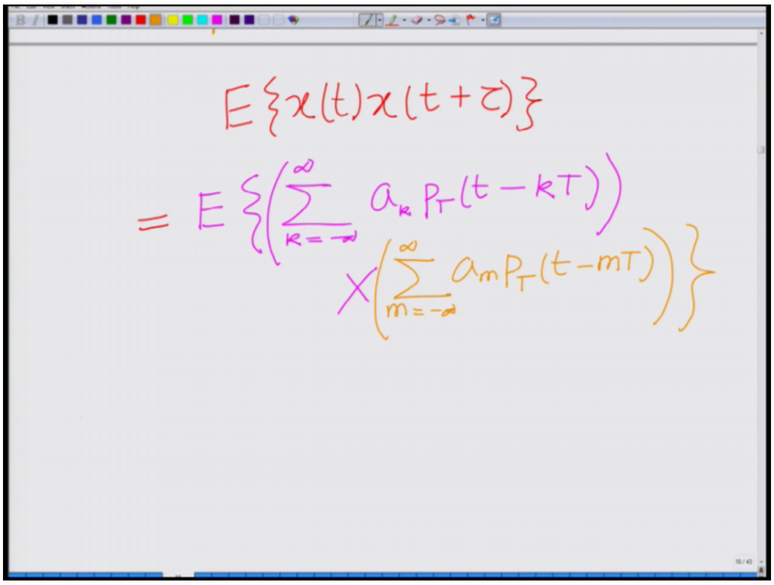
Principles of Communication Systems - Part II
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

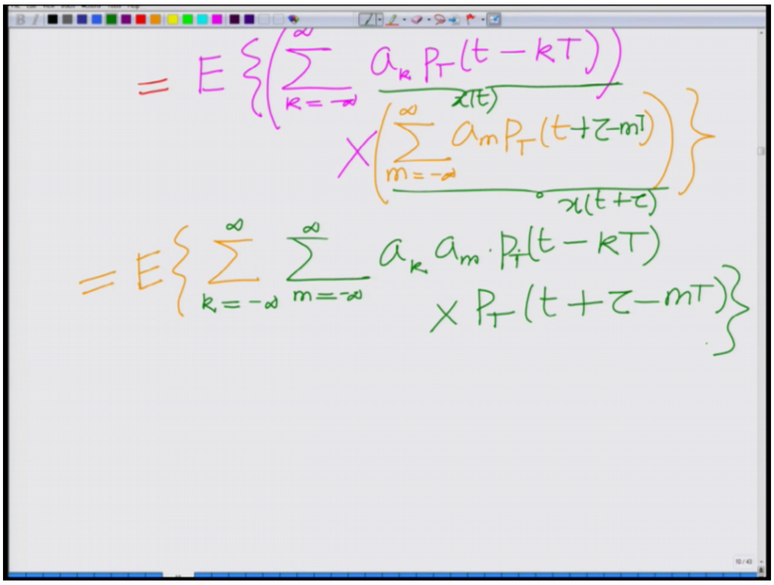
Lecture – 03
Spectrum of Transmitted Digital Communication Signal, Autocorrelation Function and Power Spectral Density

(Refer Slide Time: 00:29)



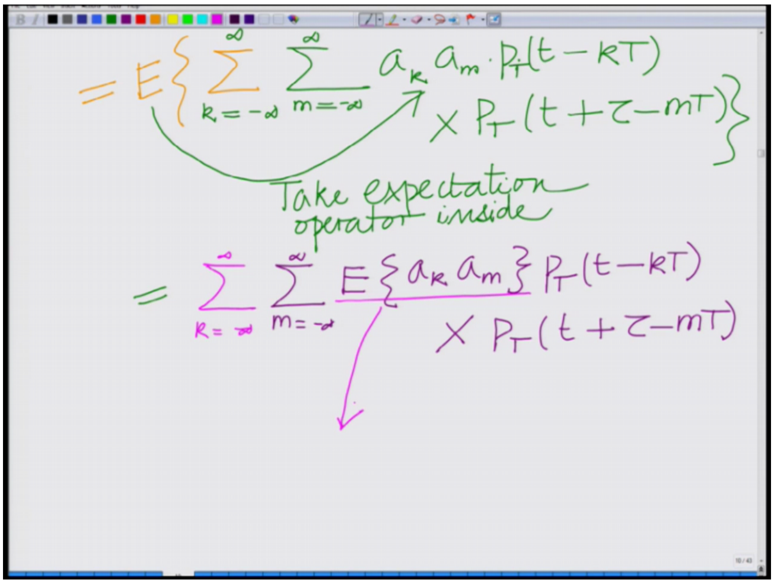
Hello, welcome to another module in this massive open online course. So, we are looking at the autocorrelation function of the transmitted digital communication signal that is expected value of x t into x t plus tau. So, we want to the compute the autocorrelation function expected value of x t into x t plus tau. Now, substituting the value of x t, I have expected value of well x t is nothing but summation k equal to minus infinity to infinity a k P t minus k t times well x t plus tau. So, summation m equals minus infinity to infinity a m P T minus m T close the brackets.

(Refer Slide Time: 01:35)



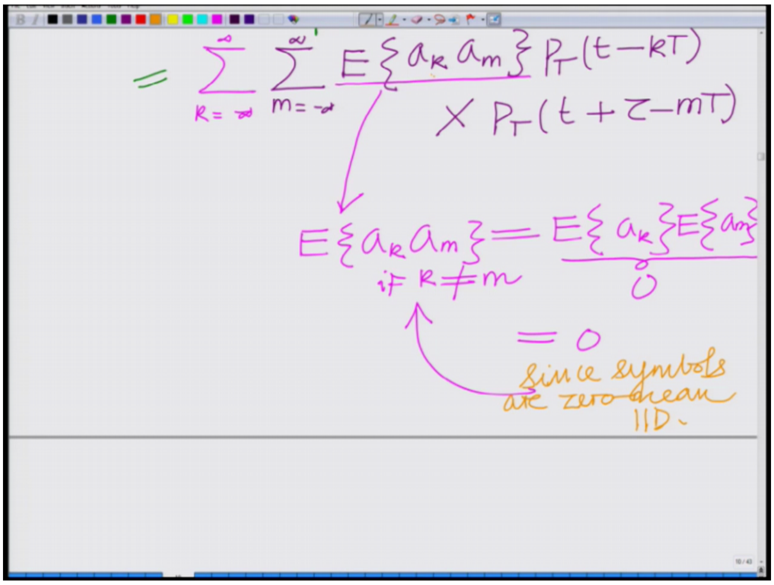
Now, we expand the summation that is expected value of well summation k equal to minus infinity to infinity summation m equal to minus infinity to infinity a k a m P well of course, this is x t plus tau, so this has to be t plus tau minus m t. So, this is your x t this whole thing, and this is your x t plus also P T of t minus k T into P T of t plus tau minus m T.

(Refer Slide Time: 02:41)



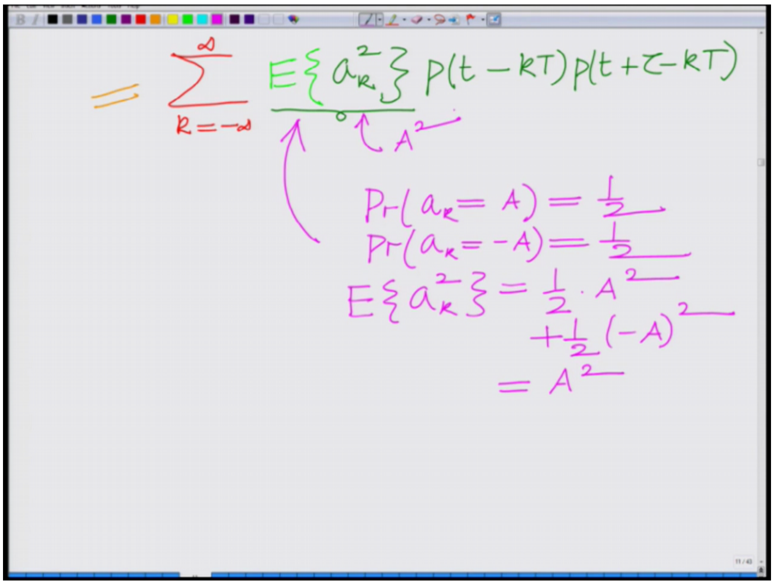
Now, if I take the expectation operator insides, so if I take the expectation operator inside the summation, we have taking the expectation operator inside, we have well summation k equal to minus infinity to infinity summation m equal to minus infinity to infinity expected value of a k into a m times P T minus k T times P T t plus tau minus m T. Now, if you can look at this quantity expected value of a k into a m. Remember we already said the symbols are independent identically distributed.

(Refer Slide Time: 03:56)



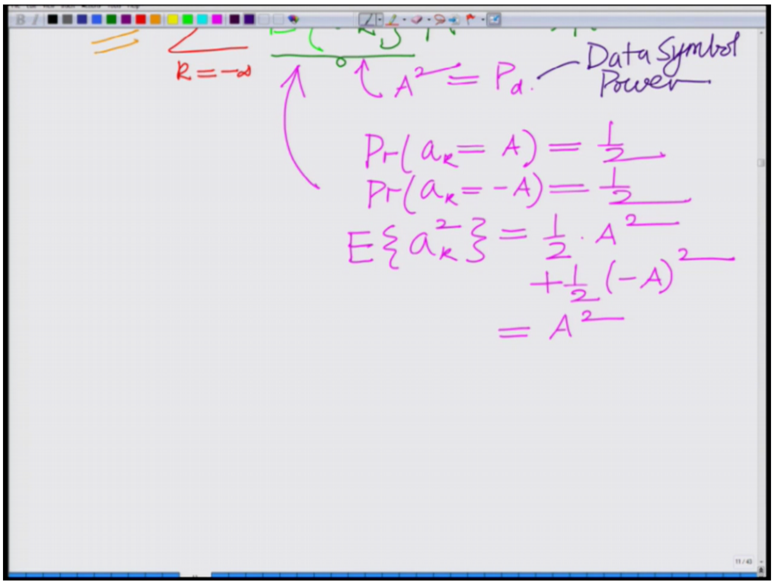
Therefore expected value of a k a m, if k is not equal to m, we have seen this expected value of a k a m; if k not equal to m is expected value because symbols are independent this is expected value of a k into expected value of a m this is equal to 0. So, this is equal to 0. Since symbols are the digital symbols are IID 0 mean, since symbols are 0 mean IID, therefore, expected value of a k a m 0 but if only if k is not equal to; if k is equal to m then it becomes expected value of a k square and that is not equal to 0. So, only those terms will survive in this summation in which k is equal to m; all the other terms will vanish because expected value of a k times a m is 0.

(Refer Slide Time: 05:05)



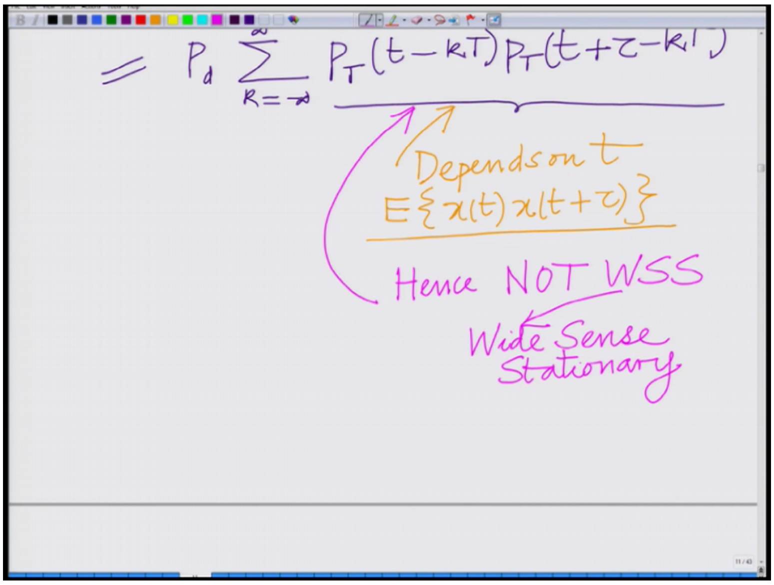
So, therefore, the summation equivalently reduces to expected value or summation k equal to, so there is only now k equal to minus infinity to infinity because only terms corresponding to k equal to m will survive. This will become expected value of a k square right into P well t minus k T into P t plus tau minus k T. Now if you can look at this quantity expected value of a k square that can be simplified as follows, we know that probability a k equal to A is half; we have assumed symbols are equiprobable probability a k equals minus A is also half. So, expected value of a k square equals half into A square plus half into minus A square both of these are A square minus A square is also A square, so half into a square plus half into a square, which is simply A square.

(Refer Slide Time: 06:33)



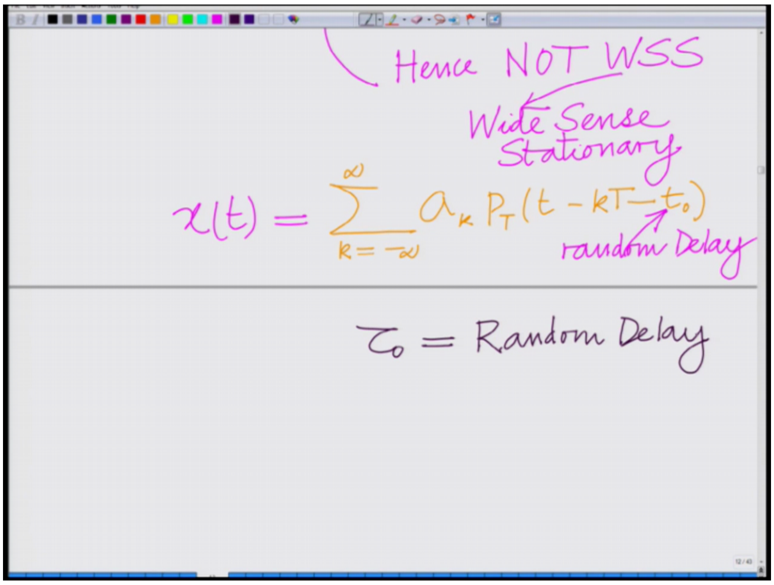
So, this quantity expected value of a k square is simply A square or we can also call this as P d which is the power of the data symbols this is P d, this is simply your data symbol, this is simply the power of the data symbols.

(Refer Slide Time: 06:53)



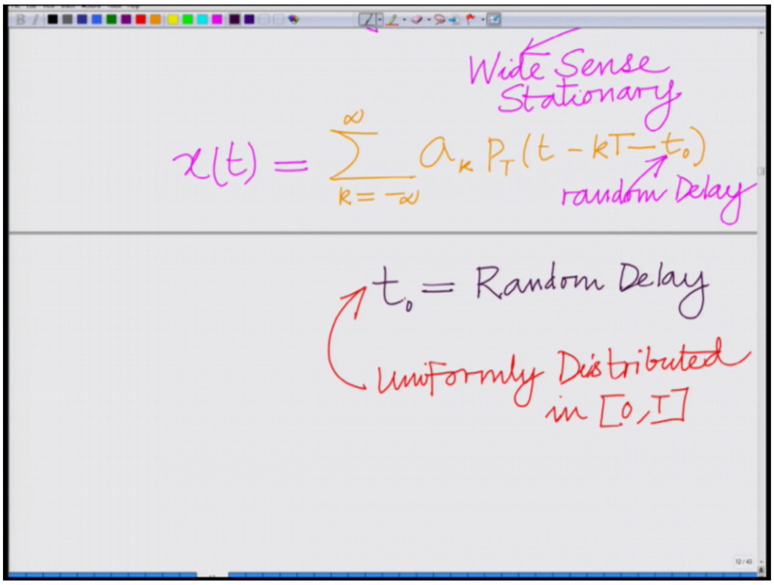
Therefore, this reduces to P d times summation k equal to minus infinity to infinity P T t minus k T P T t plus tau minus k T. Now, if you can see this quantity here, this is our autocorrelation function. Now, if you can this see this quantity here, this quantity here depends on t. When we have expected value of x t into x t plus tau, we know that when expected value of x t into x t plus tau depends not only on tau the time shift out, but also depends on t then the resulting random process is not wide sense stationary. Therefore what we can see here is that this digital communication signal is not wide stationary. Since, it depends on t hence it is not wide sense stationary that is basically wide sense. We would like to therefore convert this right because remember we wanted white sense stationary signals are very helpful to us because they are suitable to be studied, easy to be analyzed and characterized. And more importantly we can evaluate their power spectral density to characterize the spectral distribution of the power. So, we want to convert this signal which is not wide sense stationary this digital communication signal, it wide sense stationary signal.

(Refer Slide Time: 09:00)



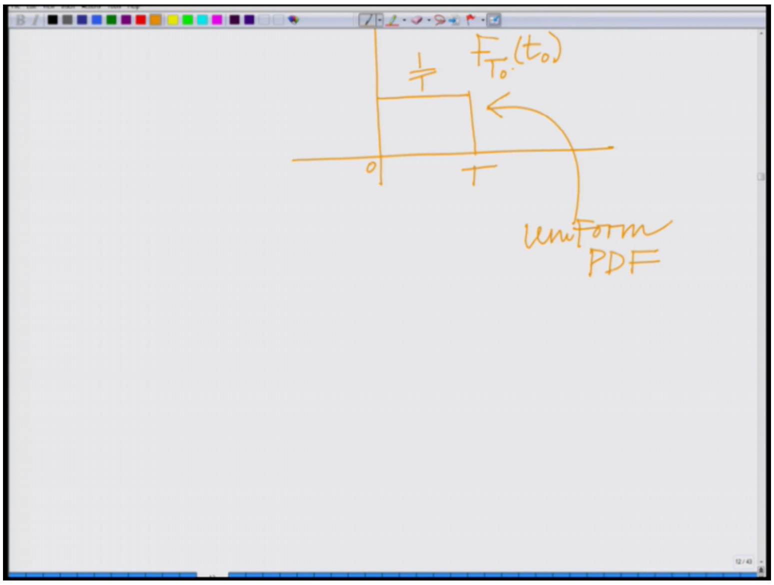
Therefore, what I am going to do is I am going to do the following is I am going to take my signal x t and I am going to move like a slight modification k equal to minus infinity to infinity a k P t minus k T capital T that is same as before. But I am also going to introduce this tau naught which is a random delay. So, this quantity tau naught is a random delay of the signal and this is uniformly distributed.

(Refer Slide Time: 09:45)



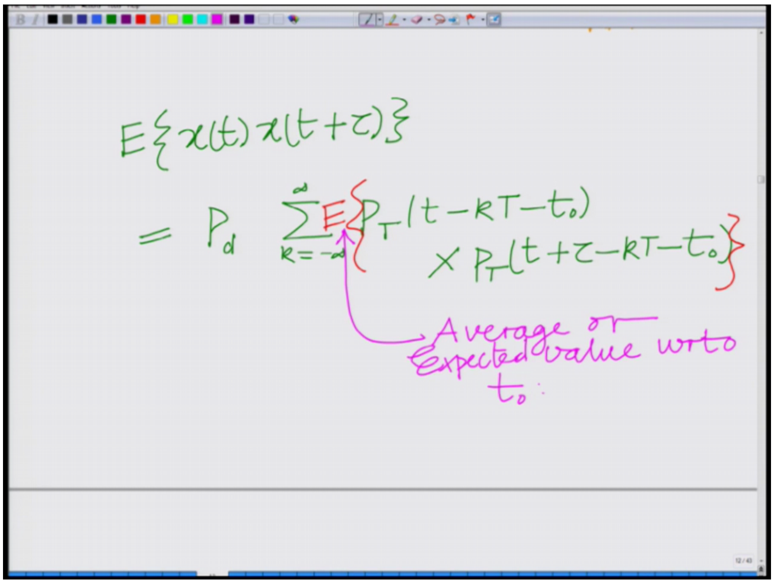
We are going to assume this is uniformly distributed in 0 to that is basically a uniform distribution. Let us so this is t naught sorry not tau naught t naught and it is uniformly distributed in 0 to T and you can see a uniform distribution in 0 to T is given as follows it is of height 1 over T.

(Refer Slide Time: 10:15)



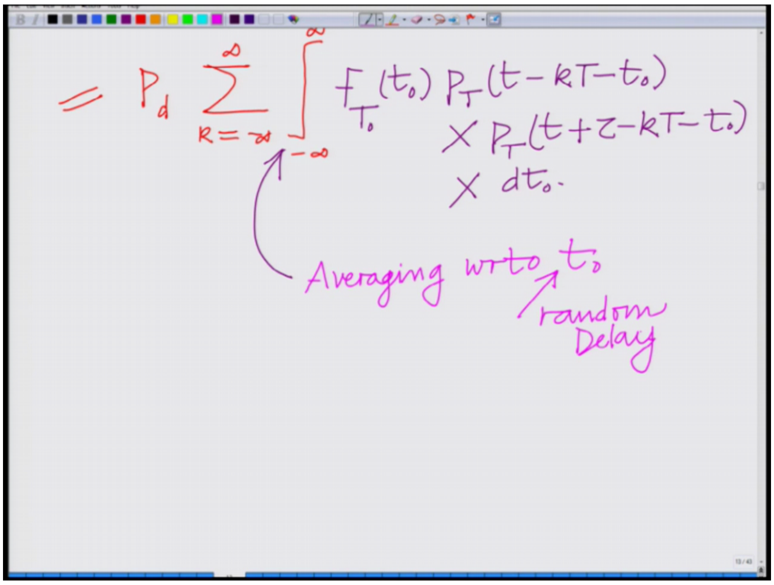
So, this is T, the distribution is 1 over T in 0 to T. So, this is your F of T naught of t naught, this is the uniform probability density function of T naught, this is your uniform probability density function. So, now what we are assuming is that this digital communication signal has a random delay T naught, which is basically distributed in uniformly in 0 to capital T, where capital T is the symbol time, 1 over T is the 1 or capital T is the symbol rate.

(Refer Slide Time: 11:14)



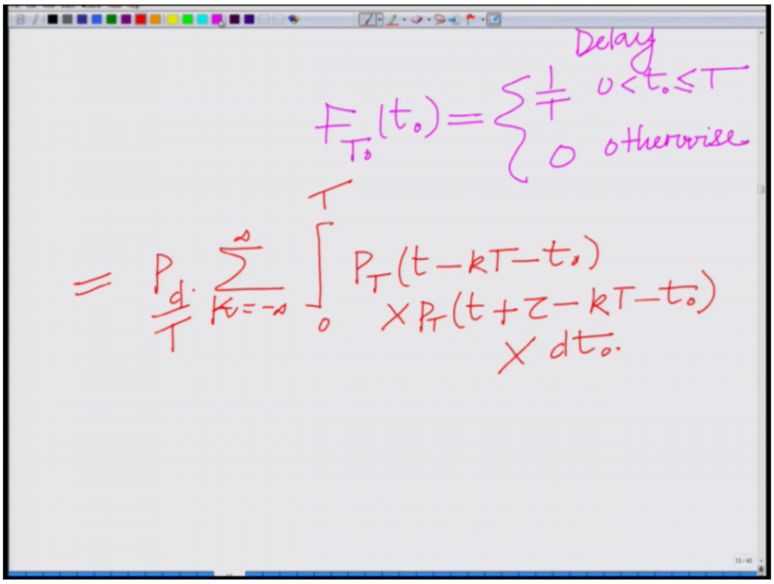
So, now we have again let us go back to our expected value of x t into x t plus tau, this is going to be well the same thing. Let us start with P d summation k equal to minus infinity to infinity P T t minus k T, now, in addition there is this delay t naught times P T t plus tau minus k T minus t naught. Now, I have to take the expected value of this because t naught is random, I have to take the expected value average with respect to average or expected value with respect to average or the expected value with respect to t naught.

(Refer Slide Time: 12:32)



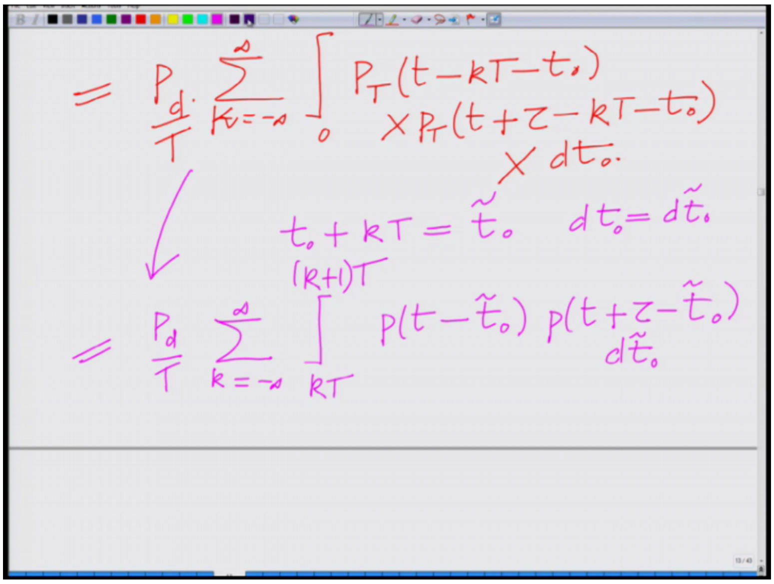
And to take the average with respect to t naught, all I am going to do is I am going to multiply by the probability of density function that is well. Let me write the complete expression I have to multiply with the probability density function of t naught that is P T t minus k T minus t naught into P T t plus tau minus k T minus t naught into d t naught. So, multiplied by the probability density functions and integrates, so this is basically averaging with respect to which is basically your random delay, this is your random delay.

(Refer Slide Time: 13:47)



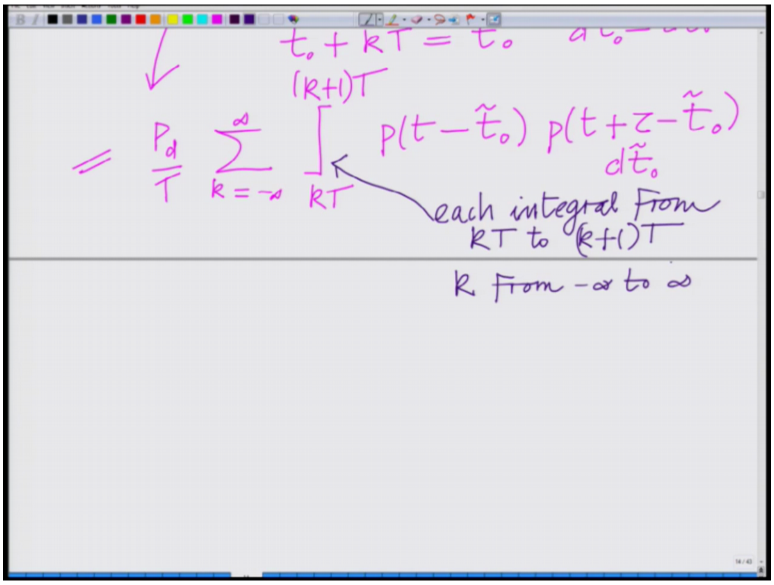
Now, F of T naught that is the uniform probability density function, so F of T naught of t naught equals 1 over T 0 less than t naught less than equal to T, it is 0 outside that is in this interval 0 to capital T, it is 1 over capital T, 1 over the symbol time T otherwise it is 0. Therefore, I can write this thing as follows therefore, I can write expected value of x t into x t plus tau is summation k equal to minus infinity to infinity integral 0 to T well 1 over T, F of T of t naught is 1 over T 1 or a constant. So, I can bring that outside 1 over T P T t minus k T minus t naught times P T t plus tau minus k T minus t naught into d t naught equals P d over T summation k equal to minus infinity to infinity.

(Refer Slide Time: 15:15)



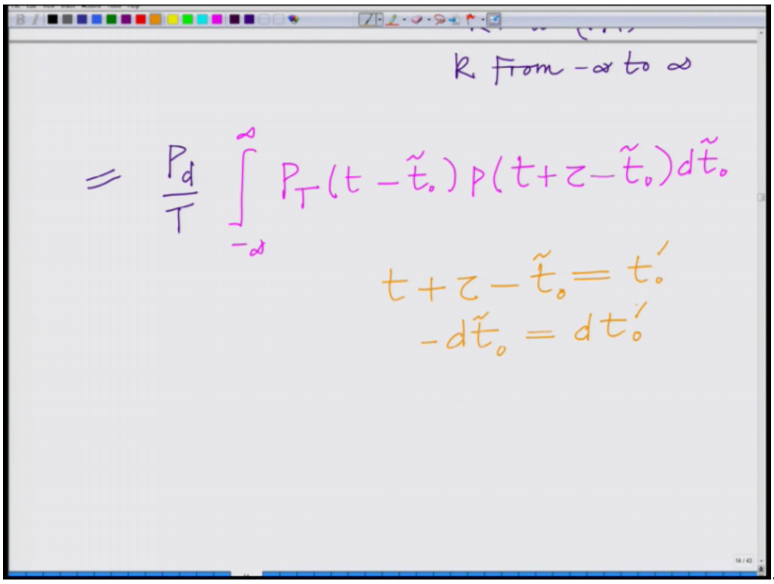
Now, what I am going to do is I am going to employ a substitution here that is let us employ the substitution t naught plus let us employ the substitution t naught plus k T equals t naught tilde. So, we have d t naught equals d t naught tilde and this becomes well P d over T times summation k equal to minus infinity to infinity well integral limits become well t naught plus k T equal to t naught tilde. So, limits become k t to k plus 1 T t naught plus k T is t naught tilde, so this is P t minus t naught tilde into P t plus tau minus t naught tilde into d t naught tilde.

(Refer Slide Time: 17:00)



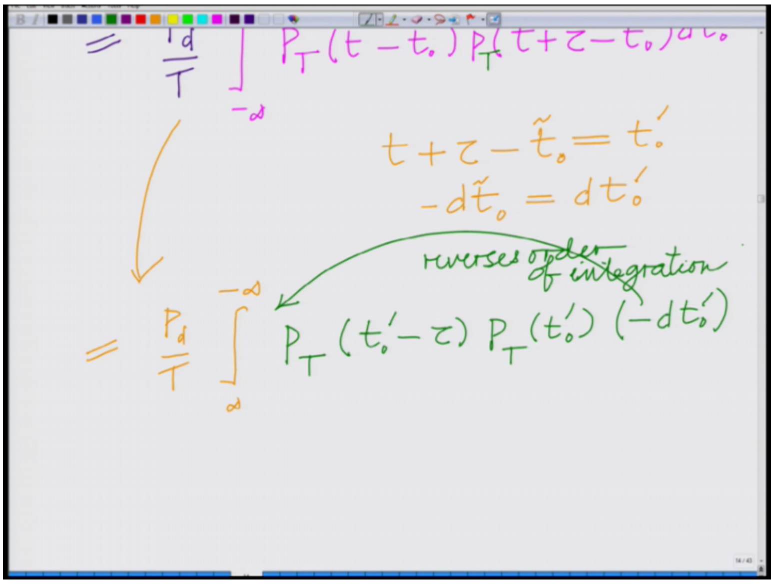
Now, if you can look at this integral each integral is from k T to k plus 1 T each integral is from k T to k plus 1 T k is from infinity minus infinity to infinity. Therefore, the entire this can be also. So, we have a sum of several integrals each integral is going from k T to k plus 1 T and k itself is going from minus infinity to infinity. Therefore, I can replace it by 1 integral going from minus infinity to plus infinity that is the whole point.

(Refer Slide Time: 17:44)



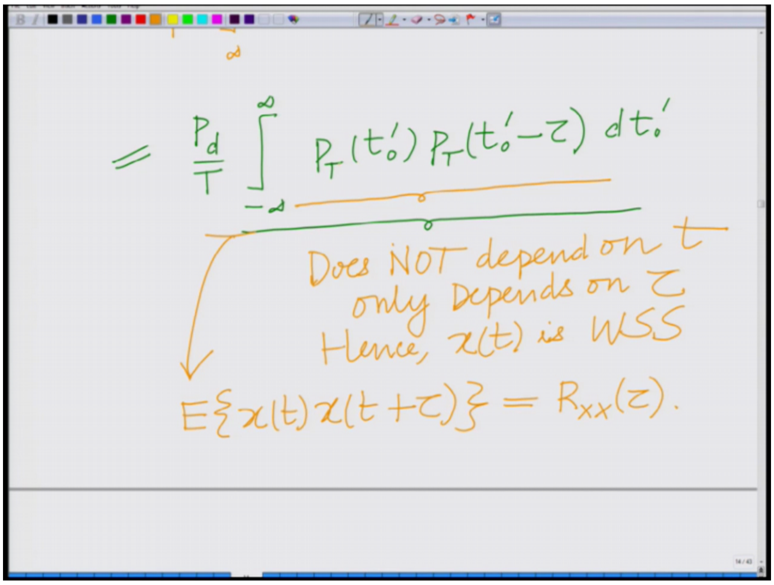
So, I get P d over T integral, I can combine the summation and integral as minus infinity to infinity P T t minus t naught tilde into P t when t plus tau t plus tau minus t naught tilde into d t naught tilde, so that is my resulting integral. Now, what I can do is I can further employ a substitution t plus tau minus t naught tilde equals t naught prime, which implies minus d t naught tilde equals d t naught prime.

(Refer Slide Time: 19:01)



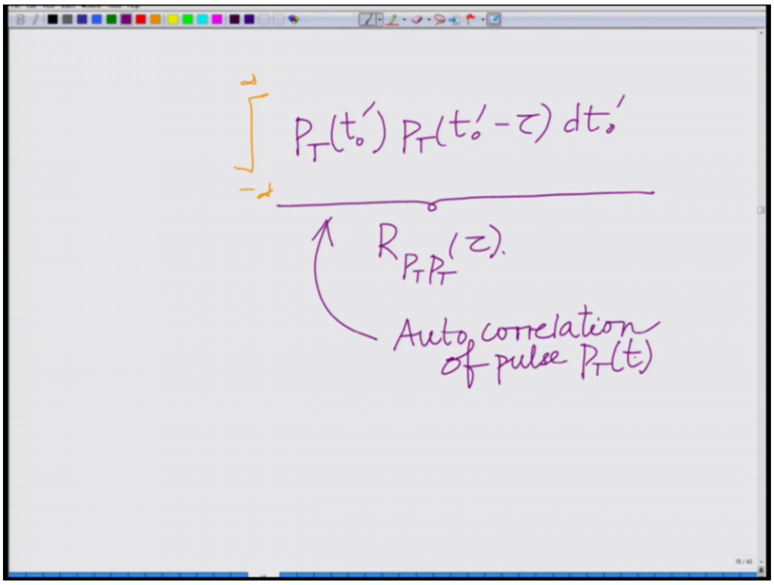
So, therefore, this integral further becomes P d over T, well now you have to be careful t plus tau minus t naught tilde where t naught tilde is minus infinity this becomes plus infinity when t naught tilde is infinity t naught prime becomes minus infinity times well P T t minus t naught tilde equals t naught prime minus tau into P T t plus tau minus t naught tilde that is t naught prime d t naught prime or d minus t naught prime, because d t naught tilde minus d t naught tilde is d t naught prime. So, this becomes minus d t naught prime and this minus reverses the order of integration. The integral to bring the minus sign outside integral from infinity to minus infinity becomes integral in minus infinity to infinity.

(Refer Slide Time: 20:19)



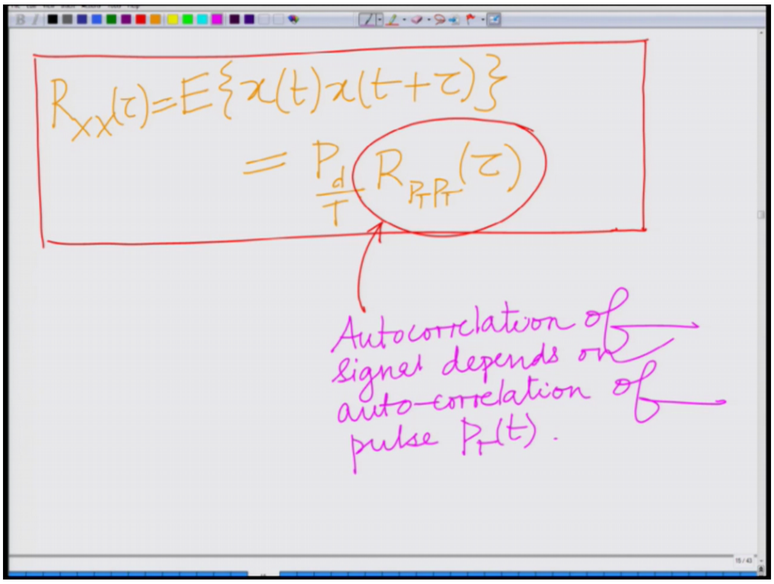
So, therefore, we have something that is P d over capital T integral minus infinity to infinity P T t naught prime minus tau into P T or let me write it the other way P T into t naught prime, t naught prime is simply a dummy p t into t naught prime minus tau. Remember t naught prime is simply a dummy variable, this is t naught. And now if you look at this you will observe something remarkable. If you look at this you will observe that this whole thing now we have successfully avoided or successfully removed, it is dependence on t which is the specific time instant. So, this does not depend on T, it only depends on tau. So, it does not depend on T only depends on tau. Hence, now x t is wide sense stationary. In fact, this is your expected value of x t into x t plus tau which is basically you can call this as R x x of tau depends only on tau.

(Refer Slide Time: 22:22)



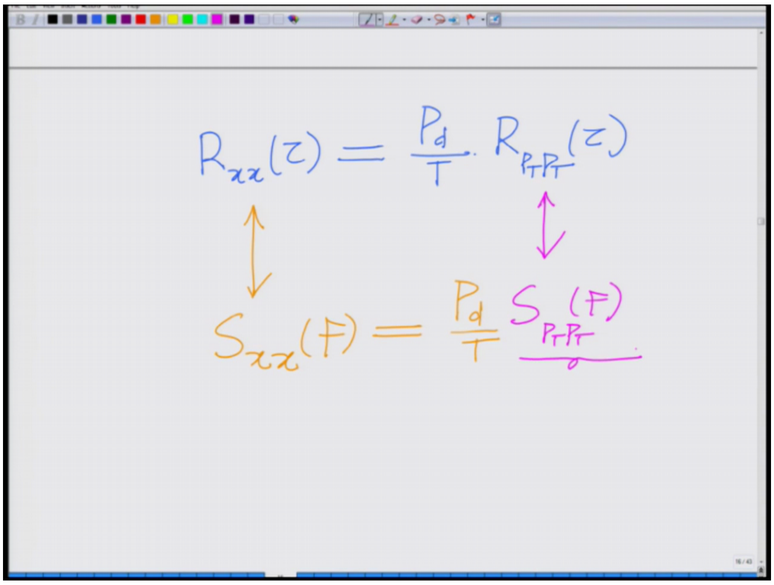
And in fact, you will observe something very interesting, if you look at this integral minus infinity P T let us write down this integral, integral minus infinity to infinity P T minus or P T or P t naught this is simply a dummy variable. So, this is t naught prime minus t naught prime minus tau or P t naught prime into t naught prime minus tau. If you remember this is nothing but the autocorrelation of the pulse, this is R P T P T of tau, this function is indeed the autocorrelation function of the pulse P T of P T. So, this is the autocorrelation of the pulse P T autocorrelation of the pulse P T. And therefore, this is nothing but P d over T times autocorrelation of the pulse P T.

(Refer Slide Time: 23:46)



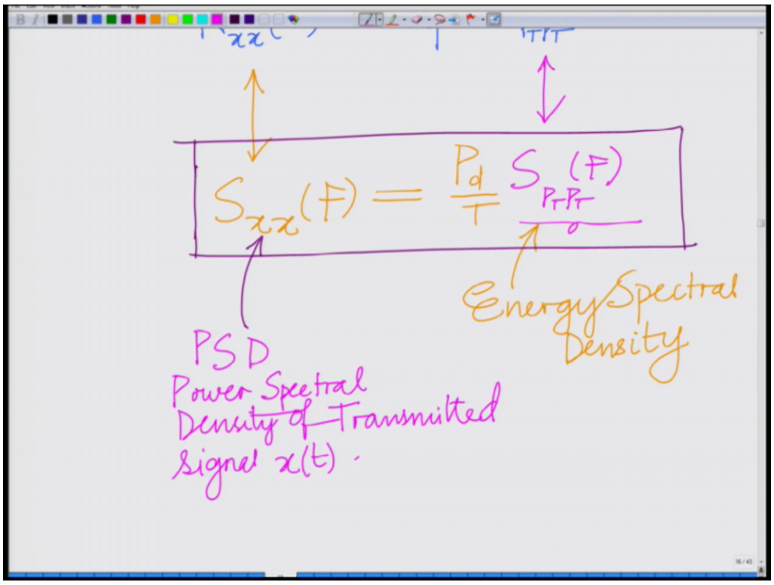
And therefore, we finally, have a very elegant result for this that is expected value of x t that is R x x tau equals expected value of x t into x t plus tau which is equal to P d over tau times simply which is equal to P d over tau which is simply equal to P d over T times autocorrelation function P T T tau of the pulse this is the very interesting result. So, what we have seen is that the autocorrelation of this thing is indeed related to the pulse and it depends on the autocorrelation of the pulse it depends on the. So, what we are able to see is something very interesting autocorrelation of the signal depends on autocorrelation function of the pulse P T of t it depends on the autocorrelation function of the pulse P t of t.

(Refer Slide Time: 25:19)



In fact, if you can now take the Fourier transform of this autocorrelation function, so we have R x x tau equals P d over T, P d over T is a constant times R P T P T of tau depends on the autocorrelation of the pulse. If you take the Fourier transform of this you will get the power spectral density that is you will get S x x of F which is equal to P d over T times the Fourier transform of the pulse which is nothing but S P T P T of F this is nothing but the autocorrelation. This is nothing, but the energy spectral density of the pulse.

(Refer Slide Time: 26:05)



Remember if you take the Fourier transform the autocorrelation function of a deterministic signal such as this pulse, you will get the energy spectral density. For a random signal it gives the power spectral density; for a deterministic signal, the Fourier transform of the autocorrelation is the energy spectrum, it is the energy spectral density. And we have this very beautiful relation where the Fourier transform of the pulse that is we have the power spectral density of the transmitted signal the PSD or power spectral density of the transmitted signal that is x t.

So, what we have able to show that the autocorrelation function that is first we have cons converted this signal into a wide sense stationary signal. And then we will be able to derive the expression of autocorrelation, we have seen that as a very good elegant expression the autocorrelation function of the digital communication signal simply depends on the autocorrelation function of the pulse. And moreover the power spectral density of the transmitted signal is simply therefore proportional to the energy spectral density of the pulse which is nothing but the Fourier transform of the autocorrelation function of the pulse and that is a very interesting result. So, we have been able to derive now the power spectral density of the signal which gives the distribution of the spectral distribution of the power of the transmitter digital communication signal. So, we stop here and look at this further in the subsequent modules.

Thank you very much.