

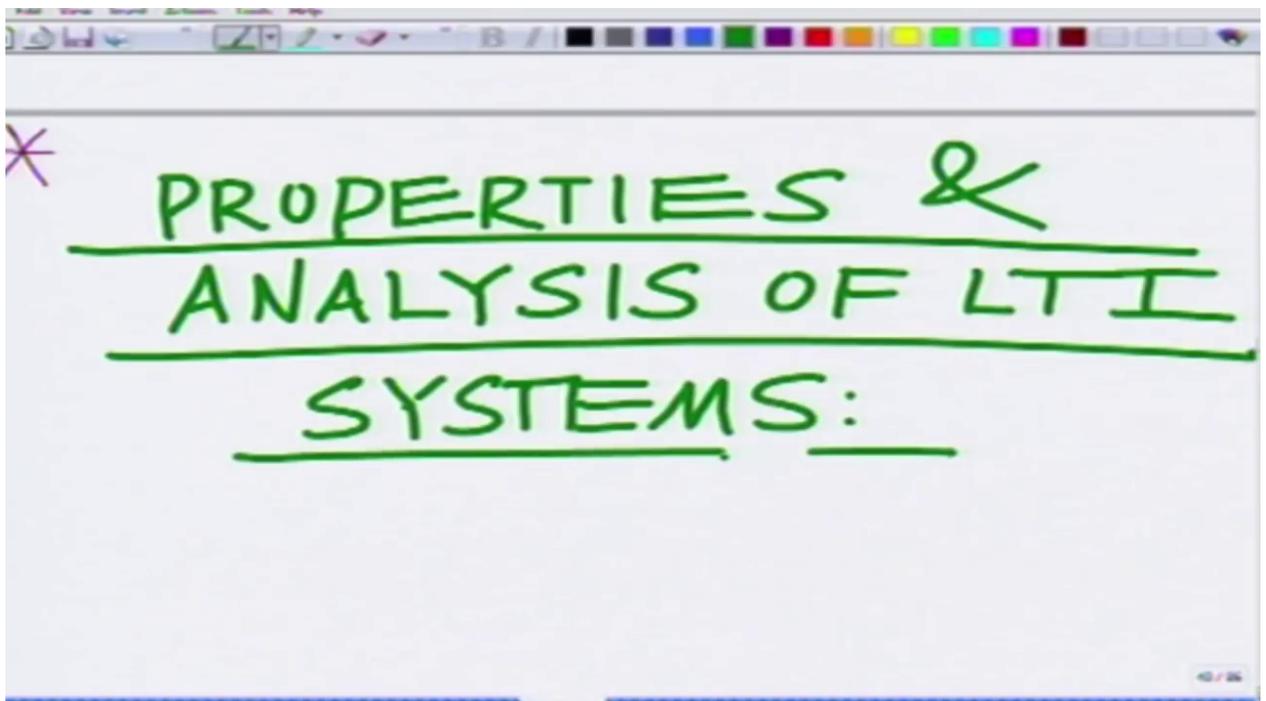
Principles of Signals and Systems
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Lecture – 11

**Properties and Analysis of LTI Systems – Memoryless Systems, Causality, Stability,
Eigen function**

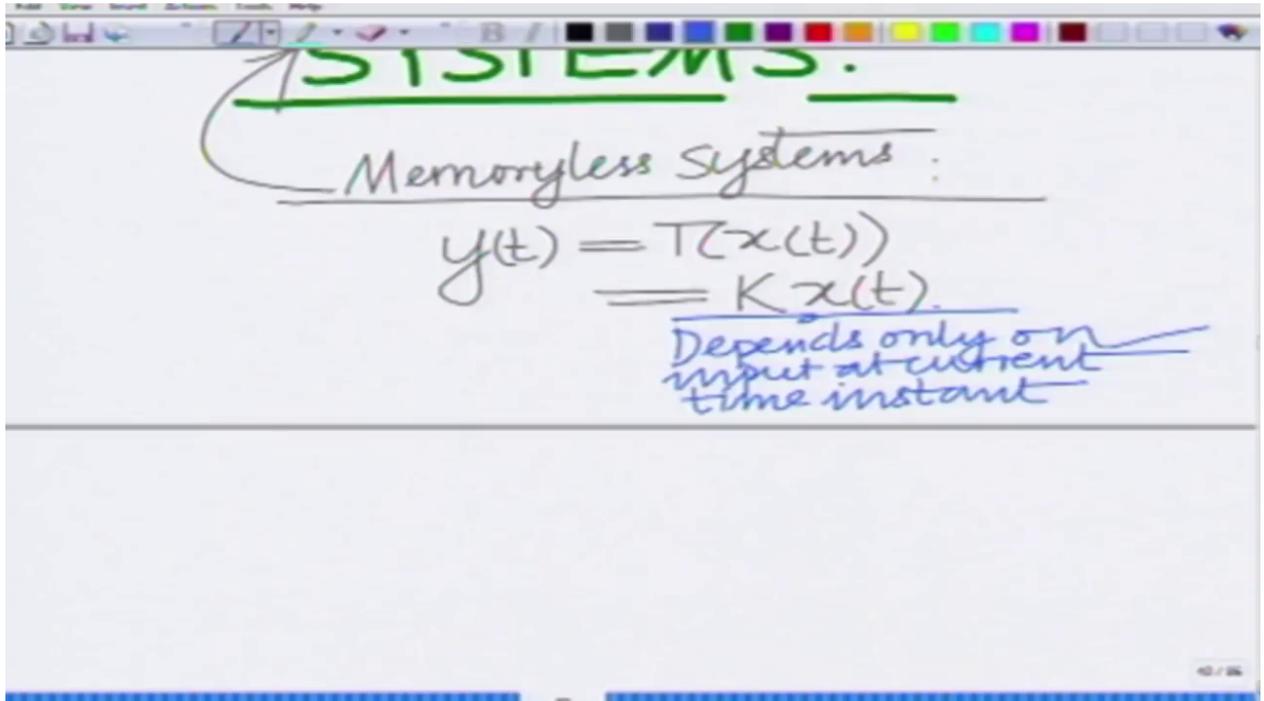
Hello, welcome to another module in this massive open, massive open online course. So, you are looking at the properties of and analysis of LTI systems, let us continue this discussion.

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So, we want to look at further properties, we are looking at further properties and analysis of LTI systems, let us specifically look at memory less systems.

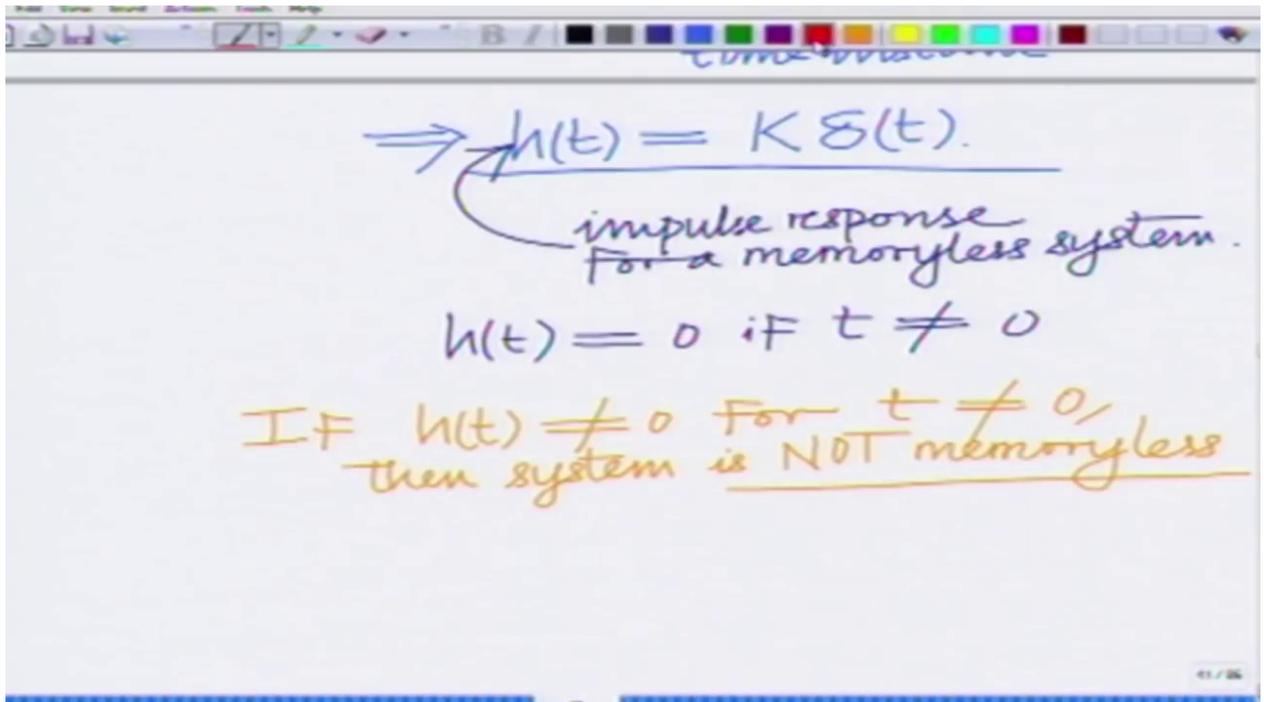
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Now remember in a memory less system we have $y(t)$ the output equals $x(t)$ is simply equal to α times $x(t)$, that is it depends only on that is some constant K times $x(t)$ which implies depends only on the current input at the current time instant correct yes.

So, it depends only on the input at the current time instant depends only on the input at the current time instant all right does not depend on the input of the past or the future time instant all right. So, that is known as a memory less system does not have memory and hence the corresponding impulse response of the system would therefore, be $y(t)$ equals. So, the implies impulse response $h(t)$ would be K times $\delta(t)$ correct because the output depends only on the current input.

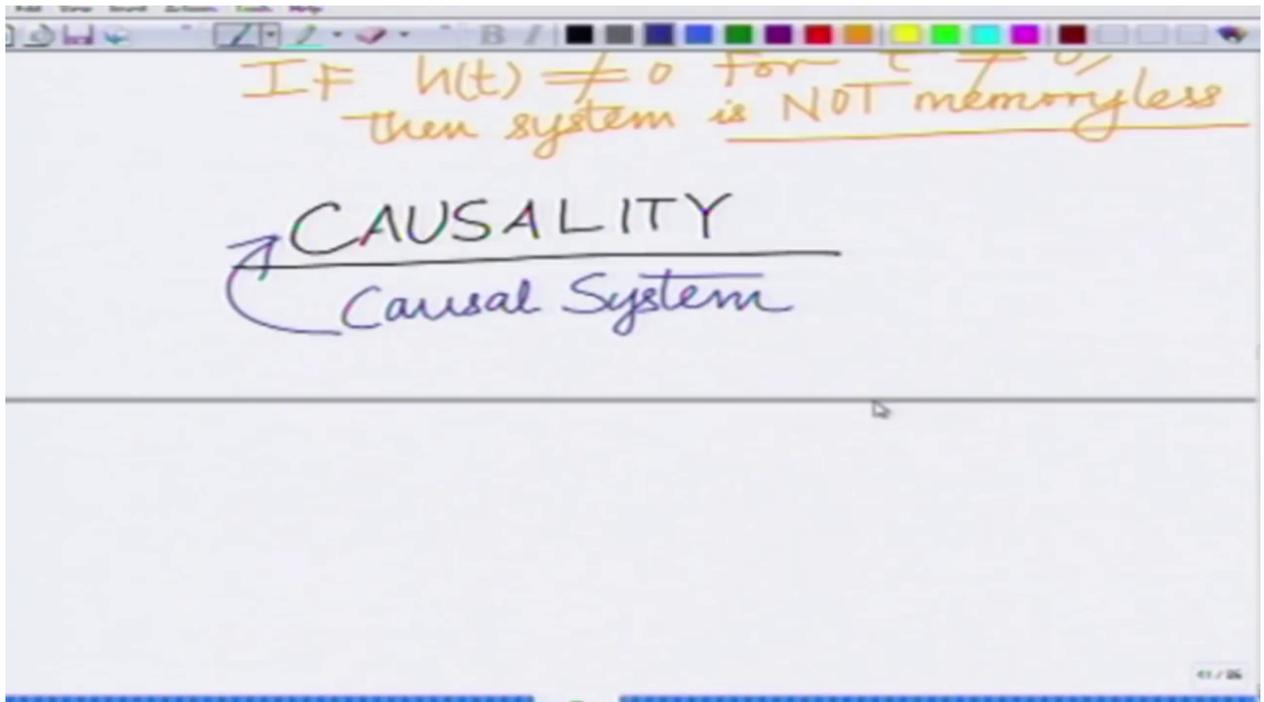
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So, the impulse response will simply be a constant times delta t since y t, this k types is the output is k times any input the impulse response is simply the constant. So, this is the impulse response, this is the impulse response for a memoryless system.

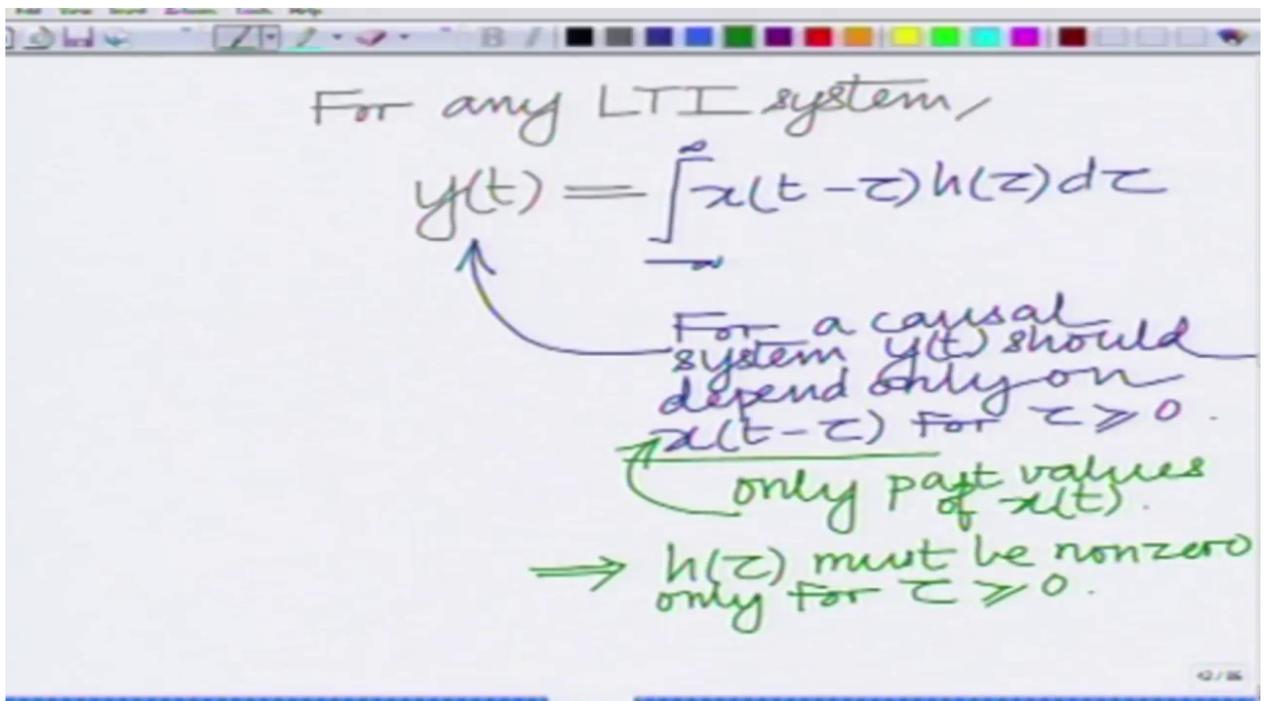
So, it basically says that h of t equals 0 if t is not equal to 0. So, its non 0 only for t equal to 0 is h of z is equal to 0 for t equal to t not equal to 0 or in other words if, another way of saying this is if h of t is not 0 for t not equal to 0 then system is not memory less than system will have, system is not if h of t is non 0 for t not equal to 0 then the system is not h of t is non 0 for t not equal to 0 the system is not a memory less system. So, that is a characterization of the of memoryless systems in terms characterization of a memoryless system in terms of its impulse response ok.

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Now, causality, another important property of LTI systems if you remember is causality or a causal system, causality or a causal system.

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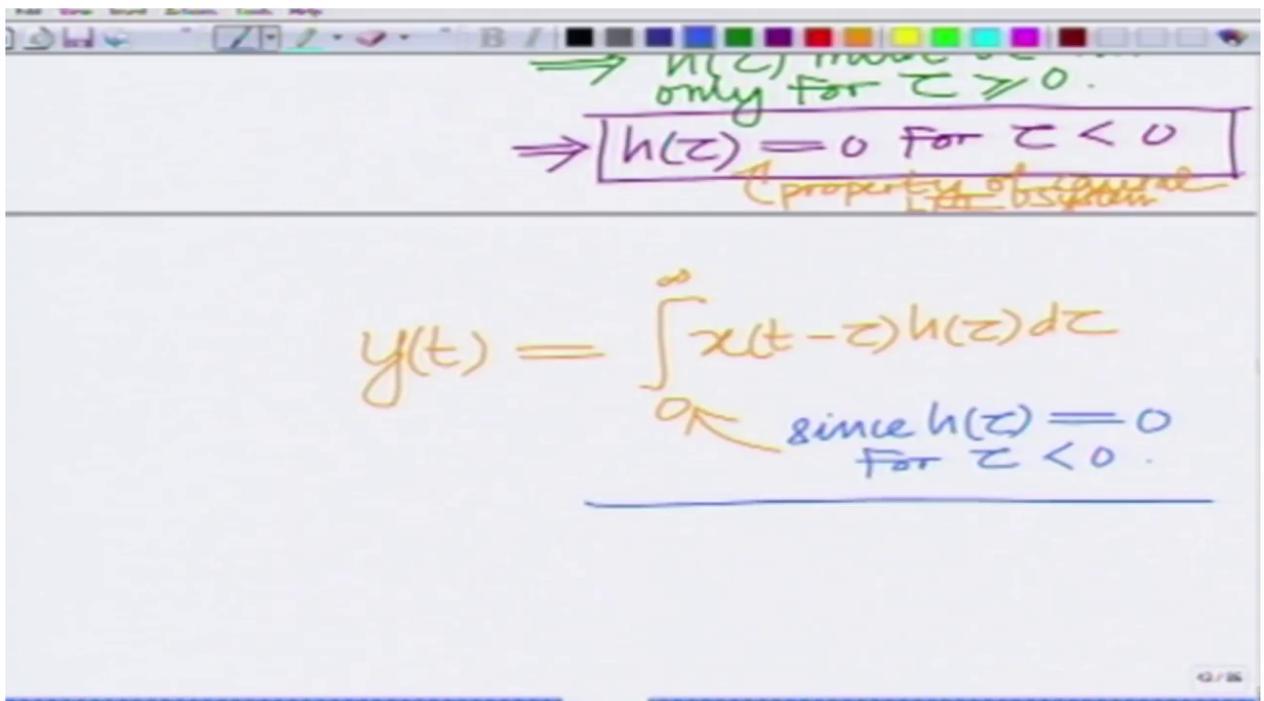


So, for a causal system, now remember for any LTI system as we have seen in the previous module we have output y of t equals input x of t minus τ times h of τ $d\tau$

this is the output expressed in terms of the input x of t all right and where h of t or h of τ is the impulse response of the LTI system. Now, this y of t for a causal system should depend only on x of t and past values of x of t . So, it should not depend on the future values of x of t that is. So, y of t for a causal system, y of t should depend only on x of t minus τ for τ greater than or equal to 0, that is only passed values of x of t and this implies. So, this implies now h of τ must be non 0 only for τ greater than equal to 0 because if h of τ you can see from this integral over here if h of τ is non 0 for τ less than 0 then this will also pick up negative values of τ , that is x of t that is x of t minus τ where τ is negative will depend on basically is a future value of the signal x of t .

So, therefore, we need h of τ to be non 0. So, h of τ not equal to 0 only for τ greater than or equal to 0 that is an important. So, h of τ must be non 0 only for τ greater than equal to 0.

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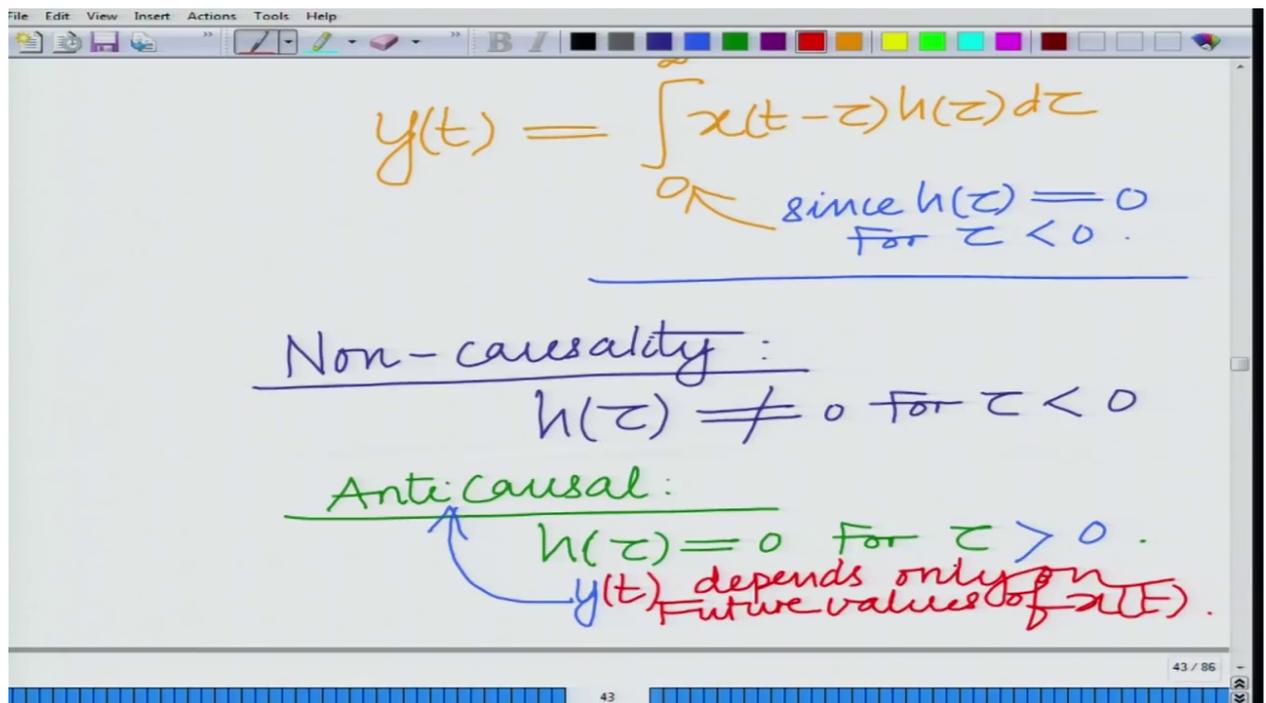


So, h of τ which also implies which implies and also implies h of τ equal to 0 for τ less than 0. So, this is basically denotes a causal that is impulse response, property of a causal this is the property of a causal LTI system that is impulse response h of τ equal to 0 for τ less than 0 and as a result this integral can be simplified simply as y of t equals minus infinity to infinity x of t minus τ h of τ $d\tau$, this integral can be simply

simplified as from 0 to infinity since $h(\tau)$ this arises since $h(\tau)$ is 0 since $h(\tau)$ equal to 0 for τ less than 0 ok.

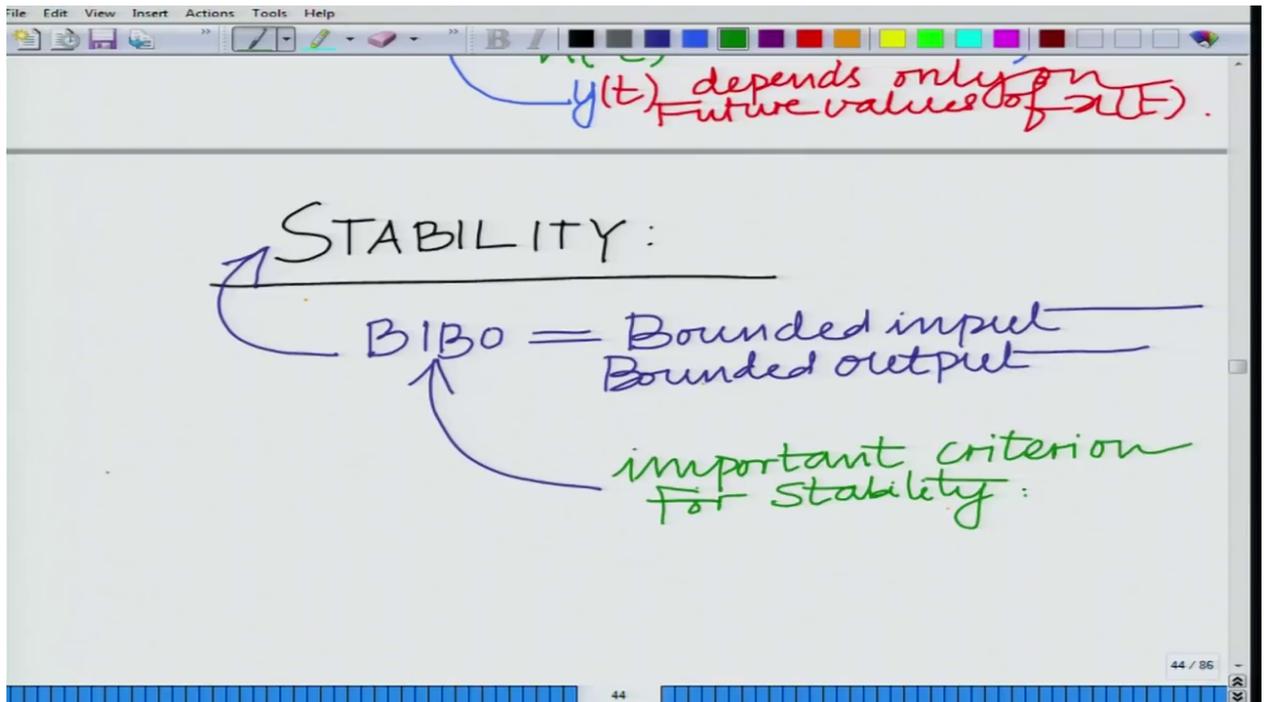
So, that is the property of a impulse response this is the property of the impulse response of in causal LTI system or a causal linear time invariant system. So, now, and therefore, the system is now let us look at another property, non causality.

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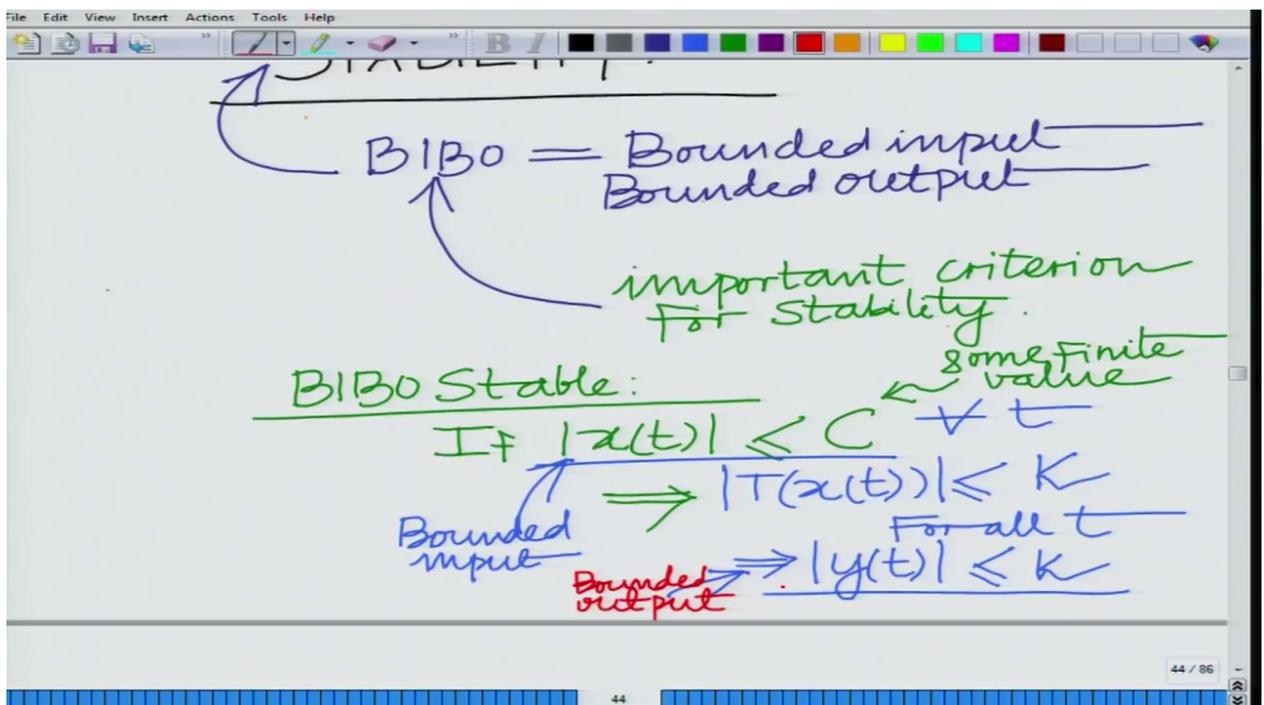
Non causality system is non causal if $h(\tau)$ not equal to 0 for τ less than 0 or for some τ less than 0 and $h(\tau)$ the system is anti causal if $h(\tau)$ equal to 0 for τ greater than $h(\tau)$ equal to 0 for τ greater than 0 and in this case anti causal system something interesting happens. Y of t depends only on future values of x of t , depends only on you can clearly see if $h(\tau)$ is 0, if $h(\tau)$ is 0 for τ greater than 0 it happens that y of t , y of t depends only on future values of x of t such a system is known as an anti causal system, right. H of τ equal to 0 for τ or h of t equal to 0 for t greater than 0 or h of τ equal to 0 for h of τ equal to 0, for τ greater than 0 or h of t equal to 0 for t greater than 0 alright such a system is known as anti causal system, so a causal systems, non causal systems and anti causal systems. Let us now look at another important aspect that is the stability.

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In particular let us look at BIBO stability, which you already described previously that is bounded input, bounded output this is an important, one of the most important criterion important for or important criterion, important for more important criterion for stability and this is characterized as follows.

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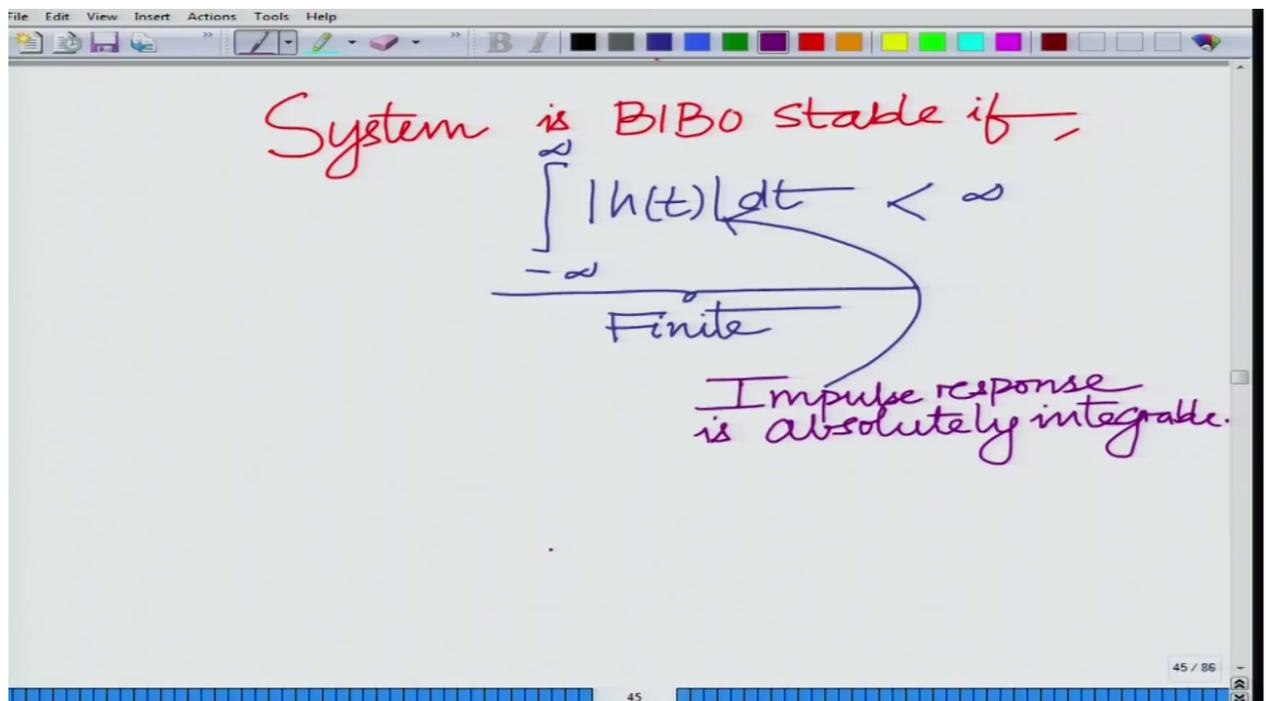


That is BIBO implies or we say system BIBO stable, if input is bounded magnitude of $x(t)$ less than or equal to c some finite value. C is some finite value that is if magnitude of $x(t)$ is less than or equal to c then this implies that the system output and this is for all T , for all T system output must be less than or equal to some other constant K also for all t . That is if the input that is basically implies magnitude of $y(t)$ output is less than or equal to k .

So, this basically implies that for any bounded input that is when the input is bounded correspondingly the output is also that is any bounded input produces a bounded output. That any bound that is any bounded input to the system must definitely produce an output that is upper bounded from by certain constant value. The system if the system satisfies this property for all bounded input signals this is known as a BIBO stable system that is it is a stable system under the BIBO criteria that is bounded input bounded output stability criterion ok.

Now, we will derive an important condition on the impulse response of the system that is BIBO, BIBO stable and that can be derived as follows.

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If the system is the system is b we will show that system is BIBO stable that is bounded input, bounded output stable if the impulse response is absolutely integrable which is

magnitude integral minus infinity to infinity magnitude h of t dt is less than infinity that is to say this quantity does not blow up or this quantity is rather finite that is this impulse response, another way of saying this is impulse responses absolutely integrable. Impulse response, impulse response is absolutely the impulse response is absolutely integrable and let us justify this condition now let us assume that you bounded.

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$-\infty$
 Finite
 Impulse response
 is absolutely integrable.
 Assume Bounded input
 $|x(t)| \leq C$
 $\int_{-\infty}^{\infty} |h(t)| dt = \alpha.$

Assume we have a bounded input magnitude x t less than or equal to C and let us say also assumed magnitude minus infinity to infinity magnitude h t equals, integral of magnitude h t dt equals some constant, let us say α then what we have equals the output.

Now, we have to show that the output is bounded.

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The image shows a whiteboard with the following handwritten equations and annotations:

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \right|$$

$$\leq \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau$$

$\leq C$ Bounded input property

$$\leq C \int_{-\infty}^{\infty} |h(t)|dt = C\alpha$$

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Now, recall that the output is given for any arbitrary input signal $x(t)$ in terms of the impulse response as well $x(t - \tau)$, $h(\tau)$ therefore, magnitude $y(t)$ if you look at that the magnitude $y(t)$ that is equal to the magnitude of the integral $x(t - \tau)h(\tau)$. Now, the magnitude of an integral is less than or equal to the integral of the magnitude of the quantity being integrated. So, this is less than or equal to magnitude $h(\tau) \times |x(t - \tau)|$ magnitude $h(\tau)$, $d\tau$, but magnitude $x(t - \tau)$ this is less than or equal to C because we assume that this follows from bounded input property.

This follows from the, this follows from the bounded input property which implies this integral is indeed less than or equal to c times minus infinity to infinity magnitude $h(t)$, $d t$ this we said is α so this is equal to c times α which implies.

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$|y(t)| \leq \frac{C\alpha}{K}$
 Bounded output

$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

So, this implies basically that magnitude $y(t)$ is less than or equal to c times α which basically implies that the in output y of t is bounded. So, for what we have able to demesne to being able to demonstrate is that if integral minus infinity to infinity magnitude $h(\tau)$, $d\tau$ is a finite quantity then every bounded input will produce a bounded output for the LTI system.

So, this output magnitude $y(t)$, o this is a bounded output and this is your constant k that we are talking about initial is magnitude $y(t)$ less than equal to k . So, any bounded input produces a bounded an output. So, the condition for BIBO stability is as we have derived this is the condition for LTI system to be, this is the condition for condition for LTI system to be BIBO stable.

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$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

Condition For LTI system to be BIBO stable.

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This is the condition for the BIBO condition for bounded input, bounded output stability of an LTI system.

Now, another important aspect that we can take a look at is that of the Eigen functions of an LTI system, Eigen functions of an LTI system.

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EIGENFUNCTIONS OF LTI SYSTEMS:

Consider $x(t) = e^{\alpha t}$

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$
$$= \int_{-\infty}^{\infty} e^{\alpha(t-\tau)}h(\tau)d\tau$$

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What are the Eigen functions? So, another aspect that we would take a look at are Eigen functions of LTI systems and for an Eigen function of LTI system consider to illustrate this property, consider $x(t) = e^{\alpha t}$ where α is some constant. Now, for an LTI system we know output $y(t)$ equals in terms of the impulse response for any arbitrary input signal $x(t)$ is given as $\int_{-\infty}^{\infty} x(t-\tau)h(\tau) d\tau$. Now, I am going to substitute the expression $e^{\alpha t}$ for $x(t)$. So, this gives $e^{\alpha t} \int_{-\infty}^{\infty} h(\tau) e^{-\alpha \tau} d\tau$ which you can now simplify.

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The image shows a whiteboard with handwritten mathematical derivations. The top part shows the convolution integral for the output $y(t)$ of an LTI system with impulse response $h(\tau)$ and input $x(t) = e^{\alpha t}$. The derivation proceeds as follows:

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} e^{\alpha(t-\tau)}h(\tau) d\tau$$

$$= e^{\alpha t} \int_{-\infty}^{\infty} h(\tau) e^{-\alpha \tau} d\tau$$

The integral $\int_{-\infty}^{\infty} h(\tau) e^{-\alpha \tau} d\tau$ is labeled as $H(\alpha)$. Below this, the final result is written as:

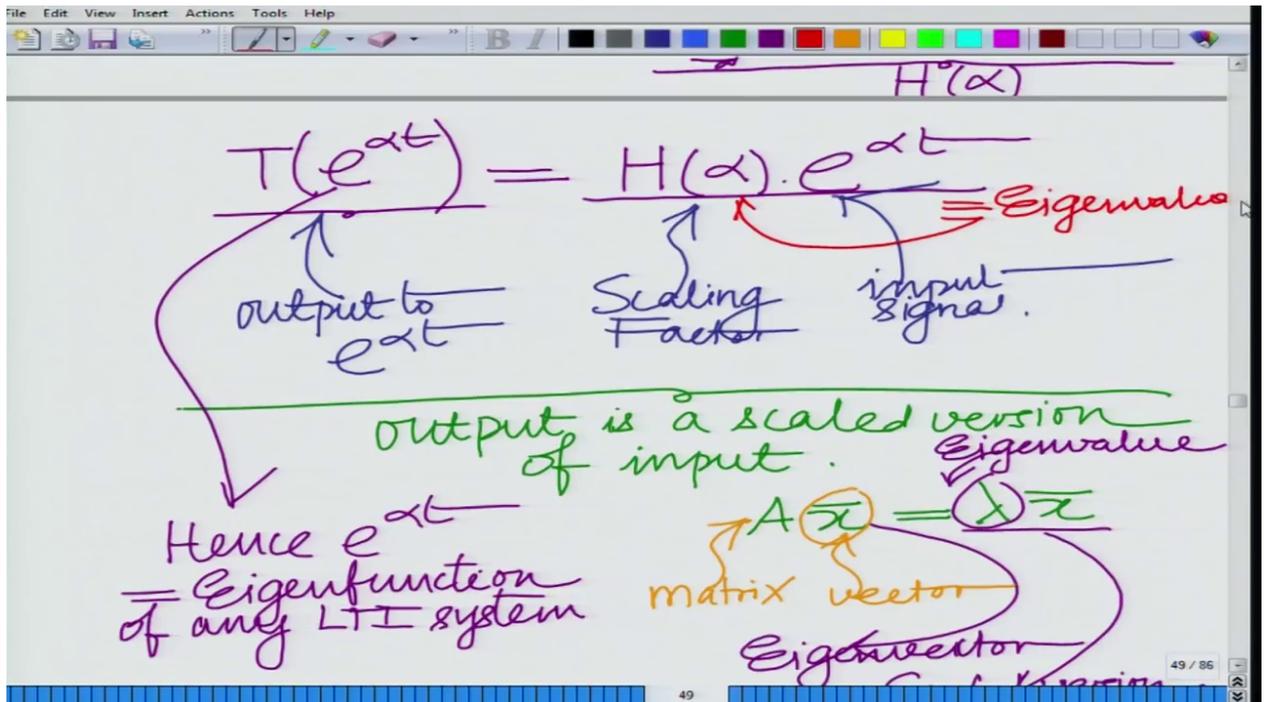
$$T(e^{\alpha t}) = H(\alpha) \cdot e^{\alpha t}$$

The whiteboard interface includes a menu bar (File, Edit, View, Insert, Actions, Tools, Help), a toolbar with drawing tools and colors, and a status bar at the bottom showing '49 / 86'.

It as remember look at this $e^{\alpha t}$ is a constant that now does not depend on the integration variable τ so that will come outside. So, this is $e^{\alpha t} \int_{-\infty}^{\infty} h(\tau) e^{-\alpha \tau} d\tau$.

Now, if you denote this integral by $H(\alpha)$ you can simply. So, it depends only on the impulse response $h(t)$ and the constant α . So, this is $H(\alpha) e^{\alpha t}$. So, what you able to show is the $T(e^{\alpha t})$ when the input is $e^{\alpha t}$ this is simply a scaled version that is output.

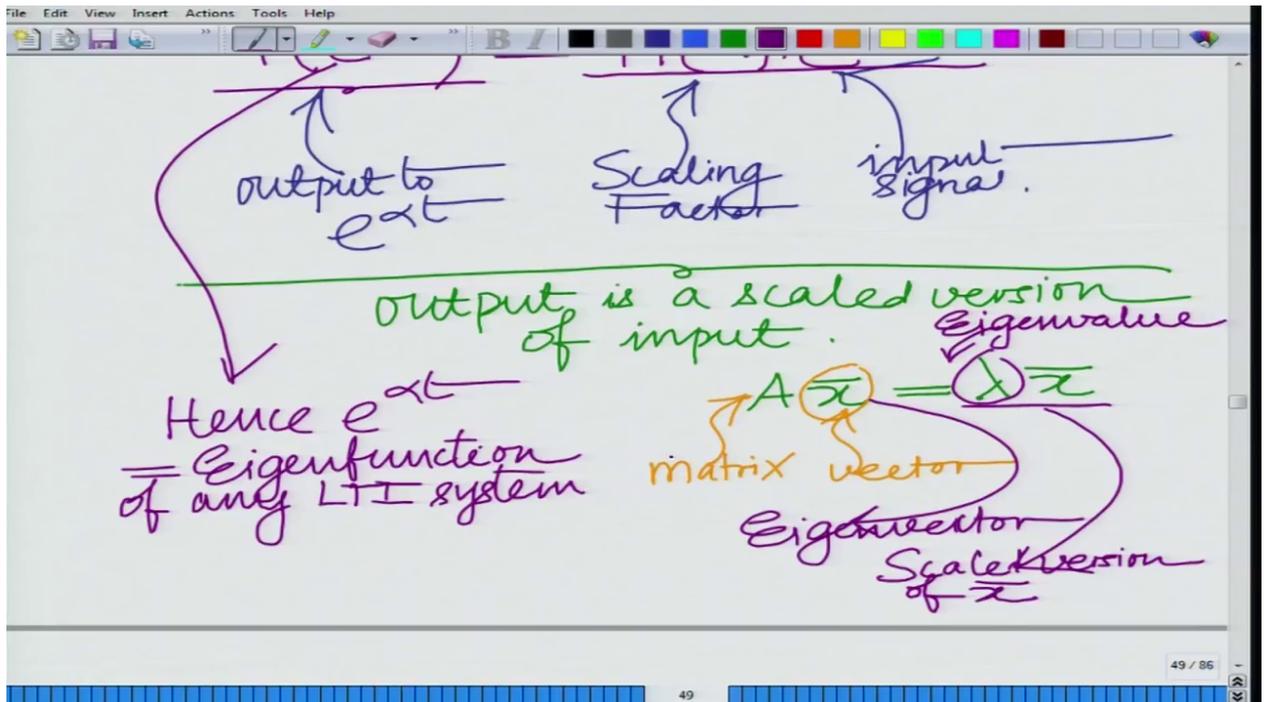
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That is if you look at this output to $e^{\alpha t}$ equals some scaling factor times again input signal, the same input signal multiplied by a cell. So, we have y of t equals some constant that is output signal t of x of t is some constant k times x of t correct. So, it is the output is simply a scaled version of the input that is a better way to say this output is a scaled version of the input ok.

So, better way to say it is output is a scaled version of the input remember in matrices wherever you have a matrix A and vector \vec{x} such that $A \vec{x}$ equals $\lambda \vec{x}$ remember. If you remember from your from your basic knowledge of linear algebra our properties of matrices whenever we have a matrix A all right and a vector \vec{x} which satisfies the property $A \vec{x}$ equals λ times \vec{x} that is if you treat this as a linear system. In fact, this is nothing, but a representation of a transformation or a linear system. So, this is a matrix.

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This is a matrix, this is a vector and a vector satisfying this property is termed as the eigenvector which means that A times x bar which is basically input to the system, input to the transformation characterized by A is simply λ times x bar scaled version of x bar.

So, this x bar is called an eigenvector and λ is called an Eigen value, λ is called an Eigen value similarly what you are observing here is that the input is $e^{\alpha t}$ output is simply a scaled version of $e^{\alpha t}$ that is λ of α times $e^{\alpha t}$. Therefore, this is known as $e^{\alpha t}$ hence, $e^{\alpha t}$ is equals the Eigen function of any LTI system and these Eigen functions play a very important role because if you look at this h

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of τ

$$H(\alpha) = \int_{-\infty}^{\infty} h(\tau) e^{-\alpha\tau} d\tau$$

Transform of $h(\tau)$.

Transforms play a very important role in Analysis of LTI systems.

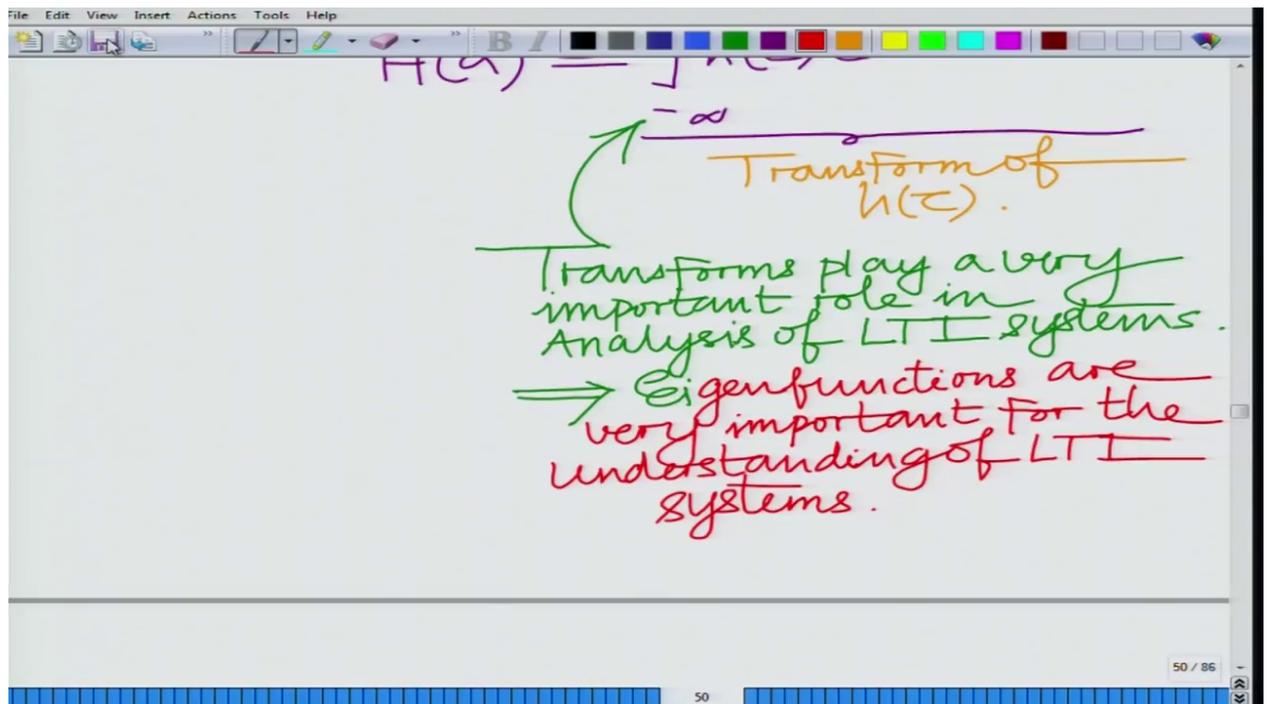
⇒ E_i

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of alpha this is equal to minus infinity to infinity h of tau e power minus alpha tau d tau this is known as a transform, this is known as a transform of the impulse response h auto and there can be several such transforms depending on the nature of alpha that is chosen. It can be the Laplace transform or the Fourier transform, it can also be in the continuous time and discrete time something, the discrete time case we will talk about that later. So, there can be various transforms and these transforms and Eigen functions are very fundamental to the analysis and to understanding the properties of LTI systems.

Therefore this property is extremely important. So, transforms which we will look at subsequently at later stages in this course, transforms play a very important role in understanding, in understanding and in analysis, let us put it this way. In analysis therefore, Eigen values are very key, Eigen values are of significant importance or let us put it as Eigen functions I am sorry the transform is nothing, but the Eigen value. In fact, you can see now from the context λx equals λI times x this is nothing, but this is equivalent to the Eigen value under this particular transformation is nothing, but the Eigen value corresponding to this Eigen function.

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So, Eigen functions are very important for the understanding of, Eigen values are very important for the understanding of, Eigen functions are very important for the understanding of LTI systems all right. So, in this module we have continued our discussion on the properties and analysis of LTI systems. So, we have looked at several properties, let us stop here and we will continue this discussion in the subsequent modules.

Thank you very much.