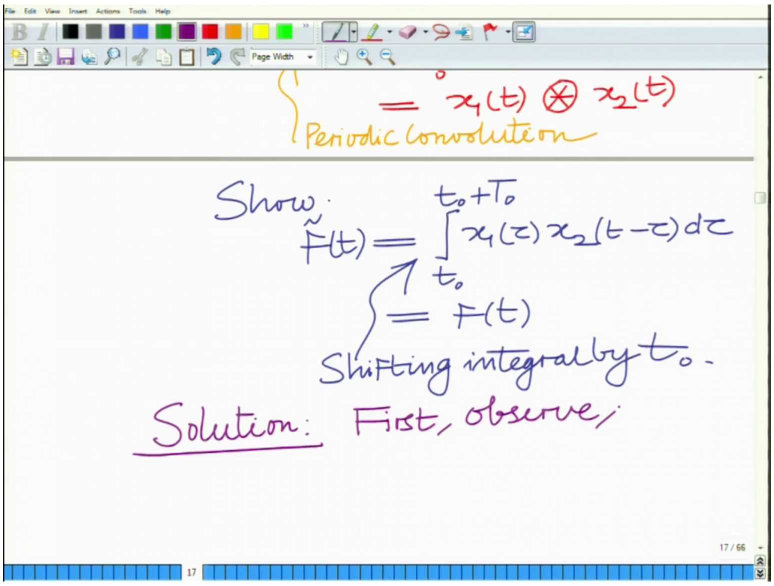

And it is desired to show that this periodic convolution remains unchanged if it if you shift it by small T naught. So, show that f of t if you shift it if you shift the integral by small T naught that is x 1 t x 2 that is if you denote this by f tilde at t x 2 t minus tau d tau, we need to show that this is equal to f t; that is, you are shifting the integral by T naught. Shifting integral by small T naught any arbitrary value small T naught. Now the solution can proceed as follows.

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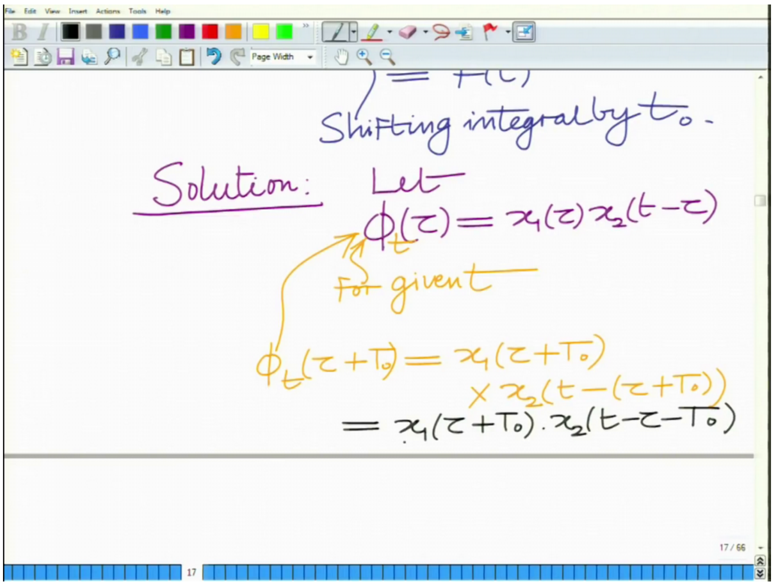

First observe that now, let us set phi of t now let phi of tau for any given value of t equals x 1 of tau x 2 of t minus tau; that is, phi of tau for now let us subscript this by t to show the dependence also on t for given for given t. Now we will show that phi of t of tau, now let us look at this phi of t of tau plus t equals x 1 of tau plus T naught in to x 2 of t minus tau plus T naught, which is equal to x 1 of tau plus T naught into x 2 of t minus tau minus T naught.

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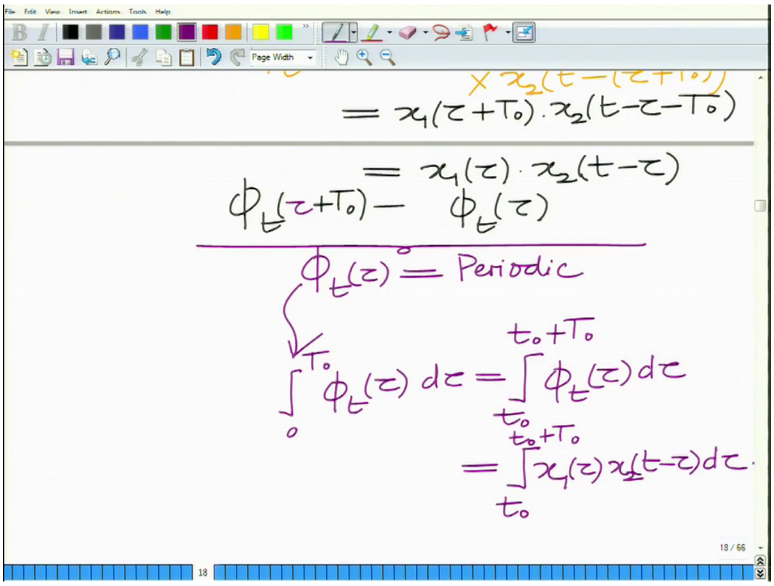

Now, look at this x 1 of tau plus T naught that is shifting by period this is equal to x 1 tau times x 2 of t minus tau minus T naught is also equal is again is equal to x 2 of t minus tau, which means this is equal to phi t of tau. So, what we have shown is phi t of t plus tau phi t of t plus T naught is equal to phi t of tau. So, we have shown that phi sub t of phi sub t I am sorry, phi sub t of tau plus T naught phi sub t of tau plus T naught equals phi sub t after which means phi sub t of tau is periodic. Which means this implies phi sub t of tau this is periodic, which implies that basically integral over any period is same.

So, you have integral 0 to T naught phi sub t of tau d t is integral, you can shift it by any constant quantity small T naught. So, integral this is equal to phi. So, t of tau dt which is equal to T naught to T naught plus tau naught to a small T naught plus capital T naught x of tau x 2 of t minus tau d tau which is equal to f tilde of f tilde of t.

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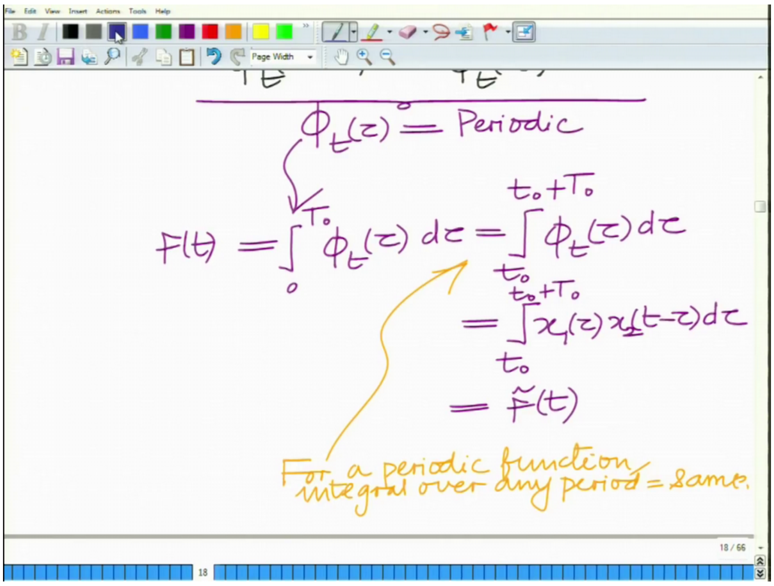

And this is f t and this follows because for a periodic function, for a periodic function integral over any period is the same that is if I have a periodic function with period T naught. Now if I consider any duration right any time interval of duration capital T naught, then the integral will be same therefore, integral 0 to capital T naught will be the same as integral any small T naught to small T naught plus capital T naught.

So, integral over any interval of duration capital T naught where T naught is the capital T naught is the period will be the same for a periodic function. So, we are using that property. So, basically shows that which shows that the periodic convolution also remains unchanged when you consider any time interval of duration capital T naught, that is the basic idea behind this a periodic convolution, which is defined for the convolution of 2 periodic functions with a common period capital T.

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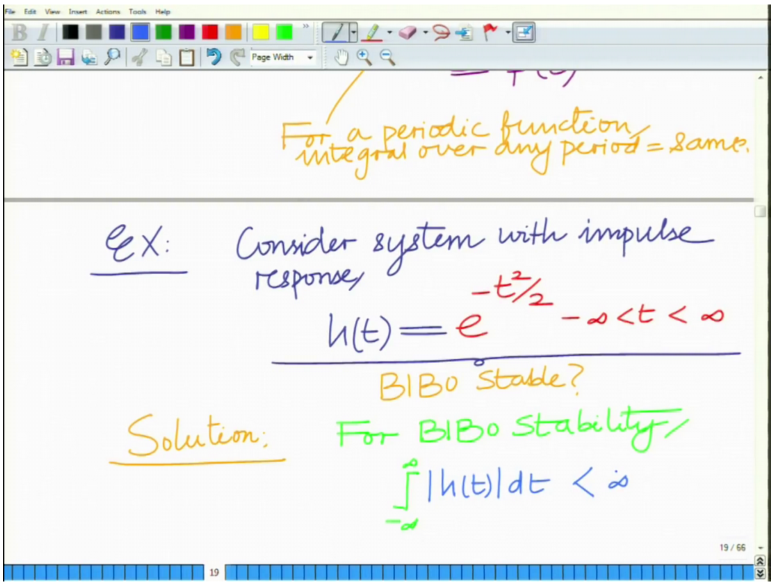

Let us look at the next example, which is consider the system with the impulse response h of t equals e to the power of minus t square by 2 minus infinity less than t less than infinity. So, we want to ask the question is this BIBO stable, is this system BIBO stable and the solution can proceed as follows. For BIBO stability remember, the condition for BIBO stability we have to have minus infinity to infinity must be finite that is the impulse response must be absolutely integrable; that is, integral minus infinity to infinity magnitude h t d t should be a finite quantity.

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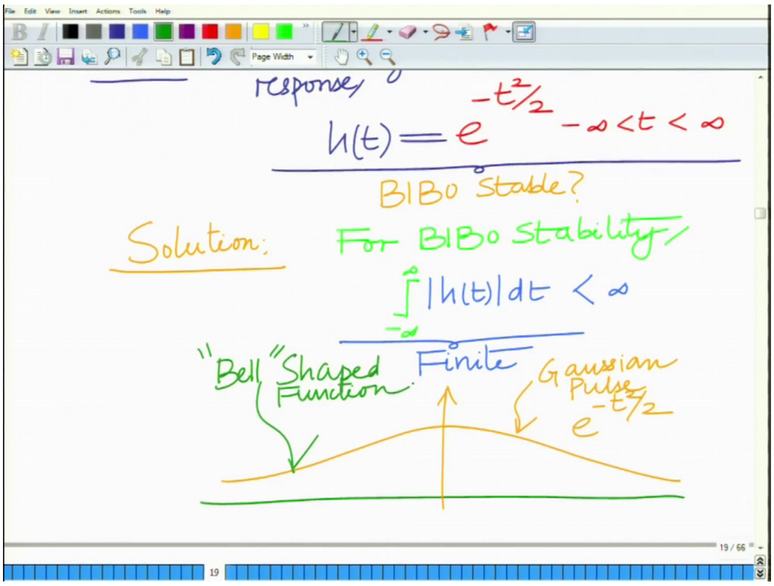

So, this quantity should be finite. Now if you look at e to the power of minus t square by 2, you will realize that it is given basically by a Gaussian pulse. Which is this is a standard signal, this is known as the Gaussian pulse. Due to the power of minus t, it has a bell shaped also known as a bell-shaped function. This is also known as a bell-shaped looks like a bell. Therefore, also known as a bell shape function. Now this is always positive e to the power of minus t square by 2.

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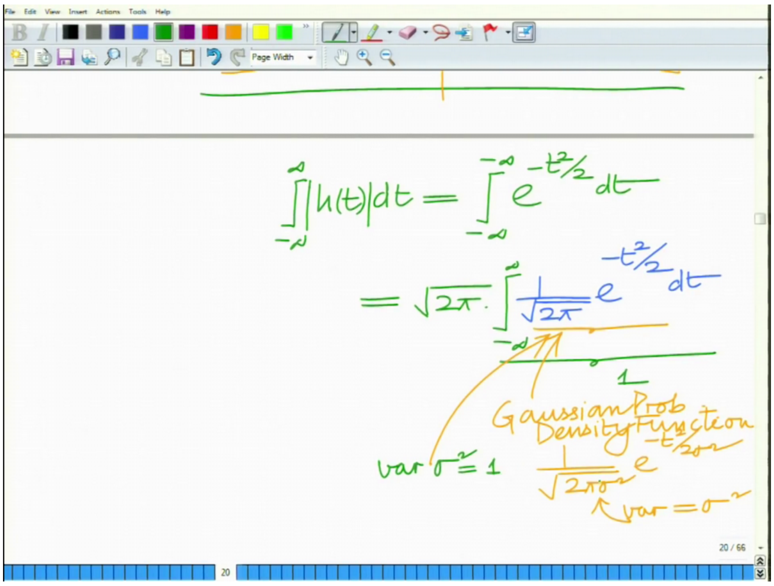

So, therefore, integral minus infinity to infinity to check BIBO stability integral minus infinity to infinity equals integral minus infinity to infinity magnitude, but this is always positive. So, this is simply h of t. Now this is also e to the power of minus t square by 2 is also you can see, now what I am going to do is I am going to multiply by square root of 2 pi and this is integral minus to infinity to infinity, 1 over square root of 2 pi, this is 1 over square root of 2 pi e to the power of minus t square by 2.

Now, you can show that this integral is equal to 1, because this integral is minus infinity to infinity of a Gaussian probability density function. So, each 1 over square root of 2 pi e to the power of minus t square by 2 this you can show is a this you can show is a Gaussian probability. This is a Gaussian probability density function that is 1 over square root of 2 pi sigma square e power minus t square by 2 sigma square is a Gaussian probability density function with mean mu, and variance equal to sigma square. And integral minus infinity to infinity Gaussian probability density function is 1. Here you can see the variance is sigma square is equal to here you can see the variance is sigma square equal to 1.

If you simply substitute sigma square equal to 1, all right. Do you get the Gaussian probability density function the sigma square equal to 1 and mean is 0. So, that gives the Gaussian probability density function 1 over square root of 2, pi e raise to minus t square by 2 all right. So, that is the Gaussian probability density function for the range t lying between minus infinity to infinity. So, this integral is one implies the net integral is square root of 2 pi which is a finite quantity.

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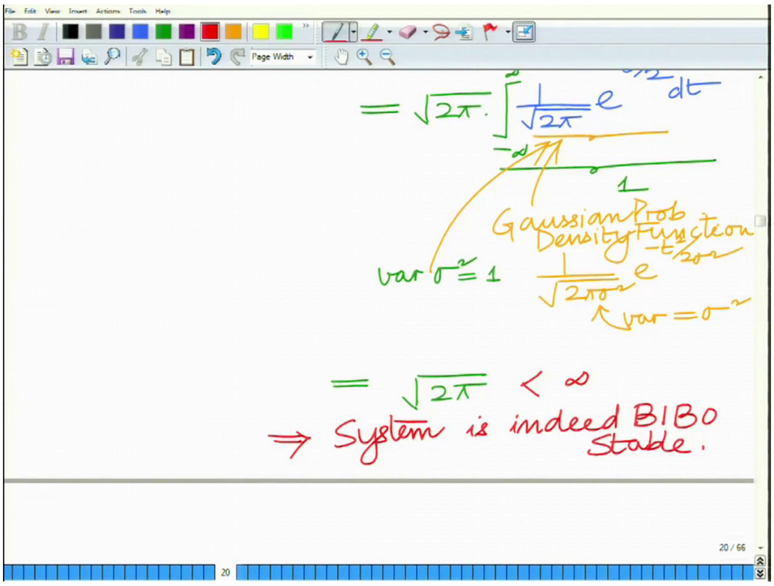

So, this is a which is a finite quantity. Hence, the system is implies a system is indeed the system with impulse response given by the Gaussian pulse is indeed BIBO stable. Because it is the integral absolute it a that is it is absolutely integrable there is integral minus infinity to infinity magnitude h of t d t is a finite quantity, all right. Let us now proceed to the next example. Which is given by the consider the LTI system given by consider the LTI system where the output y of t equals minus infinity to infinity e power minus j omega naught t minus tau x of tau d tau, this is the output.

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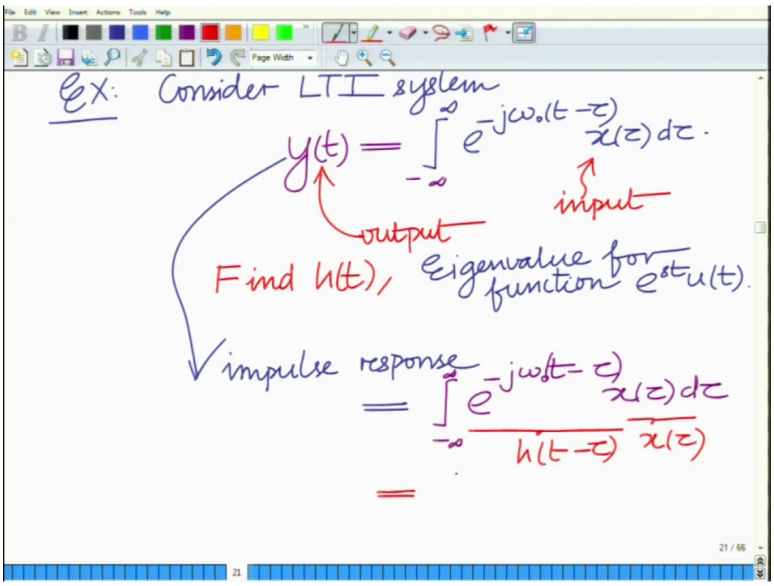

So, for input x of t, this output y of t is given by this relation. What we need to do is we need to find the impulse response. And also, Eigen value for the function e to the power of s t u t. Or let us put it this way, this is let us make it e to the power of minus t minus e to the power of minus t minus tau times. Or let us keep it this way e to the power of minus j omega naught t minus tau x tau d tau. And find the Eigen value corresponding to Eigen value for function e to the power of s t u t.

Now, it is easy to see what the impulse response is; the impulse response of this is equals well you can write this as a this is integral minus infinity to infinity minus j omega t minus tau x tau d tau. So, this is your h of t minus tau, and this is your x of tau. So, this is basically h t convolved with x t impala implies that the impulse responses; implies, that the impulse responses e power minus j omega naught t.

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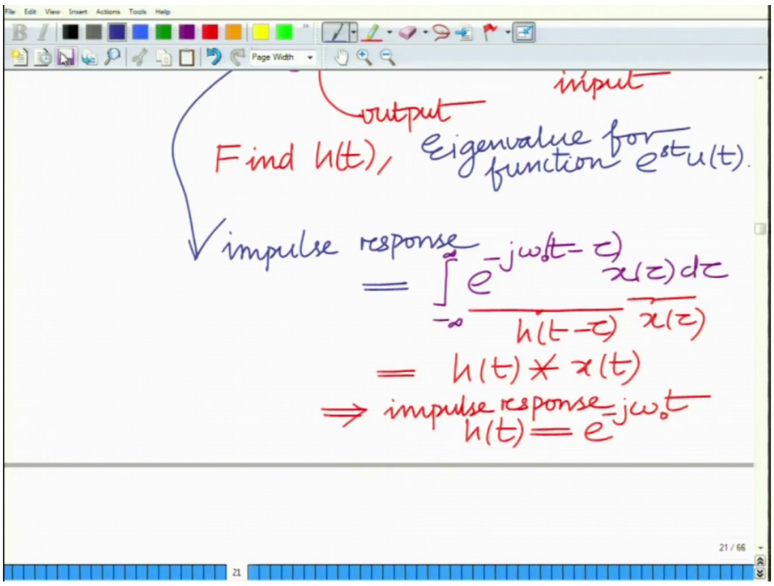

So, the impulse response of this system is e power minus j omega naught. So, all right so, we will stop this module here, we are looking at some examples of the analysis of the LTI systems. So, we will stop this module here and continue with other similar examples in the subsequent modules.

Thank you very much.