An Introduction to Coding Theory
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Module 04
Lecture Number 18
Convolutional Codes: Distance Properties

So in this lecture we are going to talk about how we are going to find out the weight

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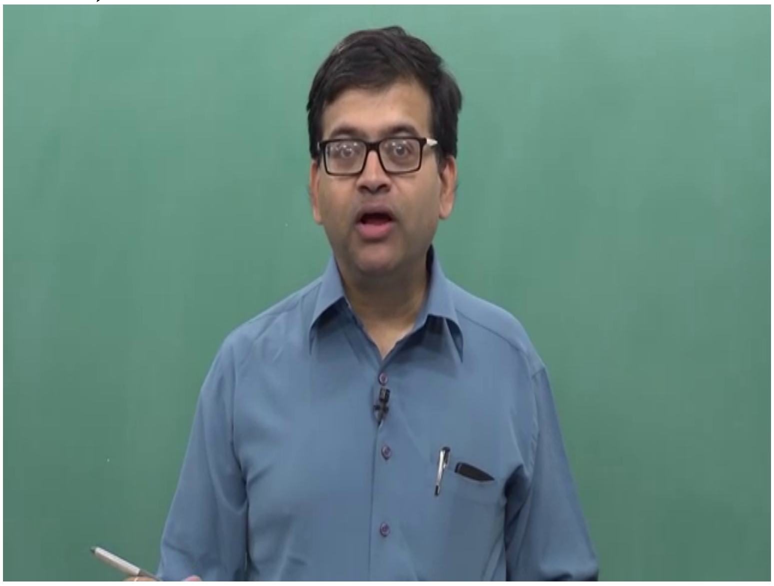

distribution of the convolutional codes and we are going to discuss about the distance properties of convolutional codes. So this

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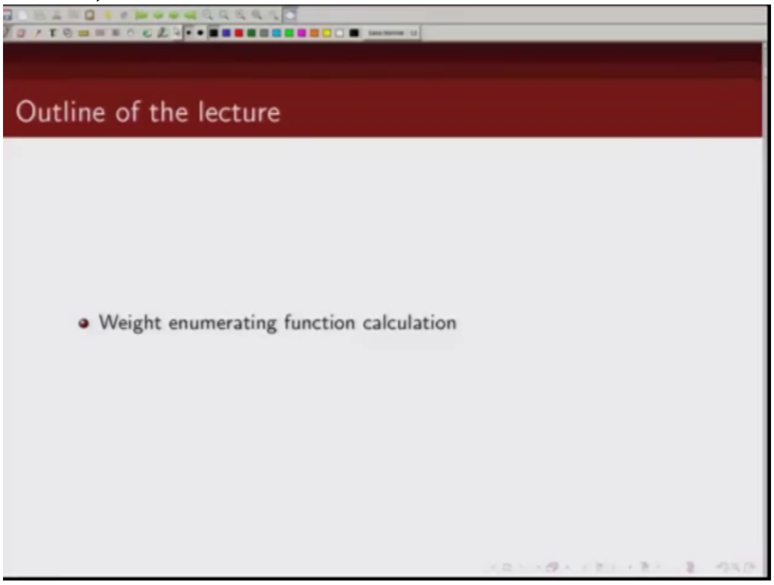

lecture deals with how we can enumerate the distance

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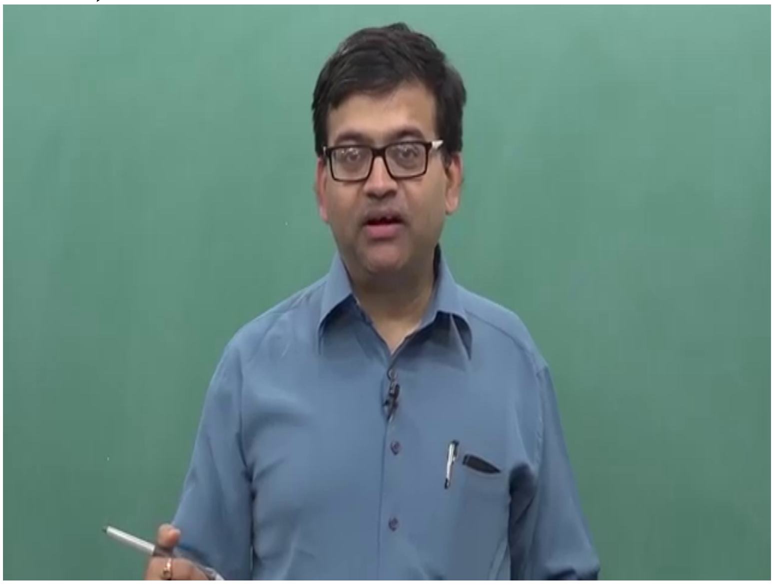

profile or the weight distribution of the convolutional code.

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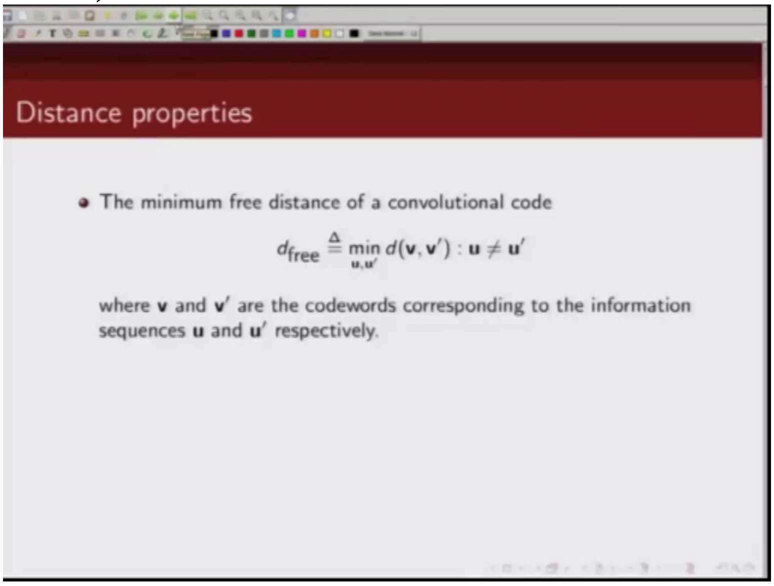

So we can define before we discuss the technique to find out the weight distribution let us define what do we mean by minimum free distance of a convolutional code. So minimum free distance of a convolutional code is defined as minimum Hamming distance between 2 codes v and v hat where v and v hat are 2 codewords corresponding to information sequence u and u hat where u and u hat are 2 different information sequences.

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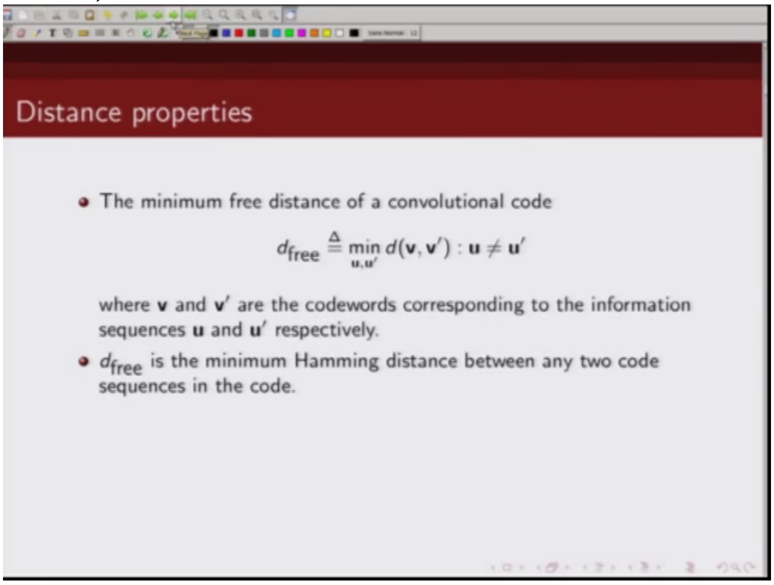

So it is basically, free distance is the minimum Hamming distance between any two codewords in the convolutional code.

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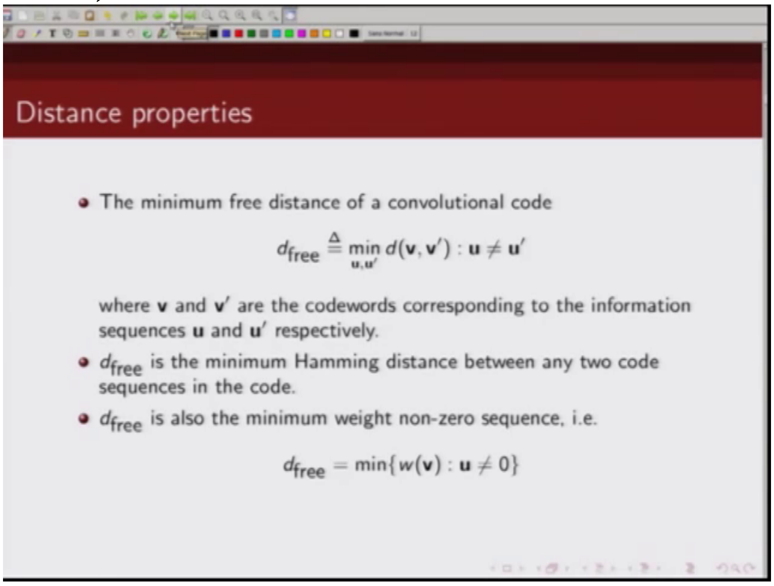

And this is same as minimum weight non-zero so it is the minimum weight of a non-zero codeword. We know

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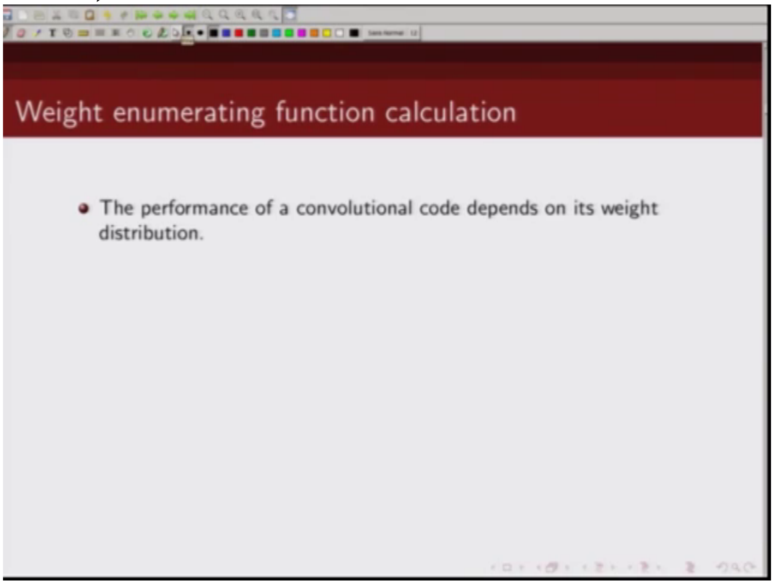

that performance of any convolutional code depends on its weight distribution and that is why we are interested to find out what is the weight distribution

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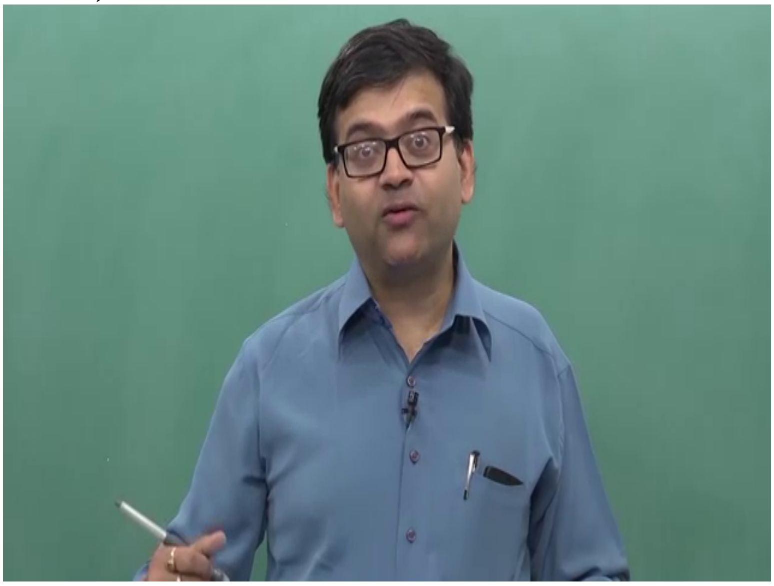

of the convolutional code. In this lecture we are going to talk about a method based on Mason's gain formula to compute the weight distribution for convolutional code.

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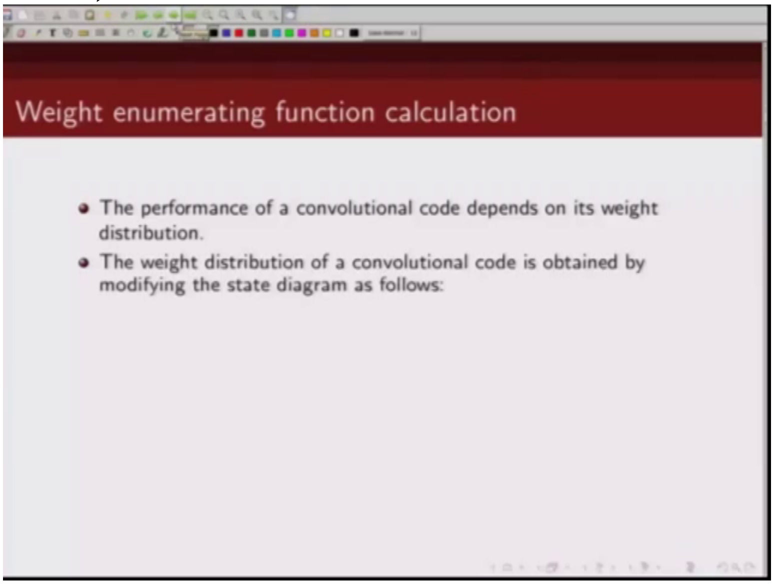

So in this we are first going to modify the state diagram of a convolutional code. And how are we going to modify the state diagram? We are going

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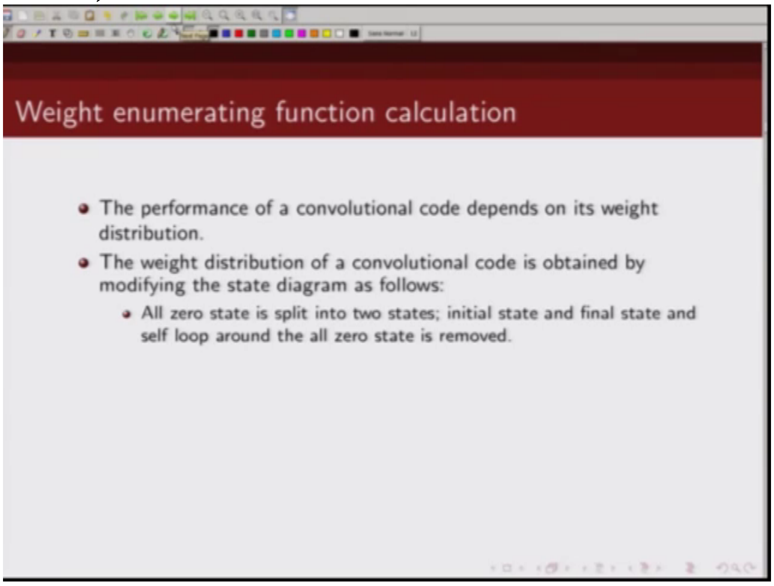

to split the all zero state into 2 states, one initial state and second final state and we are going to remove the self loop around the all zero state.

Next,

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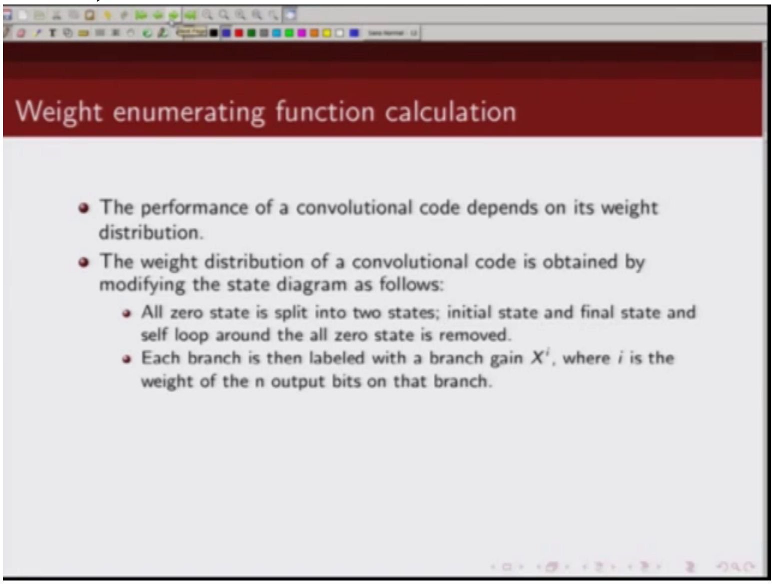

each branch which is linking from one state to another state, this branch will be labeled by the output weight of the codewords. So we are going to denote by x raised to power i where i will be

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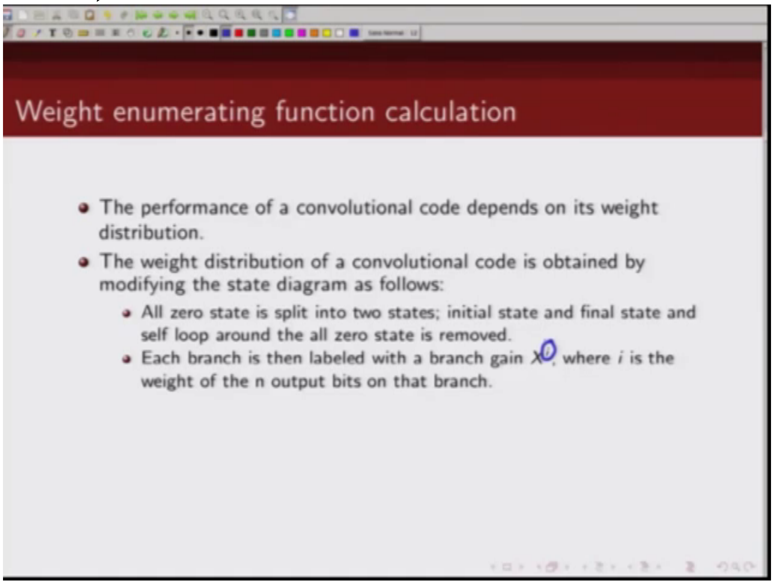

the weight of the coded bits. So for example if we make a transition from state 0 to state 0 1 when input 1 comes and output is 1 1,

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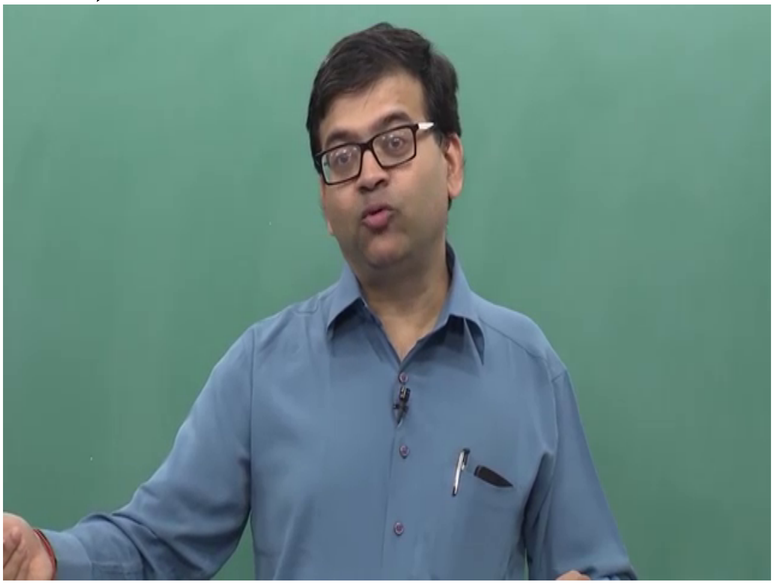

in that case

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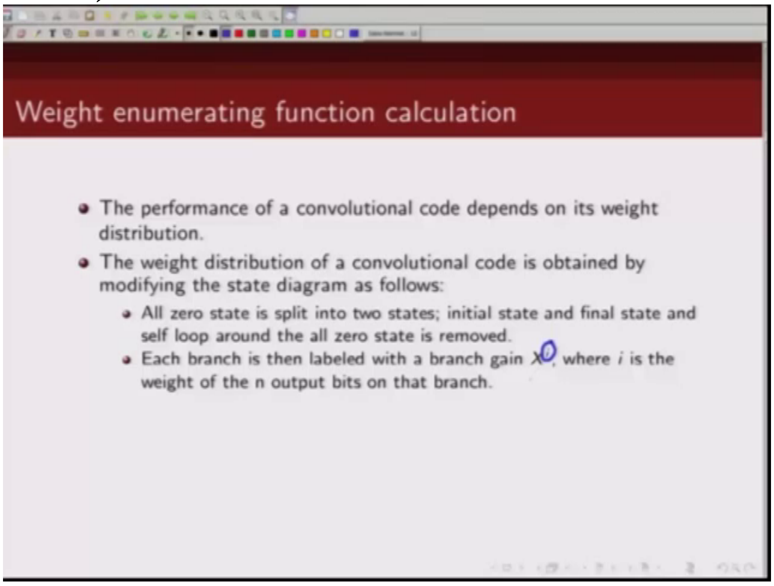

this output is 1 1, we will label that branch by x raised to the power 2.

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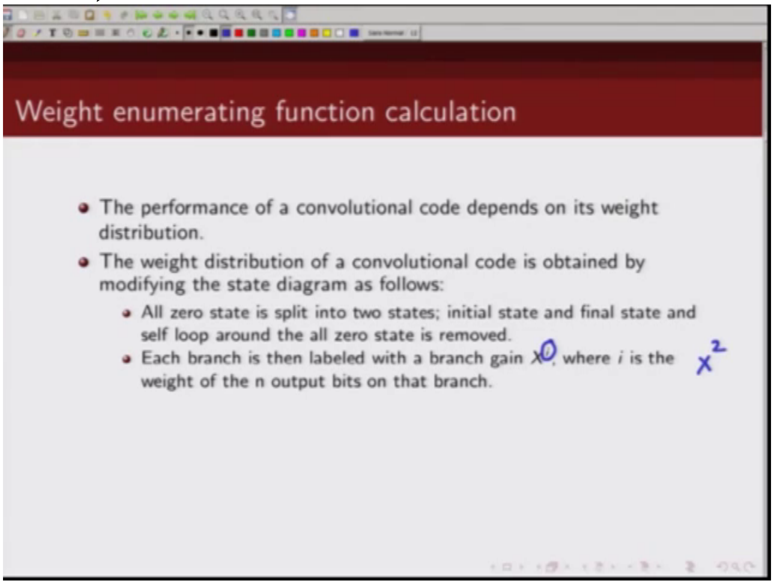

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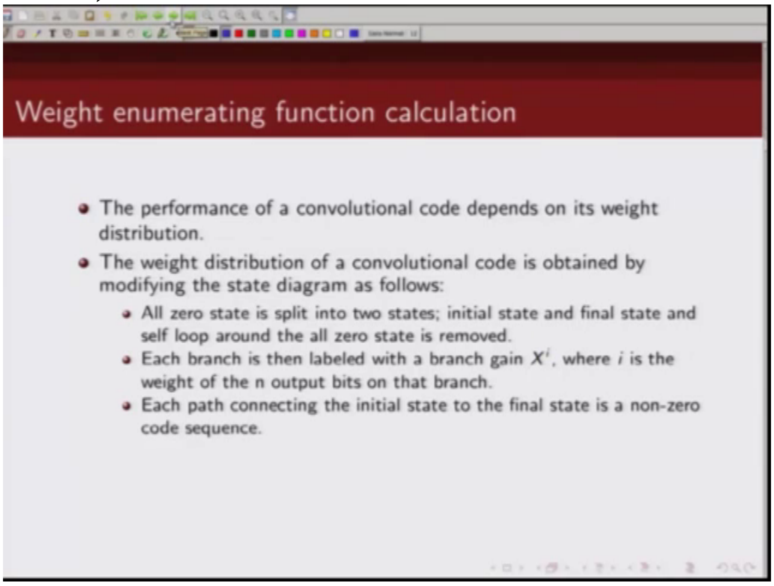

So now after we split this all zero sequence, all zero state into 2 states initial state and final state what we will have is each path starting from this initial state to the final state, that will be

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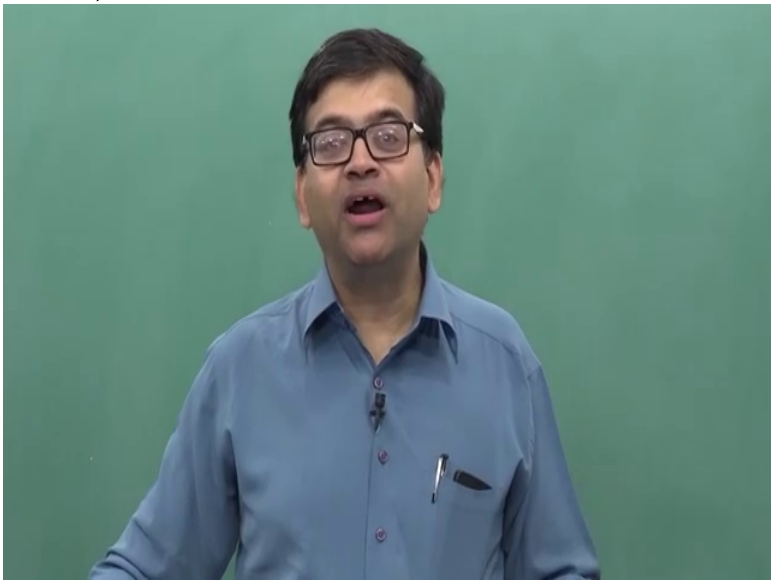

our valid codeword. So each path connecting the initial state

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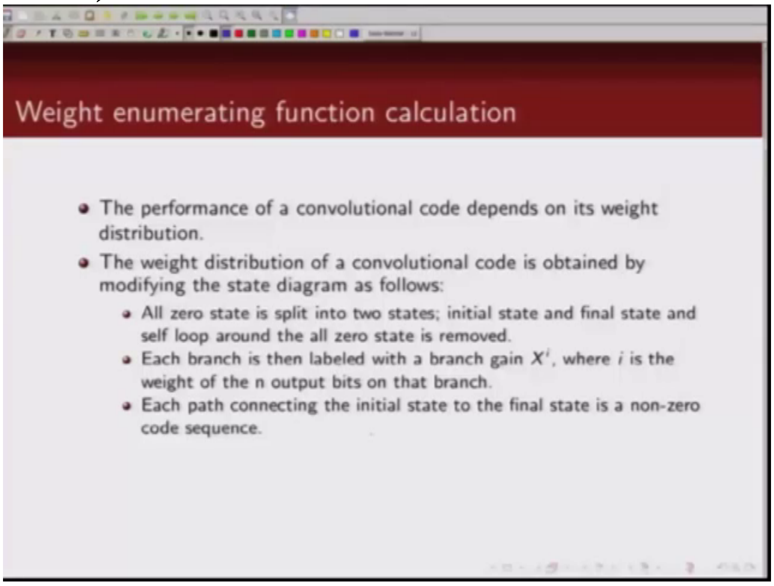

to the final state is a valid non zero codeword because we have removed the self loop

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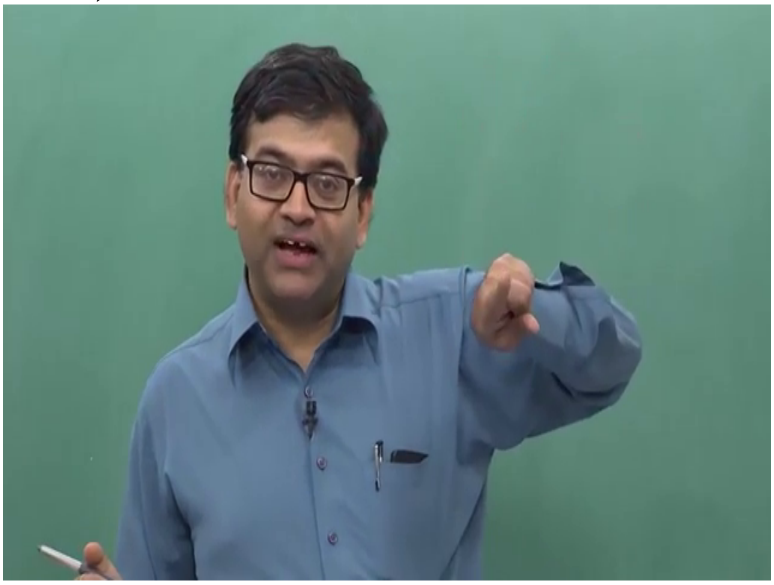

around the all zero state. So

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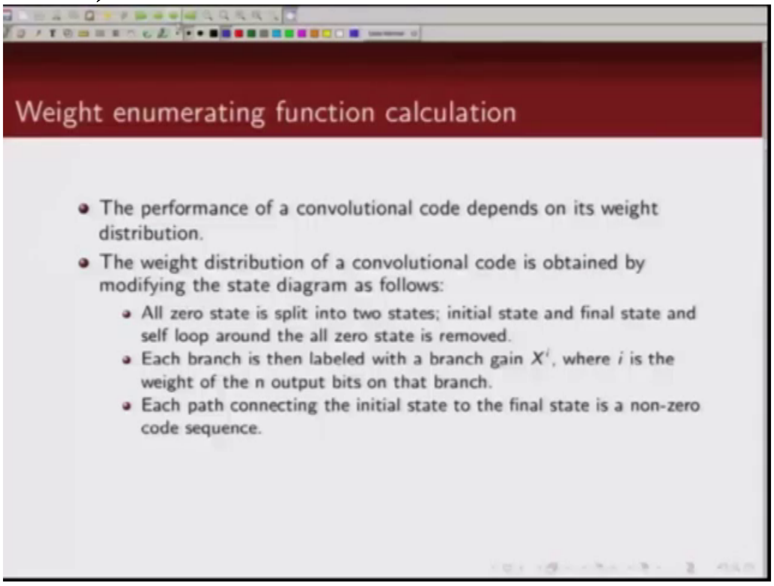

this modified state diagram is now going to show us all possible non zero codewords. We define a path gain as product of branch gains along a path

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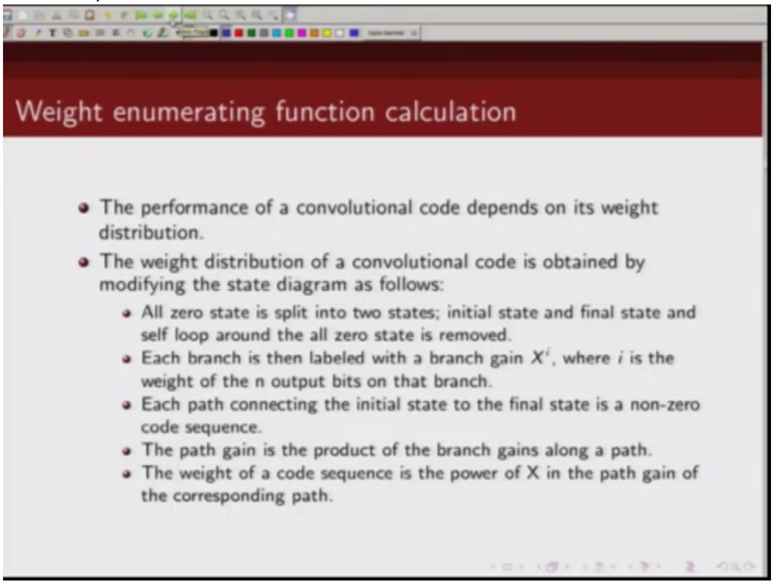

and the weight of the code sequence will be nothing but the power of x in the path gain of the corresponding path because what we are doing is, so each branch is labeled by its corresponding output

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weight. So if we look at each path going from initial state to the final state and we look at the power of x, that will give us

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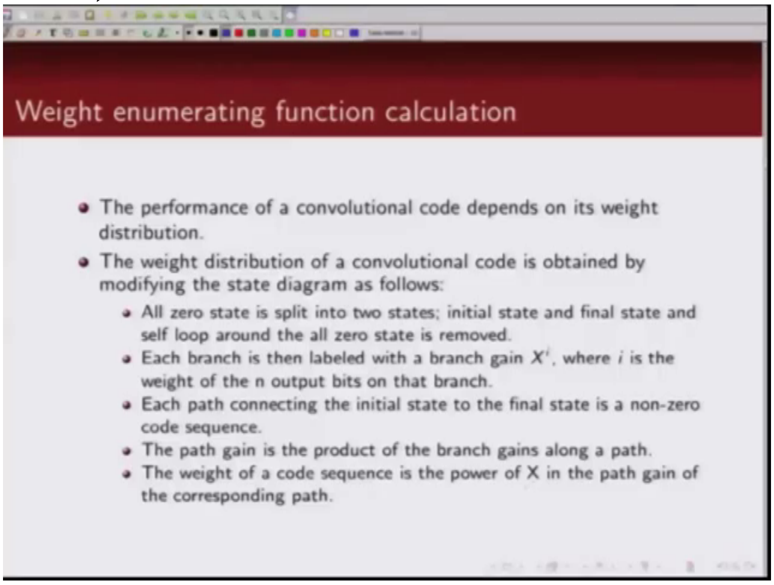

the overall weight of that particular non zero sequence. As we

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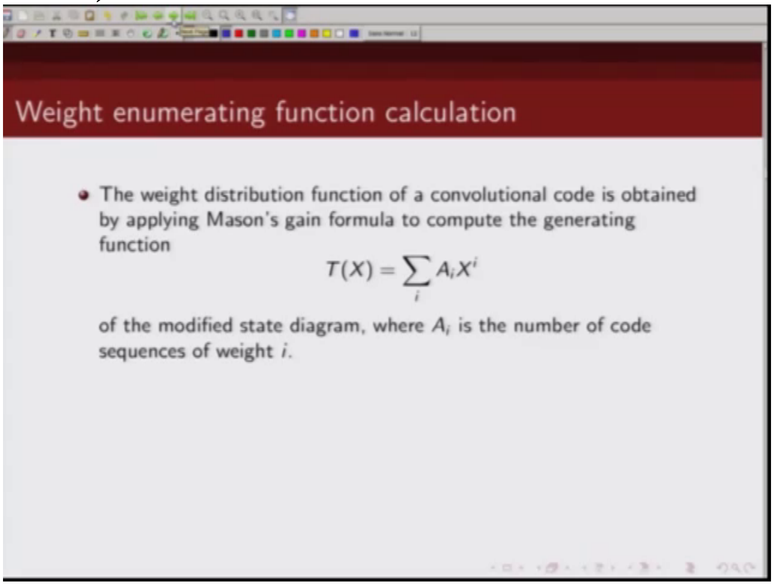

said we are going to use Mason's gain formula to compute the weight enumerating function for the convolutional code. So we are going to describe how we are going to use the Mason's gain formula. So we are representing by T of X the generating function which will basically enumerate all codewords of weight i.

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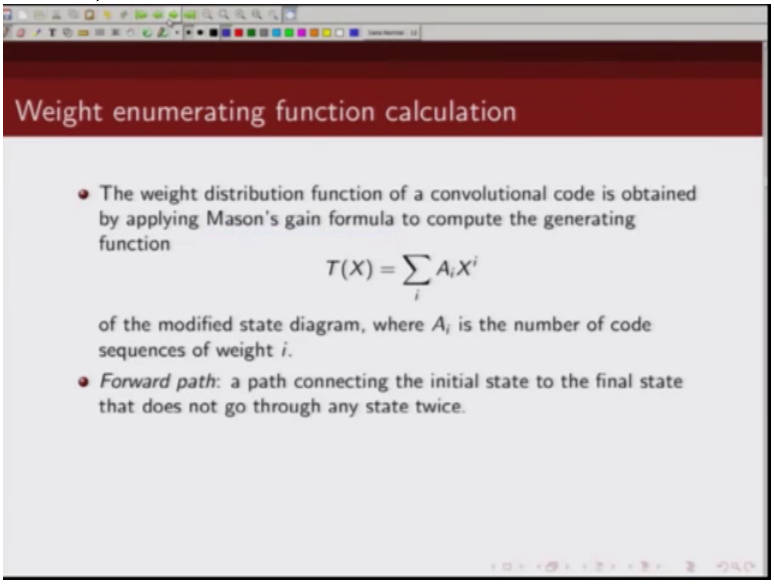

Now let us define few terms that we are going to use in Mason's gain formula.

The first term that we are going to define is what is known as forward path. So a forward path is a path from the initial all zero state to the final state, and the condition is this path should not go over any state twice. That is our forward path.

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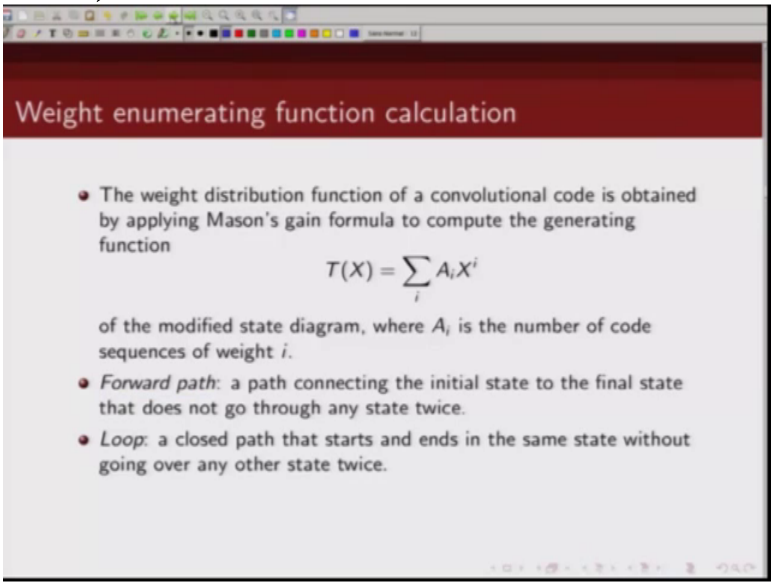

Next term that we define is basically a loop. What is a loop? A loop is a closed path that starts and ends in the same state without going over any state twice. That's a loop.

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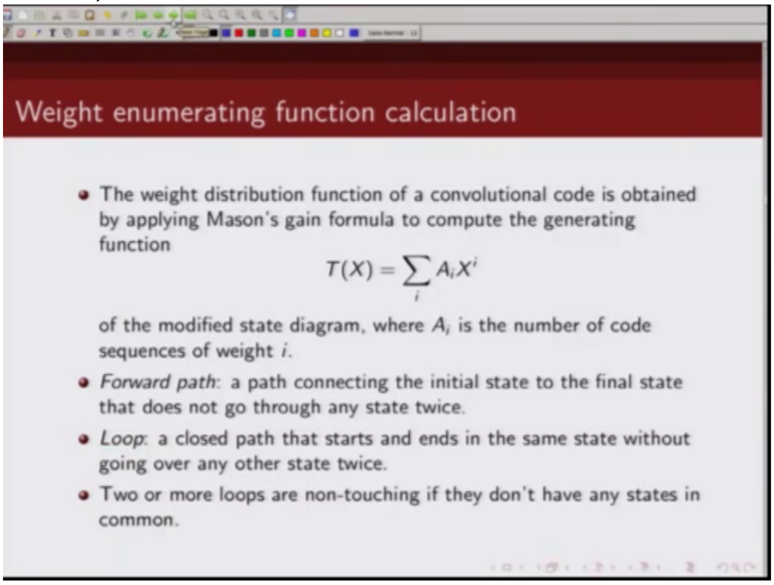

When do we say 2 loops are non-touching? We say two loops are non touching if they do not have any state in common.

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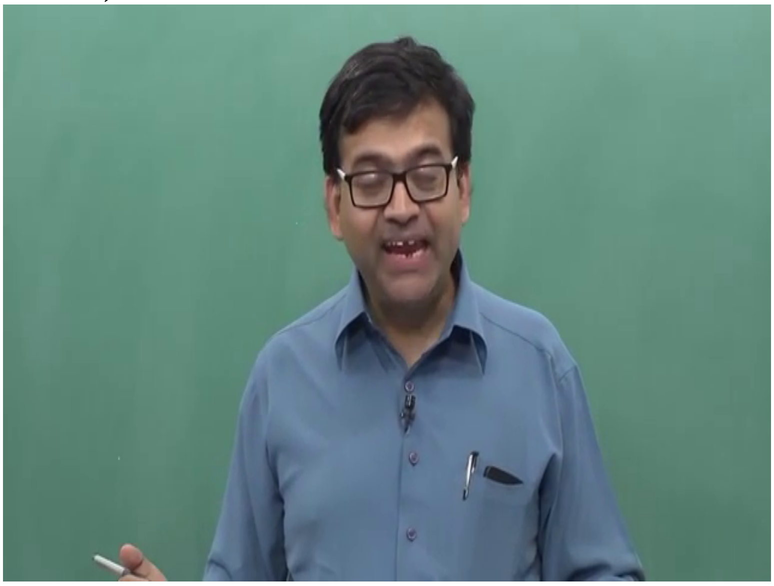

So again I repeat these 3 definitions. Forward path is a path from initial

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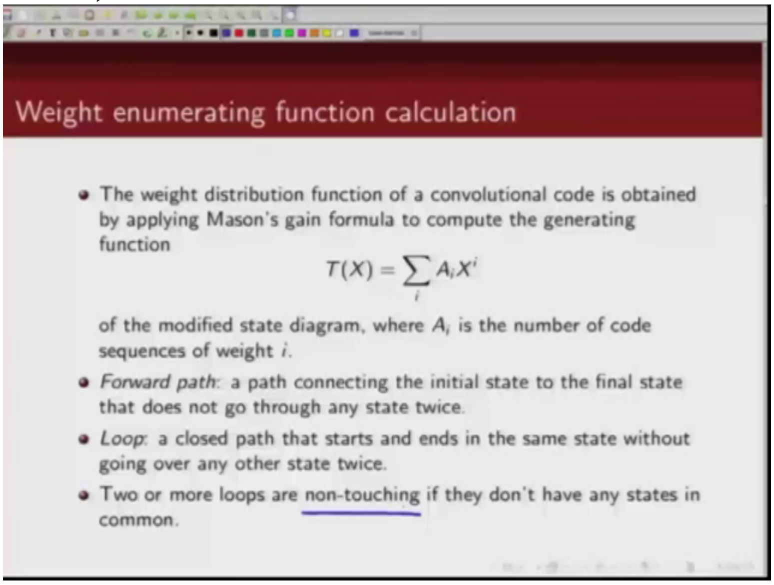

state to the final state without visiting any state twice. A loop is a closed path starting and ending in the same state without going over the same

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state twice and two or more loops are non touching if they do not have any state in common.

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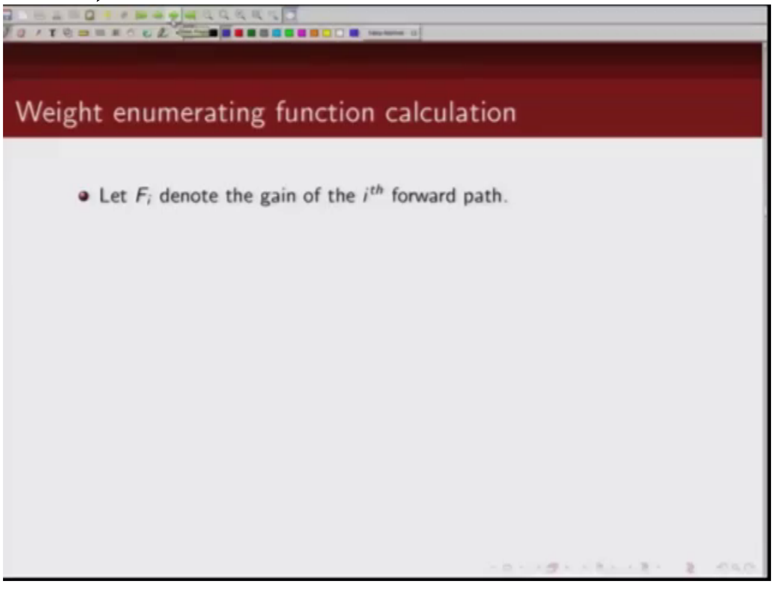

Now let us denote by F sub i, the gain of the ith forward path. And let C i

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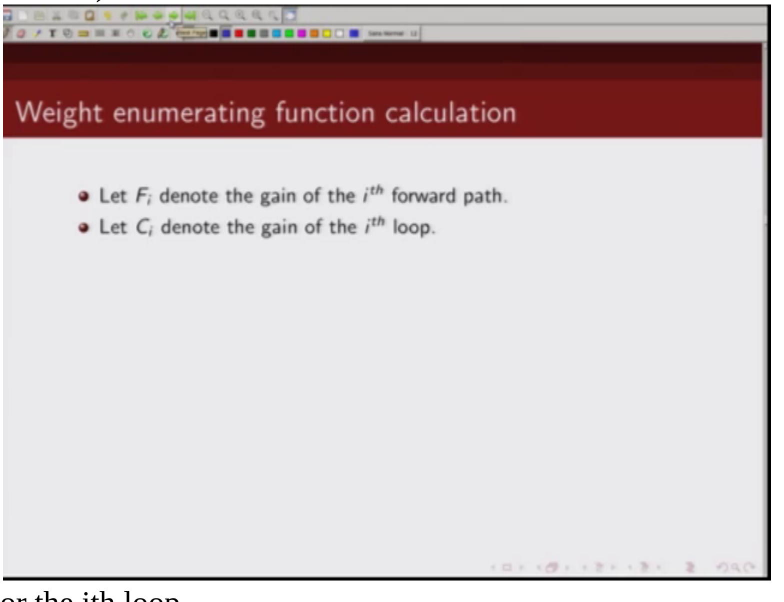

denote the gain for the ith loop.

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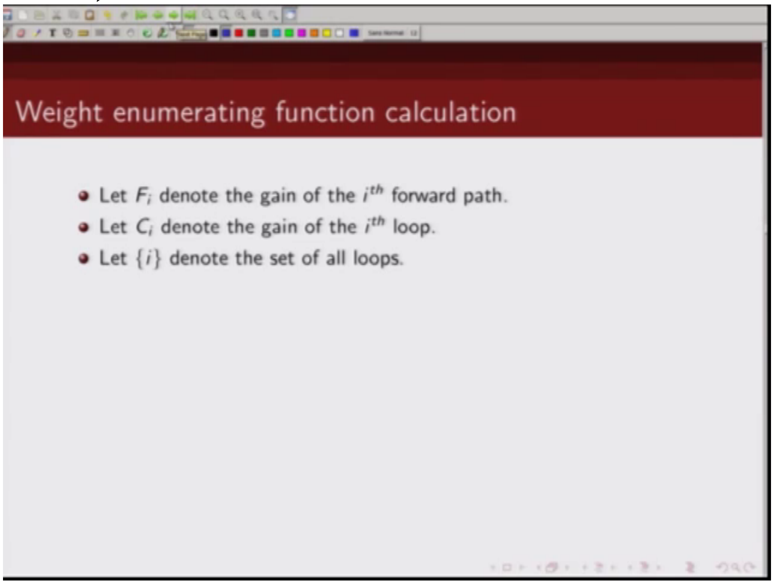

We denote by this the set of all loops. Similarly

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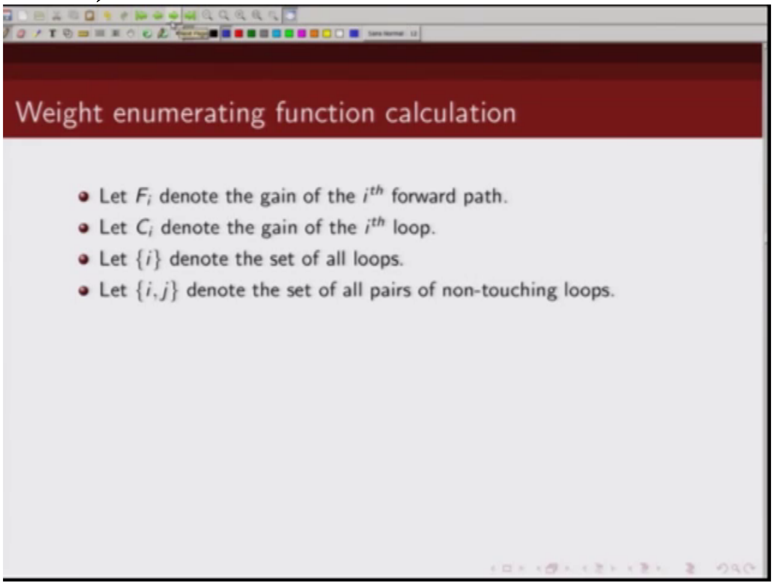

this set of i and j will denote set of all pairs of non touching loops.

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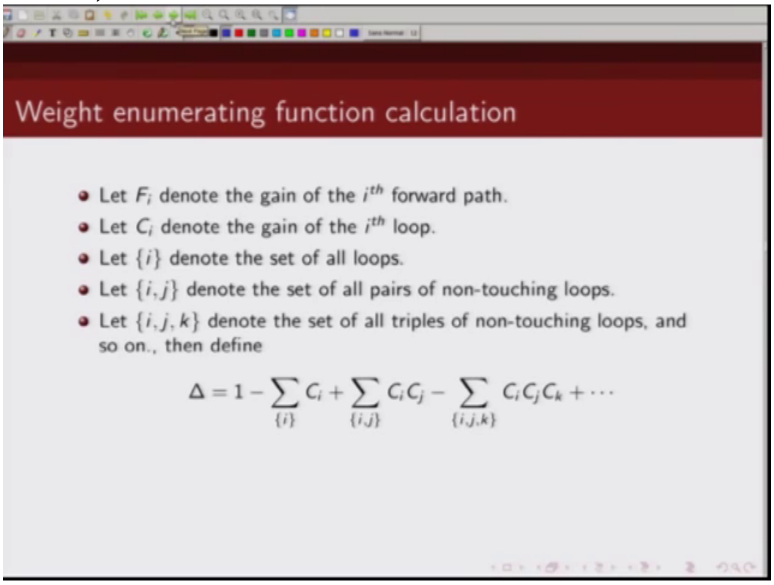

This triplet will define set of all triplets of non touching loops. So if we use this, we define a term delta which is defined as follows. It is 1 minus summation of all the gains of the loops plus product of gains of all those non touching loops minus this is product of, set of all triplets of non touching loops and it goes on like this. So that's our delta.

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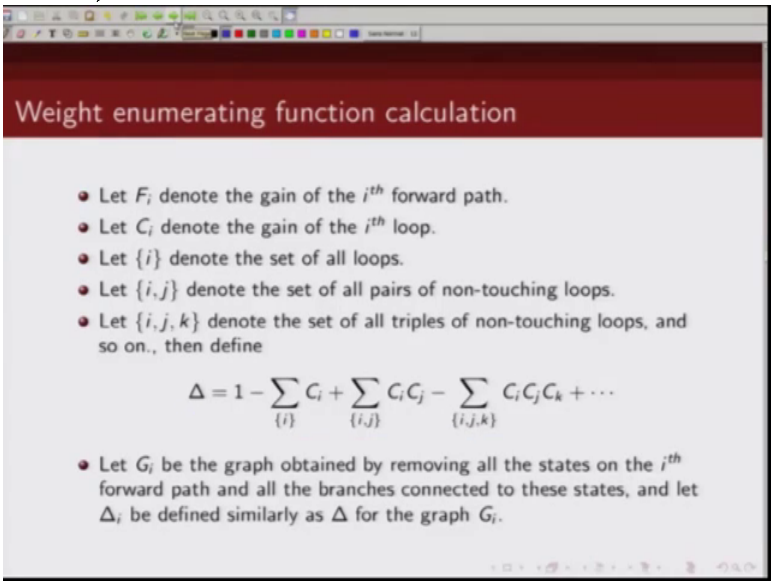

We define our graph that is obtained after we remove all states belonging to ith forward path by G sub i. So G sub i is basically the graph remaining after we remove the ith forward path and the delta corresponding to this modified graph will be denoted by delta i.

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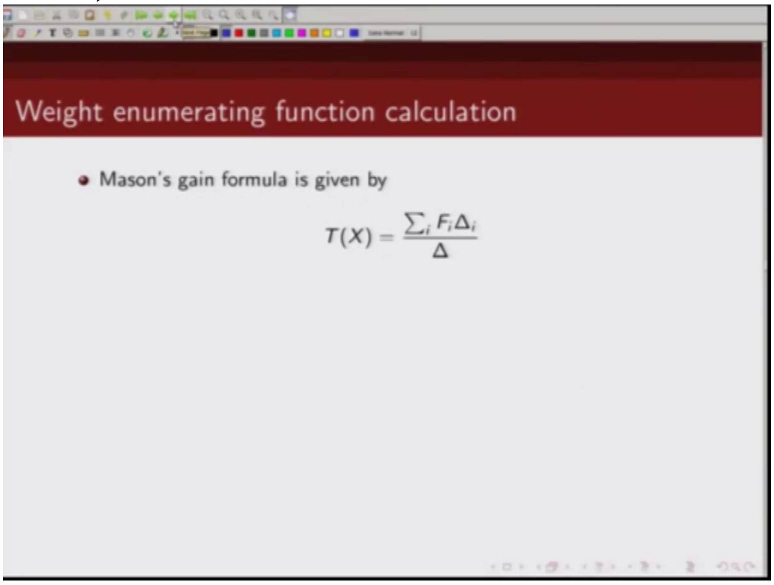

So the Mason's gain formula then says that the generator function for convolutional encoder can then be given by this expression. So this

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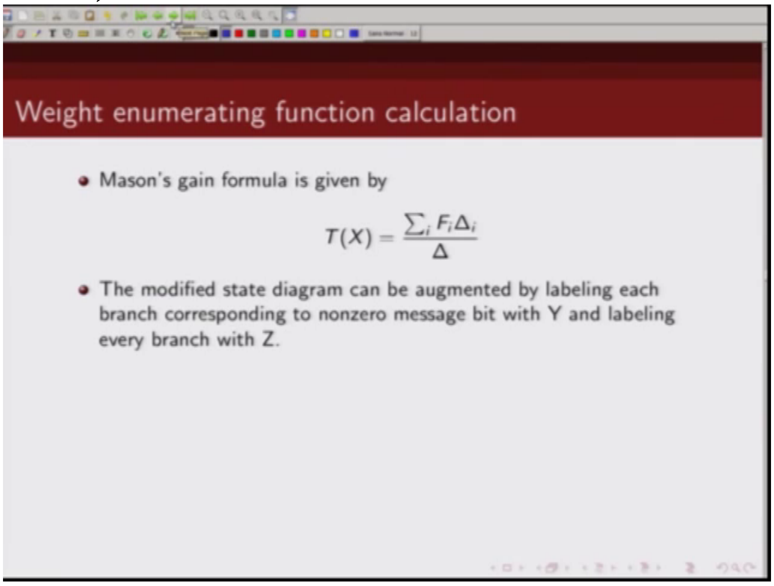

modified state diagram can be augmented to include more information. Now what we had done so far was we labeled the branches

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of the state diagram by the overall total weight. Now we can also augment this by mentioning

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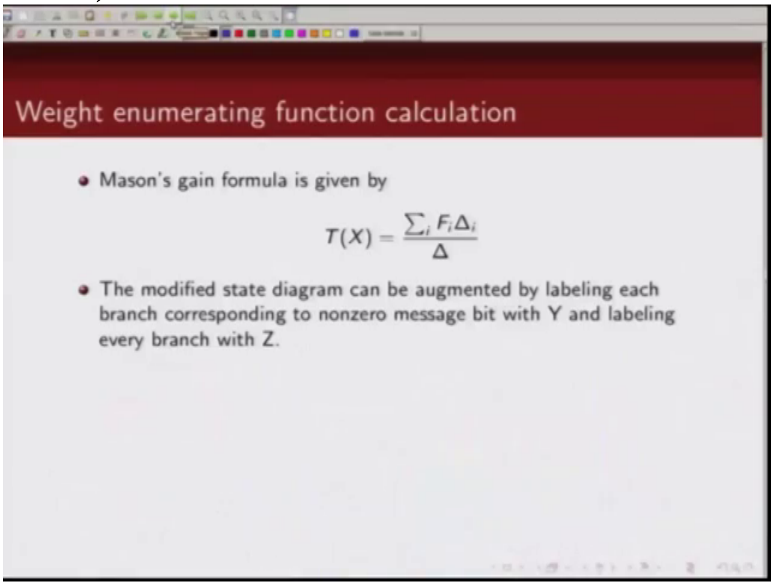

what is the input that results in that output weight. So we can label the weight of the input by y. So the power of y will denote what is the input weight and similarly we can label each branch by z. So in a path gain formula the degree of z will tell us

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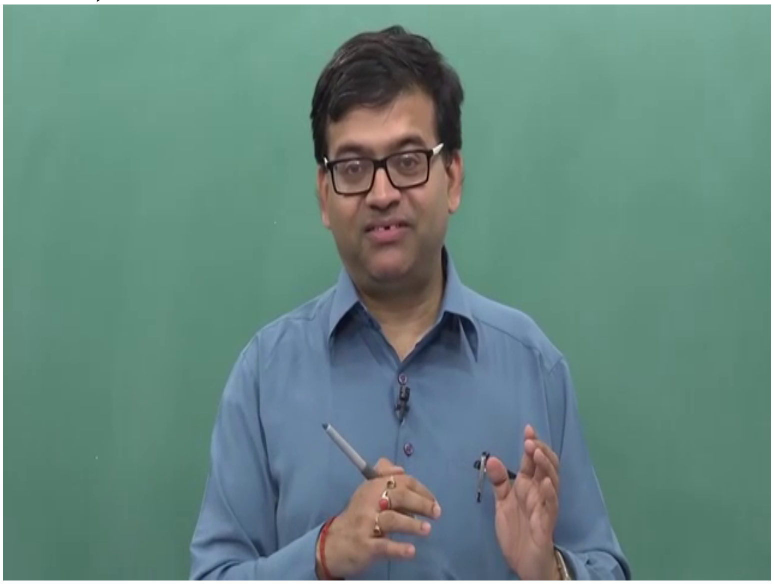

like what's the

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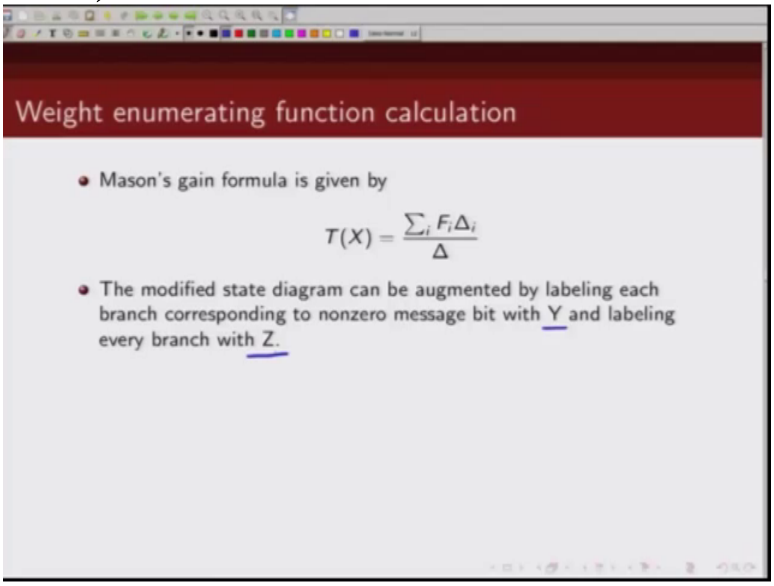

length of non zero path. So the degree of z will tell us like once it diverge from all zero state after how much time it comes back into all zero state. So we can augment our state diagram by what I call a modified state diagram by adding two additional information, one is the weight of the message bit which will be denoted by power of Y i and other is branch which will be denoted by Z.

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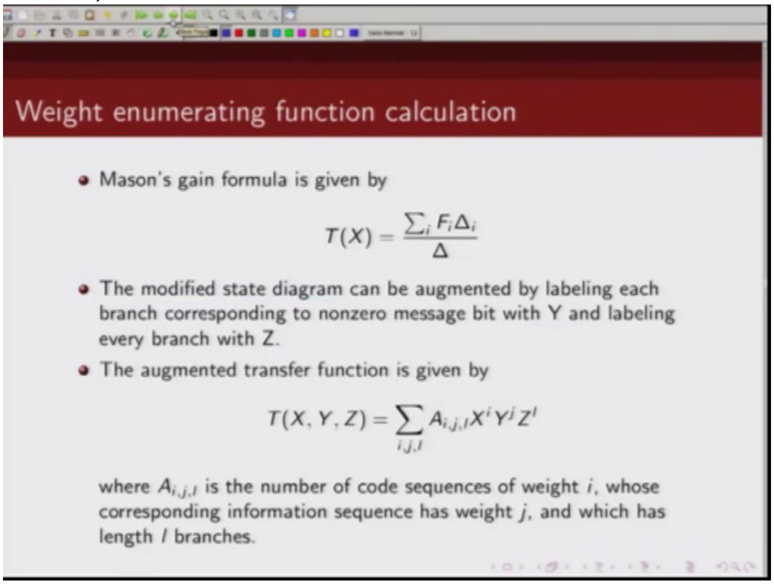

So we can then similarly define an augmented transfer function which will not only tell us the codeword weight but it also tells us what is the input weight that results in that output weight and it will also tell us length of that particular codeword. By length I mean the time it diverges from all zero state until it comes back to all zero state. So this

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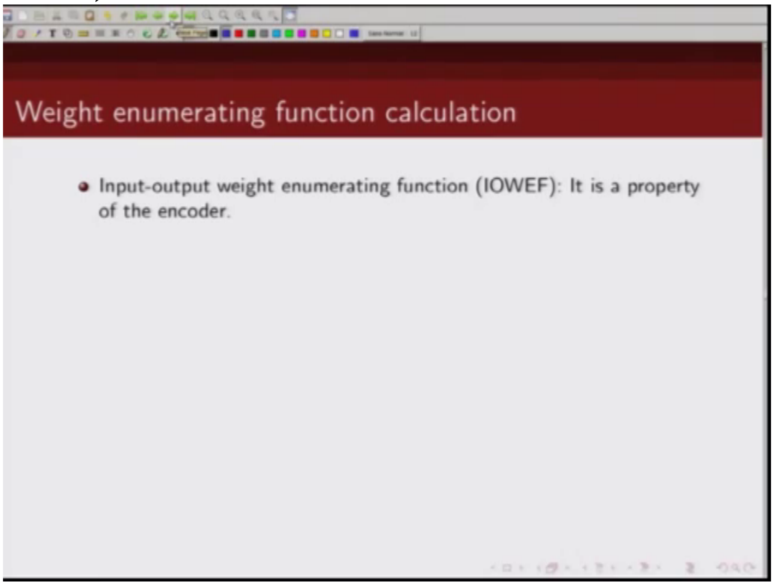

is basically what we call input output weight enumerating function. Because

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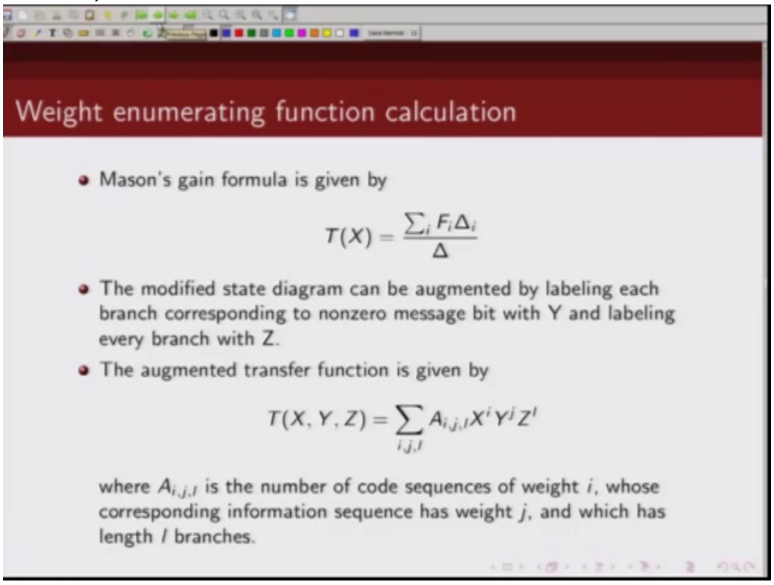

this function is enumerating for what input you get what output, Ok. So this will give us weight input output

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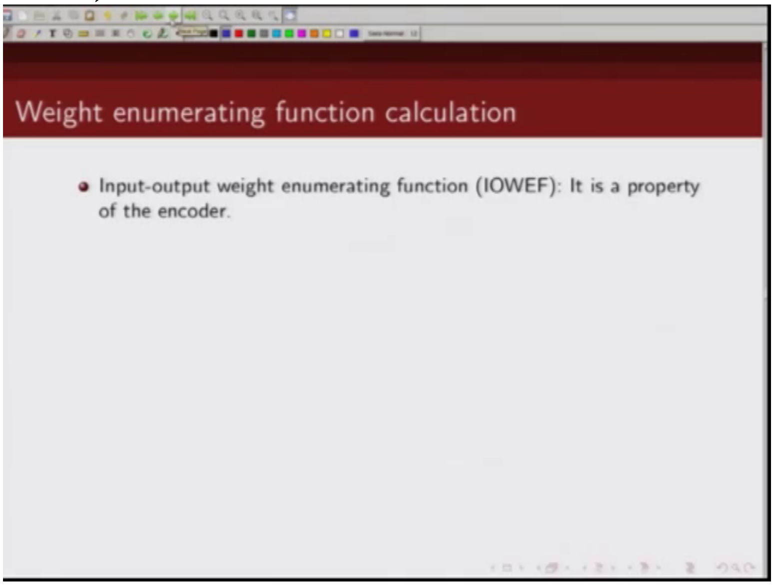

weight enumerating function. And it is a property

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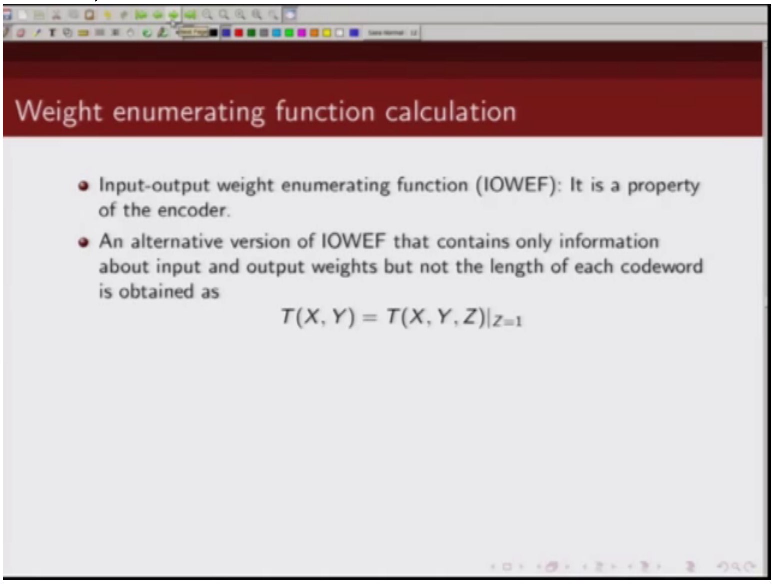

of the encoder. An alternative version of this input output weight enumerator function is one that contains only information about the input and output weight and not the length of each codeword. So if we put Z equal to 1, basically this is going to give us, it is an alternative version of input output weight enumerating function.

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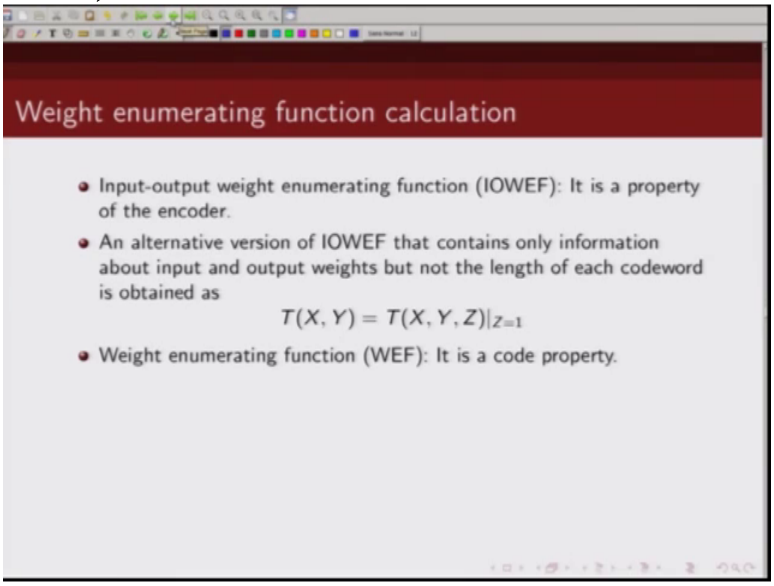

And what is weight enumerating function? Weight enumerating function will only tell us what is the overall codeword weight and this is the property of

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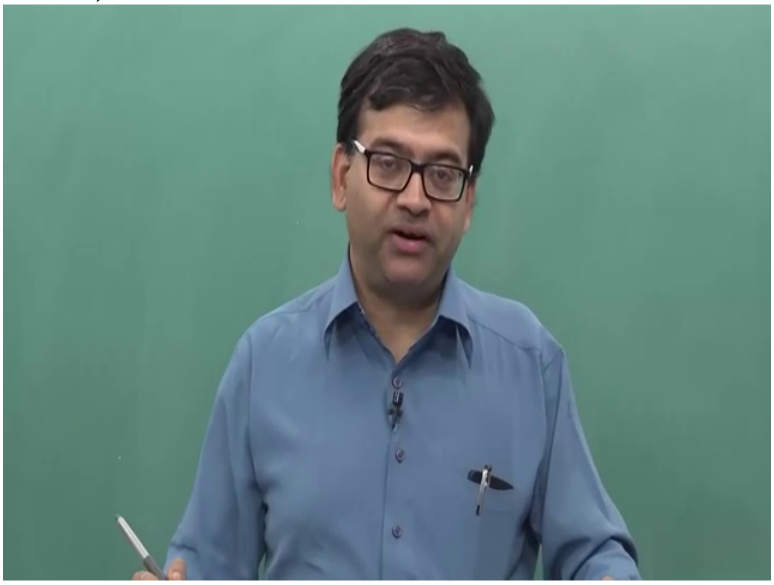

the convolutional code.

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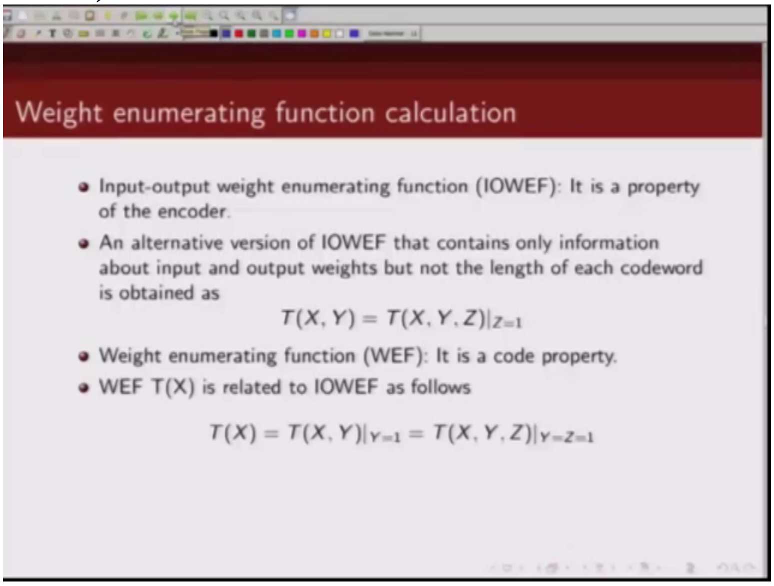

So weight enumerating function is related to input output weight enumerating function in this particular way. So if we put Z and Y as 1 in the input output weight enumerating function, we will get back our weight enumerating function. Similarly

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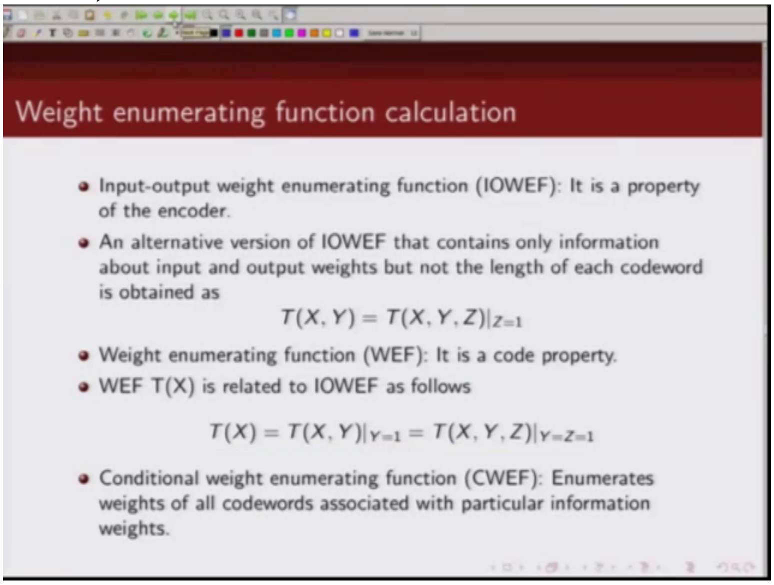

we can define what is known as conditional weight enumerating function. What is conditional weight enumerating function? The conditional weight enumerating function, it enumerates weights of all codewords associated with a particular information weight. So if you are interested in knowing what is the output weight correspond to

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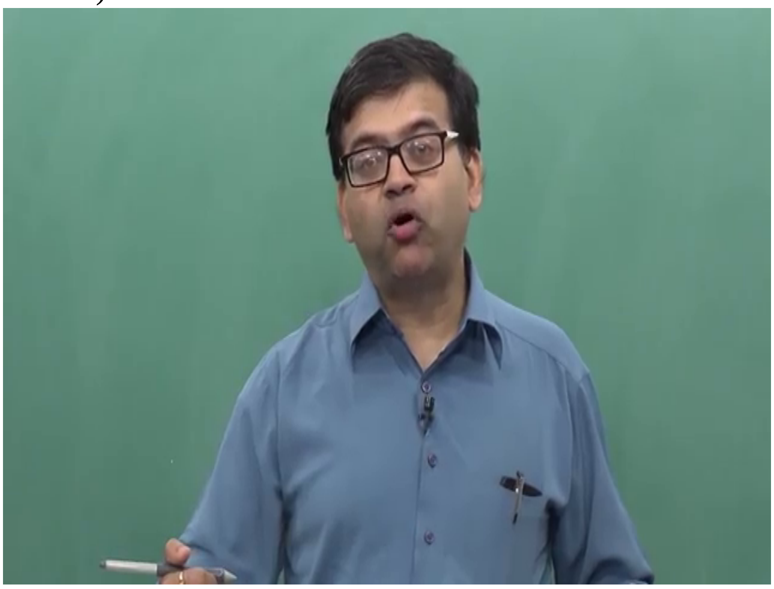

weight 4 input sequence, so from the input output weight enumerating function by collecting all terms which will have

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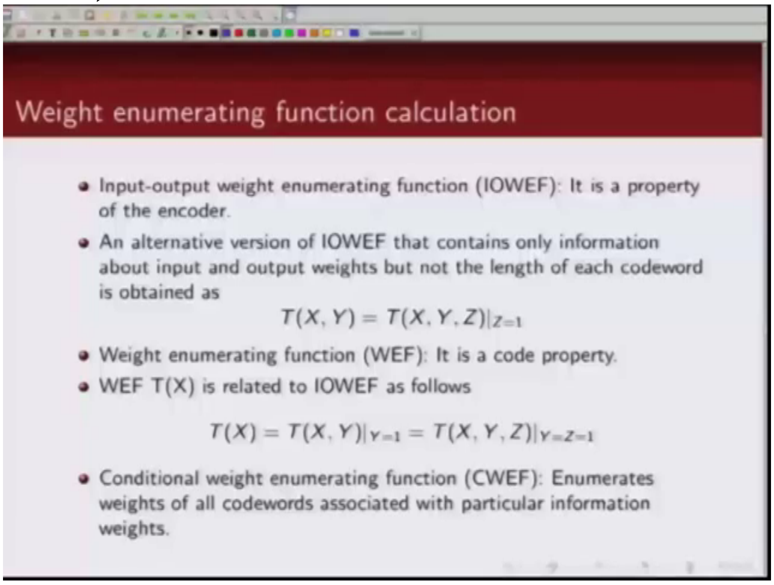

w 4, we can find what is the weight of all codewords corresponding to output weight, input weight of 4. And again we are using Y to denote input weight. So if you are interested in input weight 4, we should look for terms containing Y

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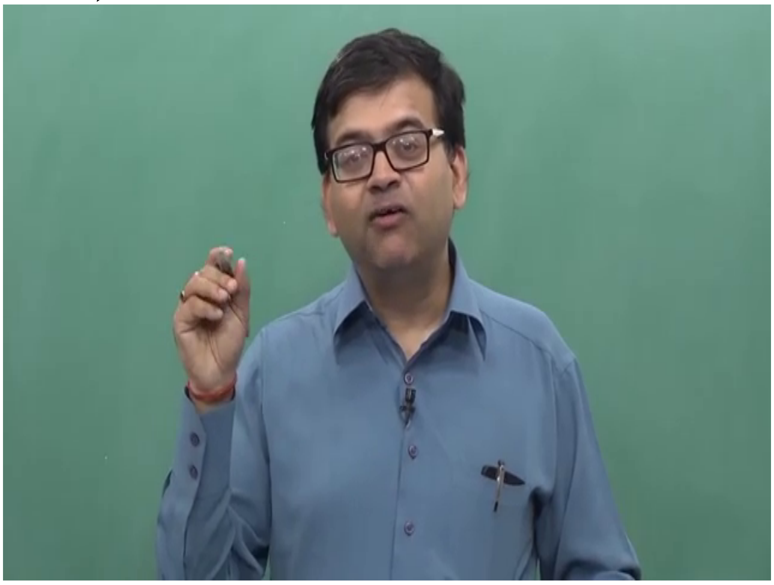

raised to power 4. So this denotes the

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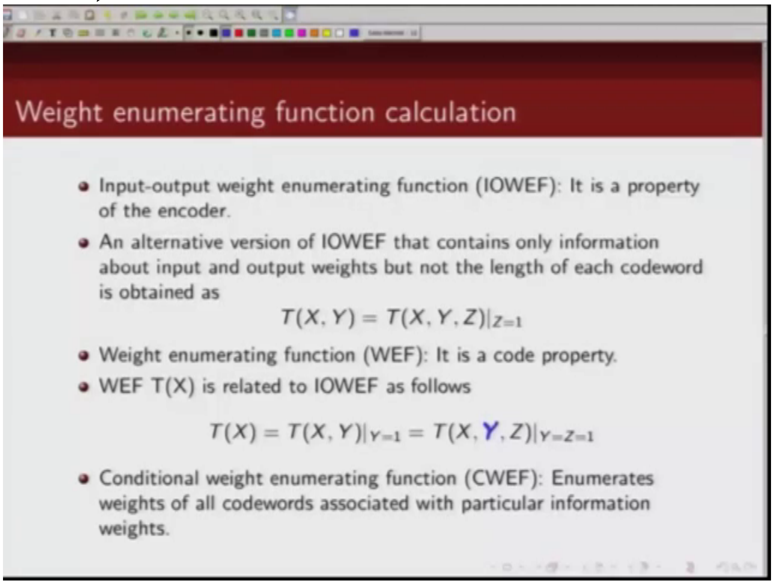

input weight, this denotes the output coded weight and this denotes the length. So as

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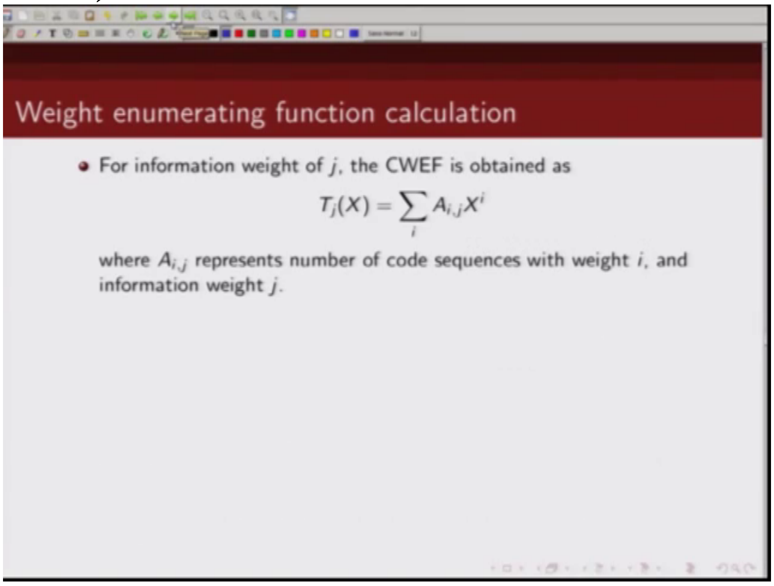

I said, for input weight of j conditional enumerating function will give us what is the output codeweight that you can achieve for a input weight of j. And we can write our

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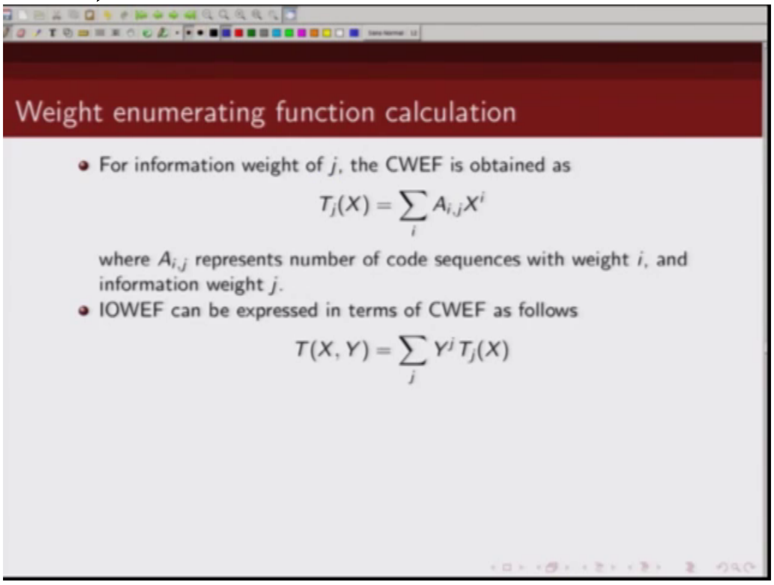

input output weight enumerating function in terms of weight enumerating function so this is basically input of weight j will result in conditional weight enumerating function and we show it for all js that will be our input output weight enumerating function.

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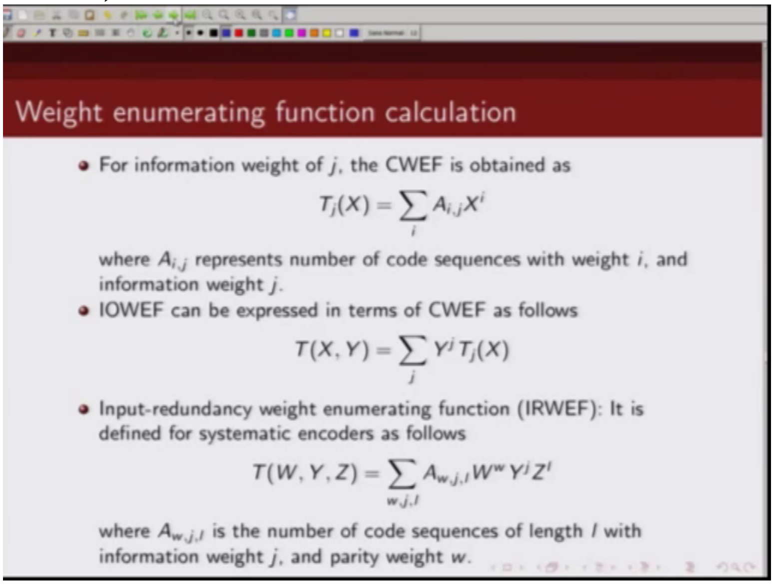

This another property which is defined for systematic encoders which is called input redundancy weight enumerating function. So here because the output weight

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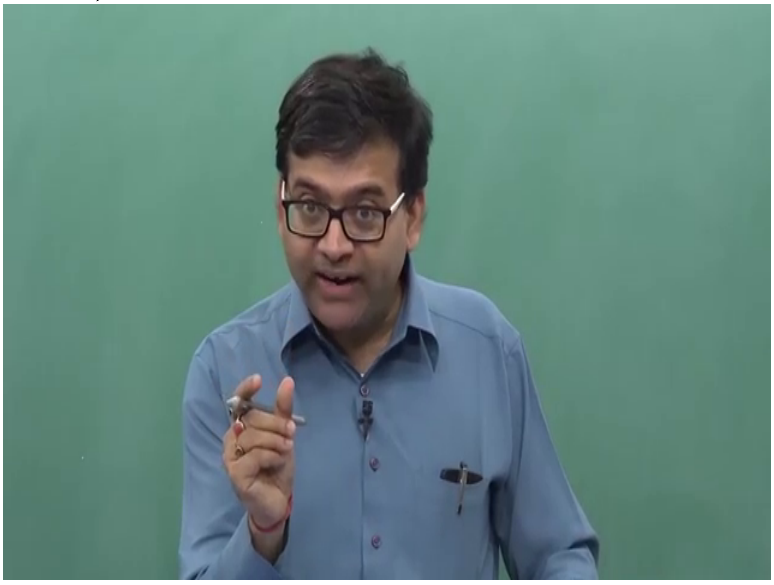

of a systematic encoder consists of weight of the information bits and weight of the parity bits. Now since

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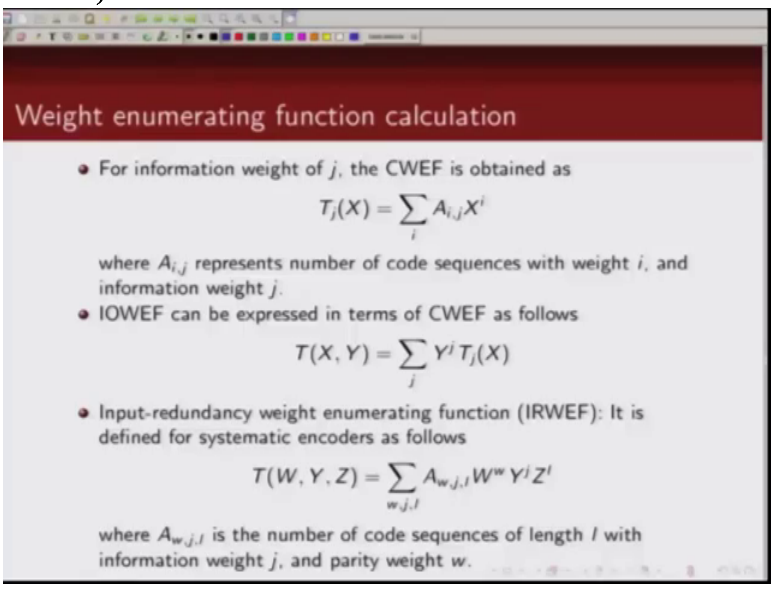

the power of Y already is denoting the weight of the information bits; so when you are asked to show the output weight, instead of saying the output weight you can just specify the weight of the parity bits because it is a systematic encoder, the overall weight will be weight of the parity bits and weight of the information bits. So overall weight would be w plus G. So input redundancy weight enumerating function is defined for systematic encoders. So where instead of specifying the overall coded bit, here you only specify the weight of the parity bits.

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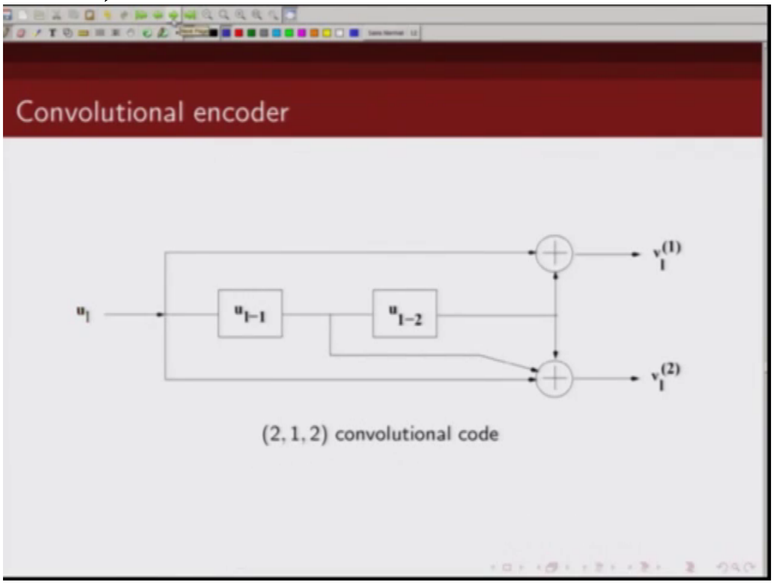

Now let's take an example to illustrate how we can find the

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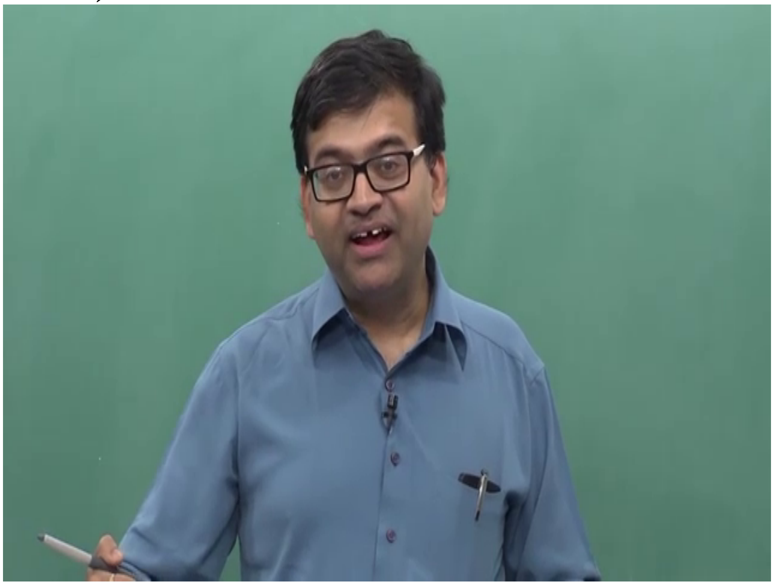

weight enumerating function of a convolutional code. So we are going to consider

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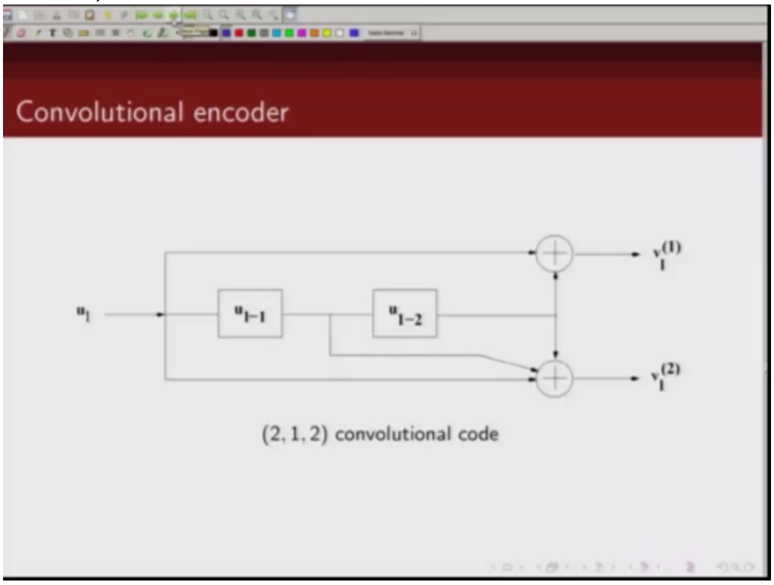

a rate 1 by 2 convolutional code whose memory is 2. So this is the convolutional code. We can see basically v l 1 is u l plus u l minus 2.

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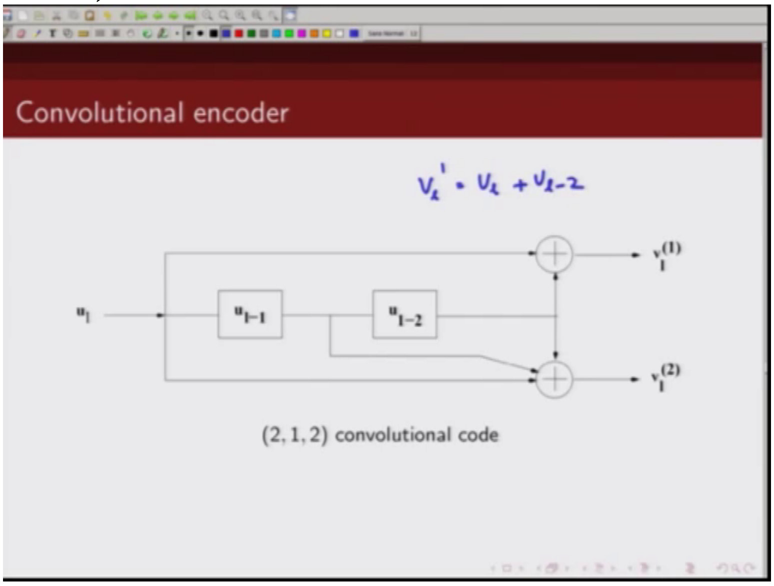

And v l 2 is nothing but u l plus u l minus 1 plus u l minus 2. This is our

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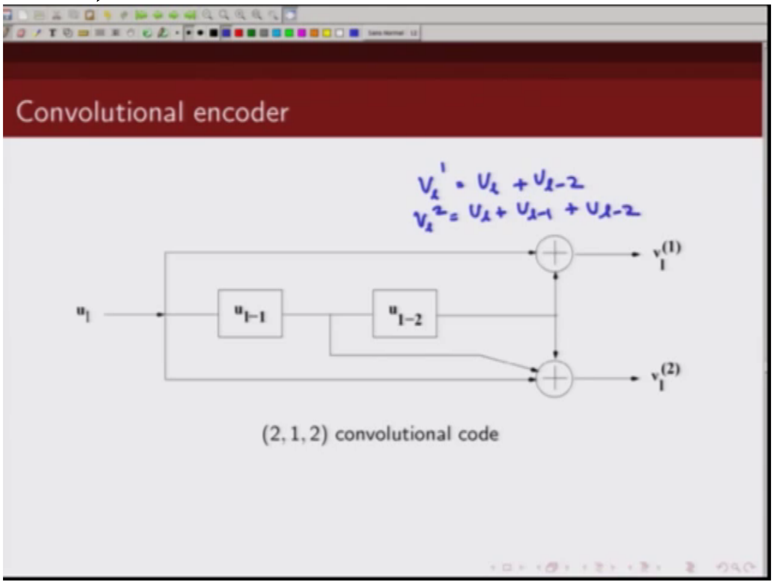

convolutional code. Now the state diagram

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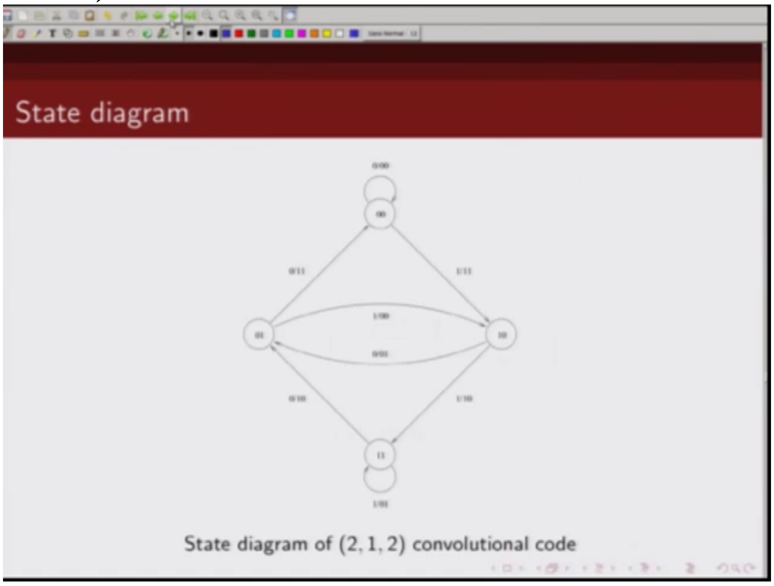

for this convolutional encoder is given by this. Now we are going to modify the state diagram for the purpose of calculating the weight enumerating function.

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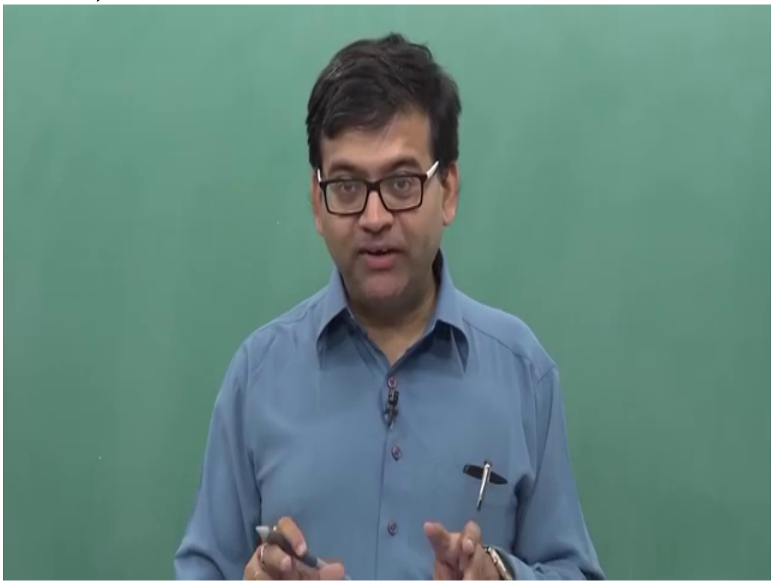

So recall what are the modifications that we have to do? We have to remove

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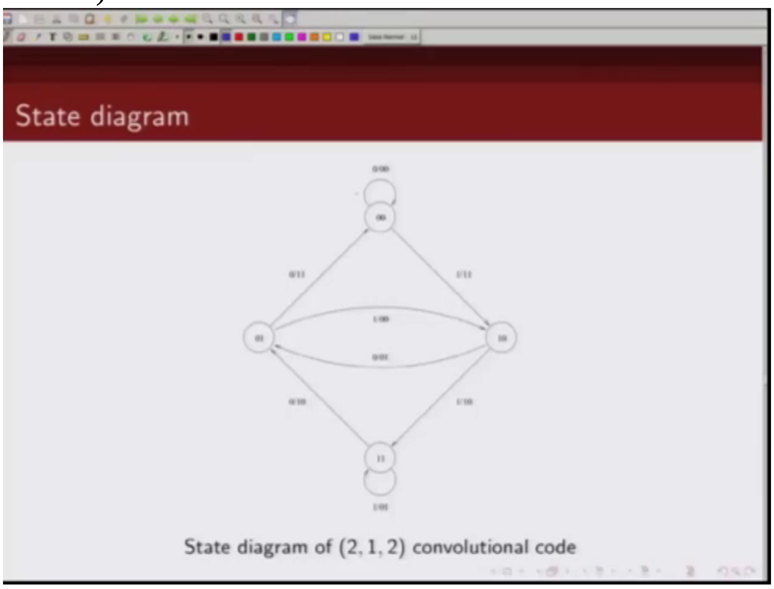

this self loop around all zero state and we have to split this all zero state into two state, initial state and final state. Next we have to label all these branches by weight of the output bit. So this will be x 2, this will be

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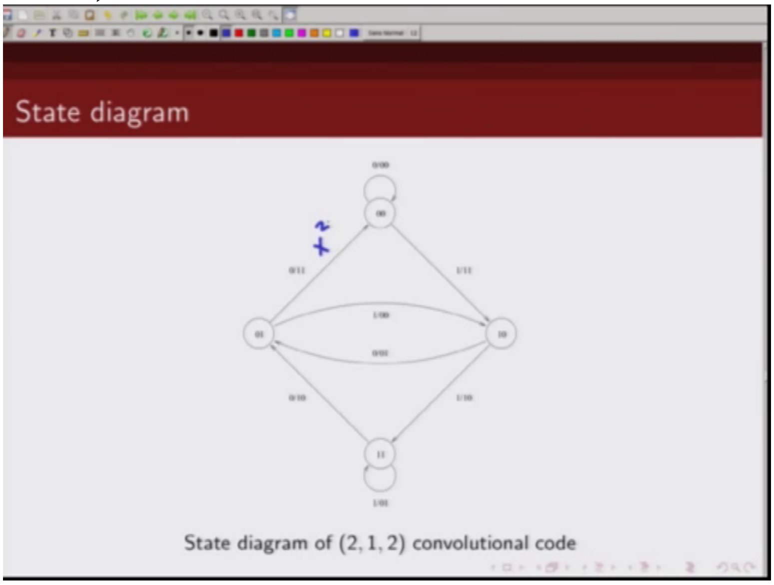

x 0 which is 1, this will be x, this will be x square. This will be x. This will be x. This will be x. If you

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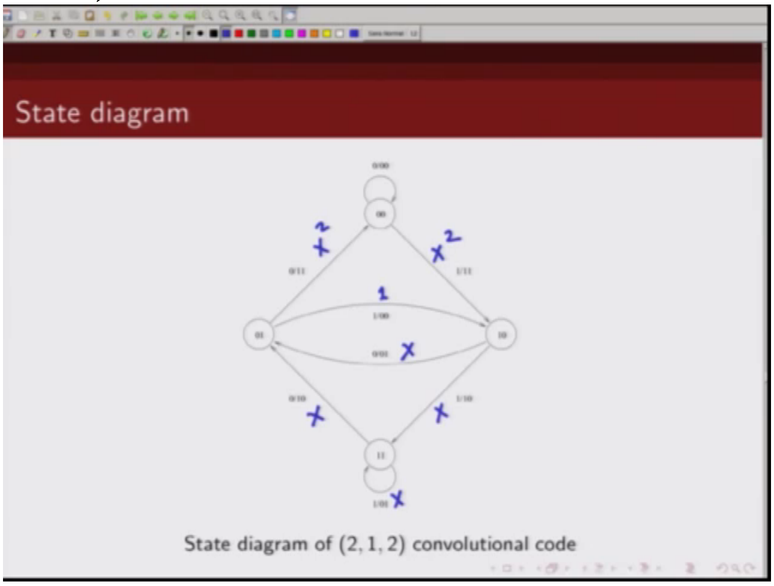

go back, this

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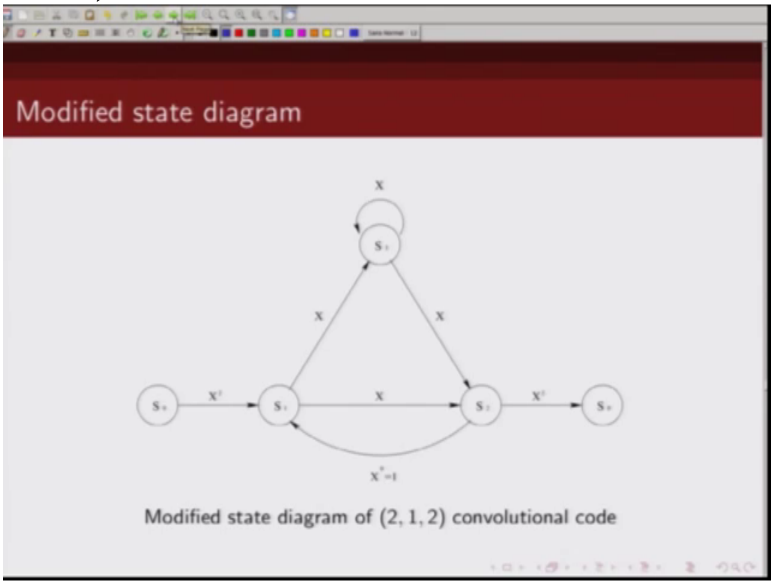

is how my augmented well modified state diagram will look like. So what I did was I have these states

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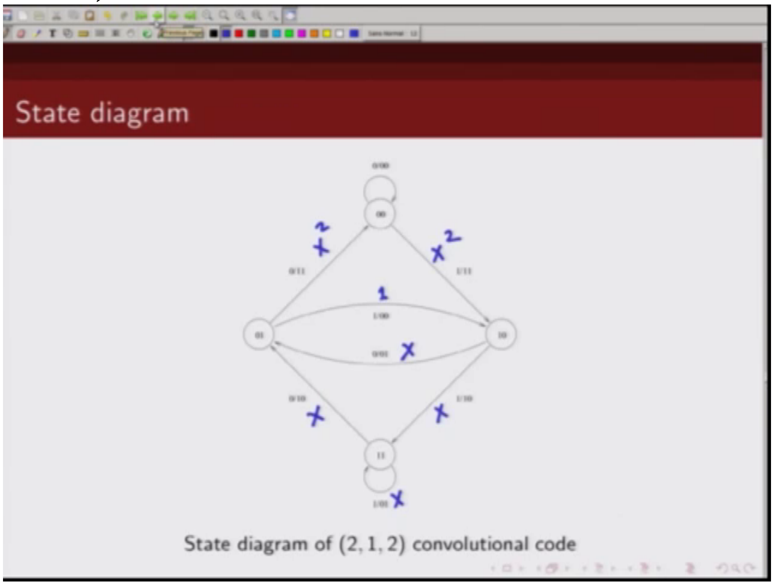

here. I split this all zero state into 2 states. So this state was split into initial

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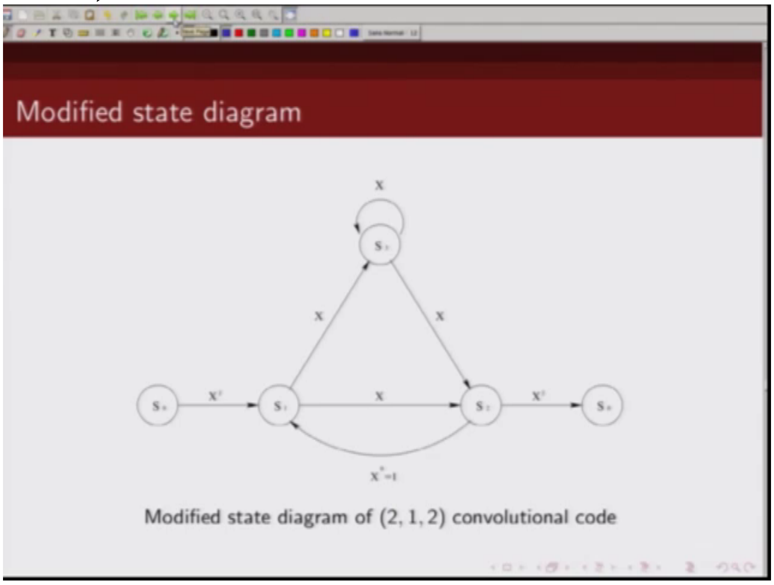

state and final state, Ok. Next what did I do?

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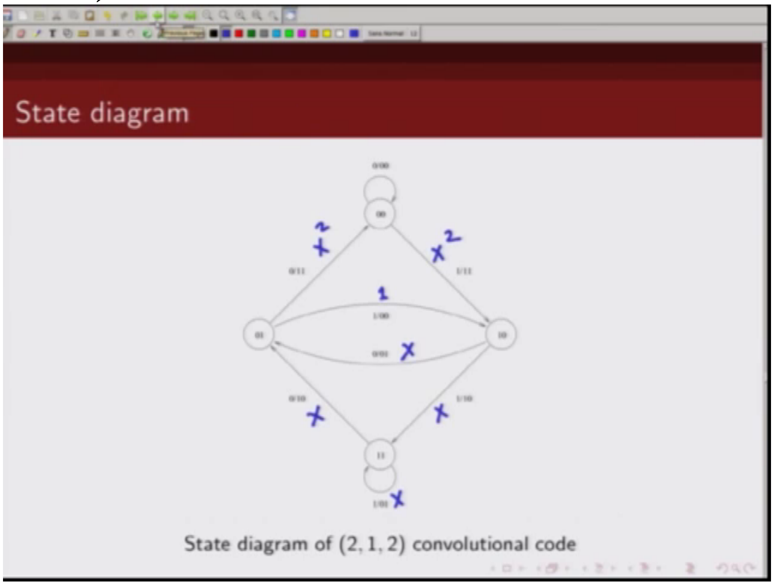

I redrew the same diagram but I labeled each transition by the weight of the output. So you can see from 0 0 I am going

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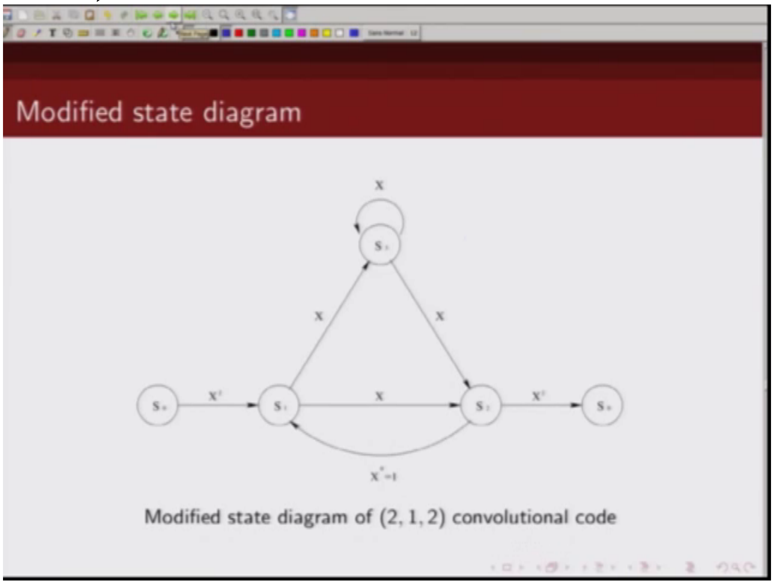

to 1 0 and its output weight is 1 1 which is x square. So let's go back here. From 0 0 I am going to this state and output is x square. This branch is labeled by x square.

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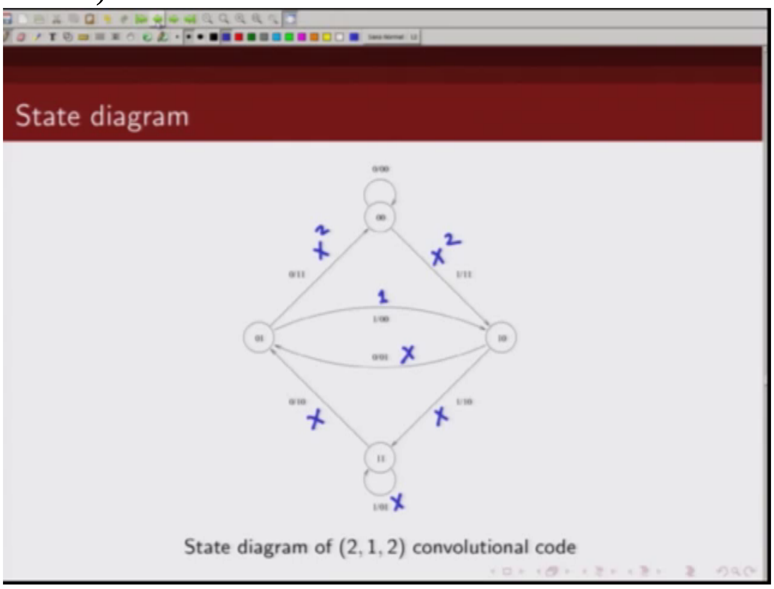

Similarly you can see from, say from this state you are going to this state and the weight is x. You can see from this I am going to

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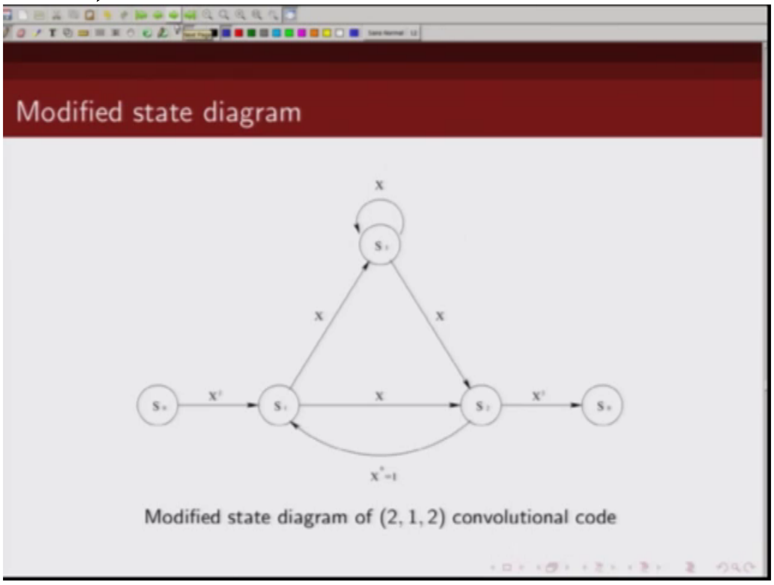

this state and the branch is labeled by x. From this you are

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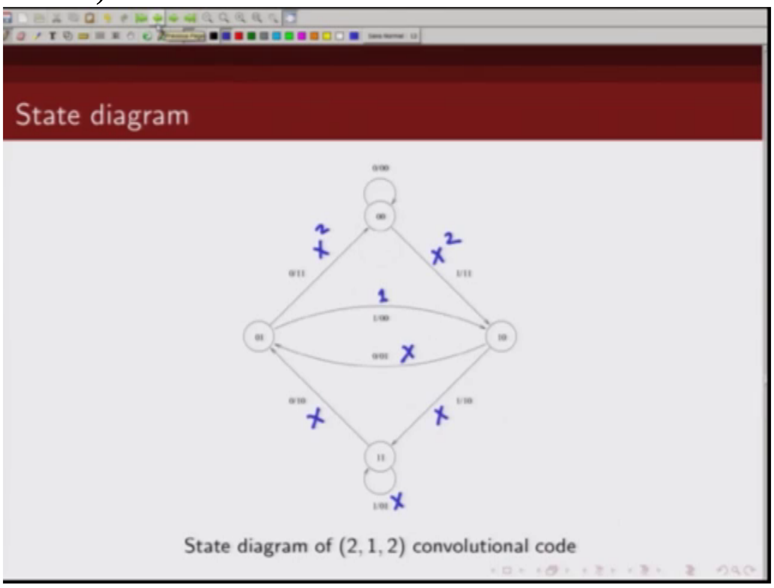

going to this 1 1 state and the branch is labeled by x.

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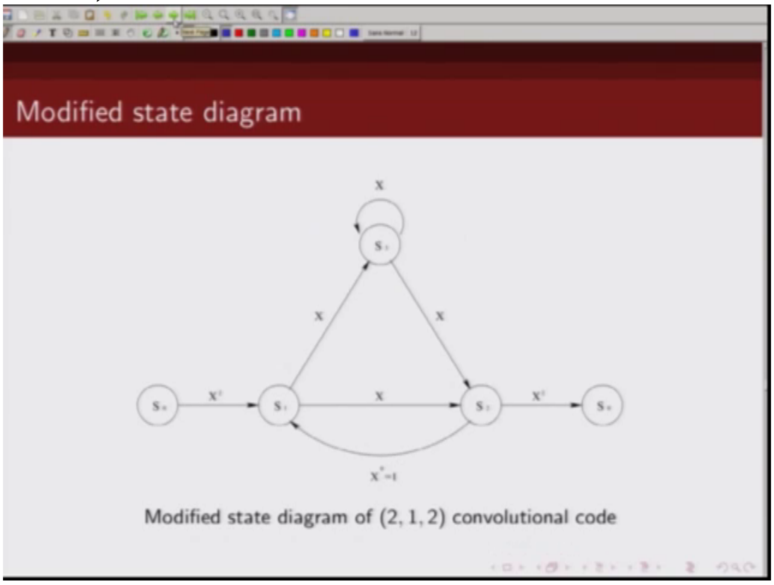

From this state you are going to this 1 1 state and the branch is labeled by x.

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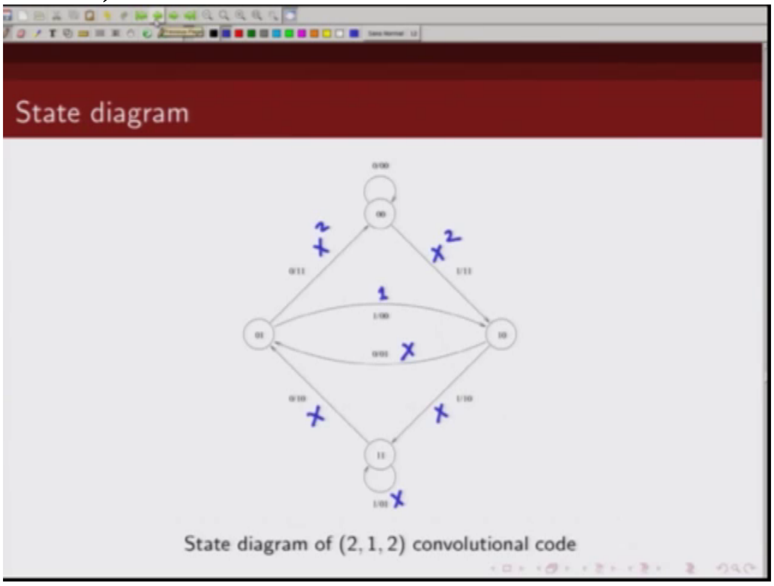

Around this state 1 1 there is a loop which has weight 1 x.

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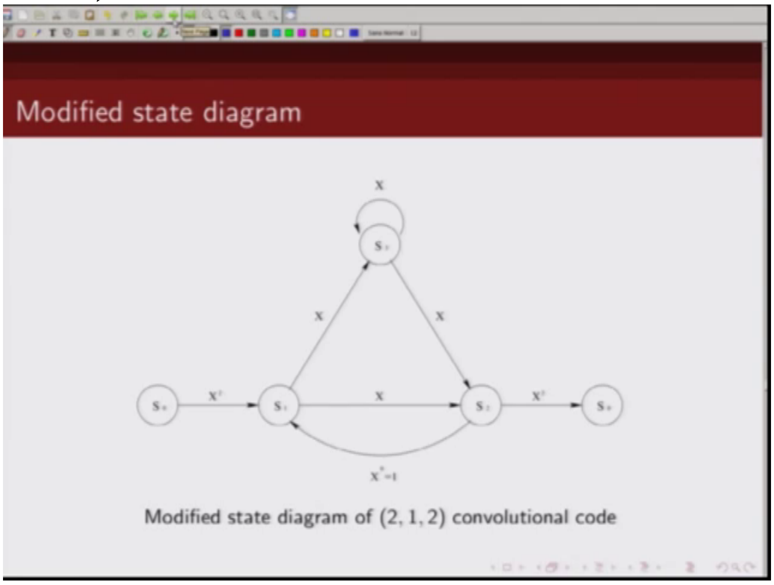

This is my loop around 1 1 which has weight x. So like that basically we modify the state diagram and this is how our modified state diagram

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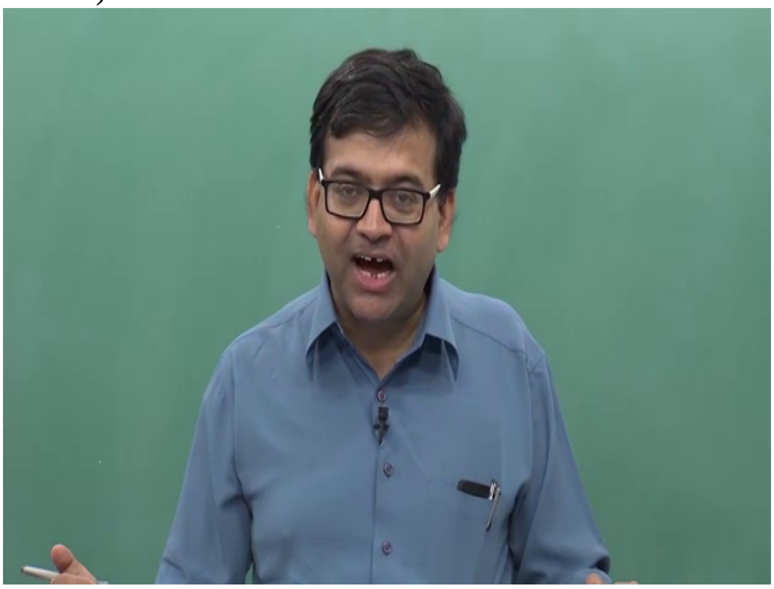

of rate 1 by 2 convolutional code that we just showed looks like, Ok.

Now the next step is

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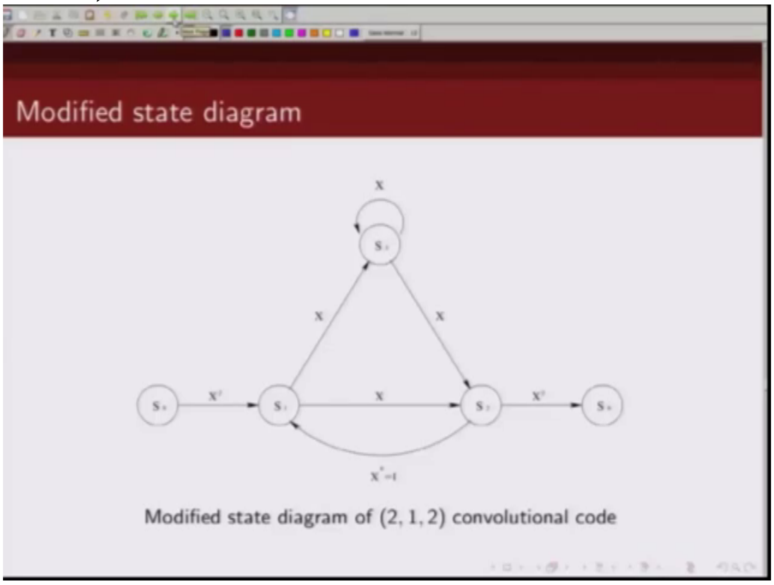

we need to find out what are the forward paths, what are the loops, what are the non touching loops and then we need to find out the

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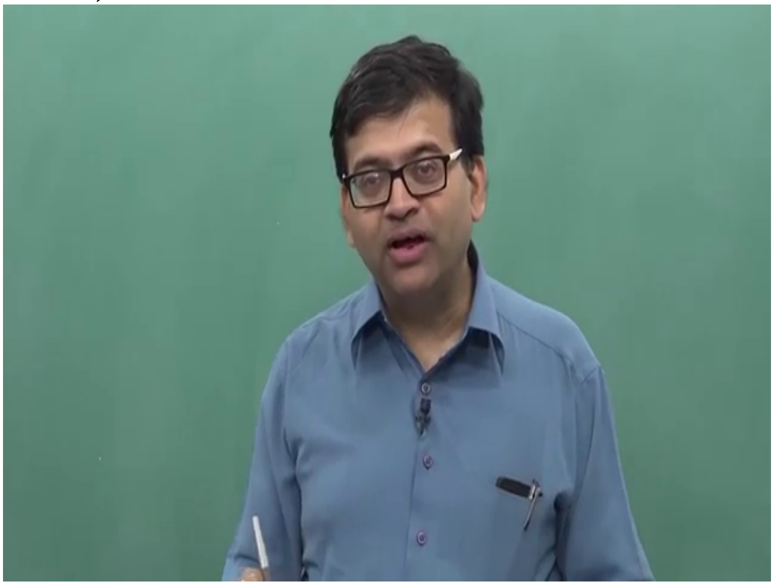

path gains along those forward paths, we need to find those deltas corresponding to these forward paths and then we need to apply Mason's gains formula to get the weight enumerating function.

So first

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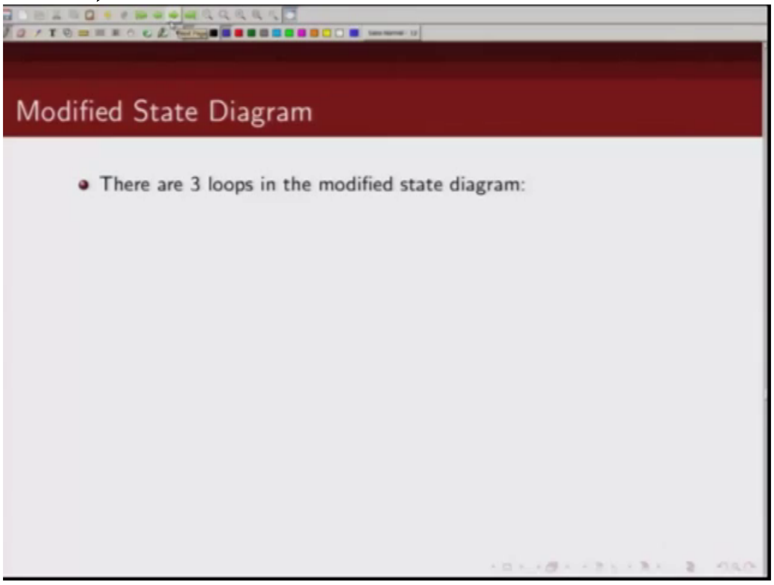

we find out what are the loops. So there are 3 loops in this. One is a

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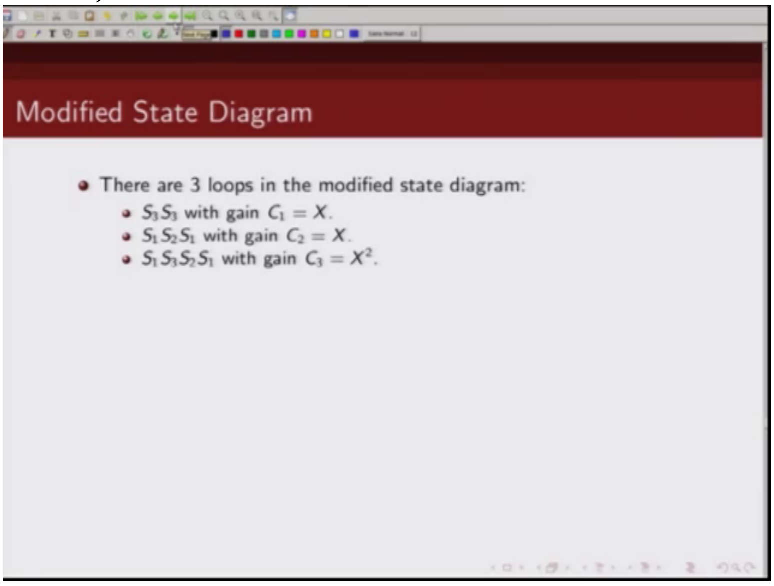

self loop around this state s 3, you can see. There is one

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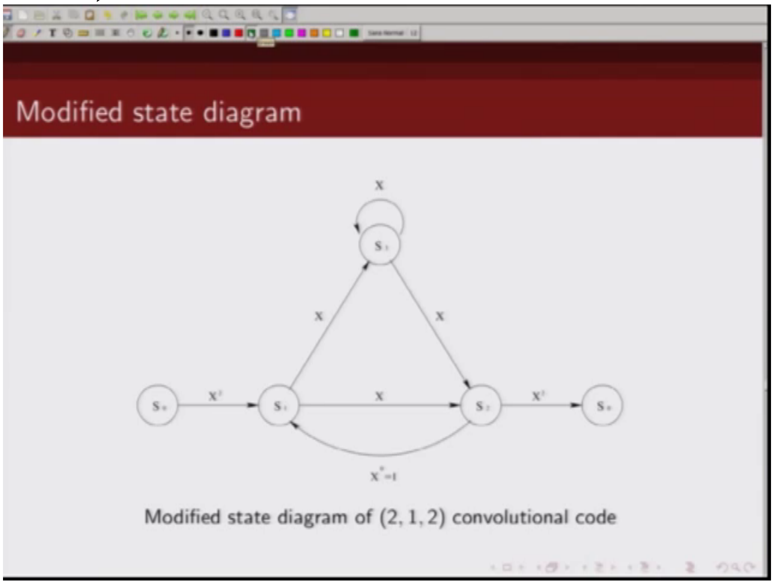

loop, right. There is another loop,

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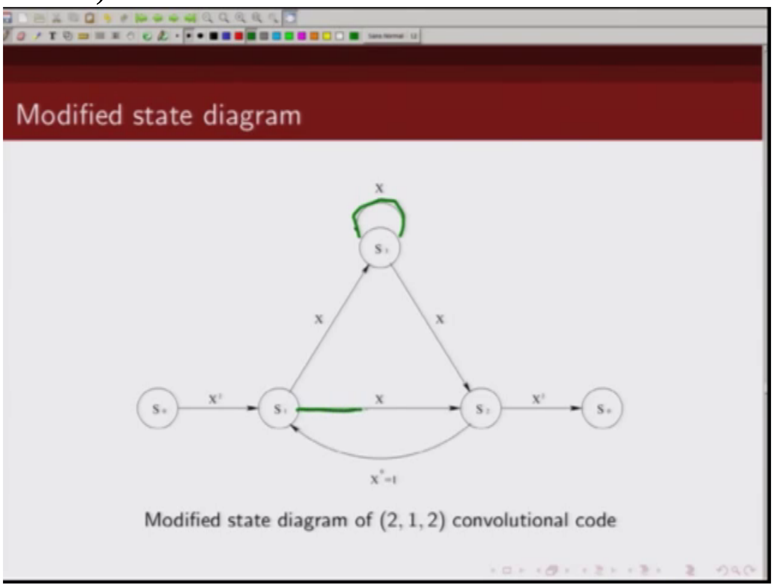

right

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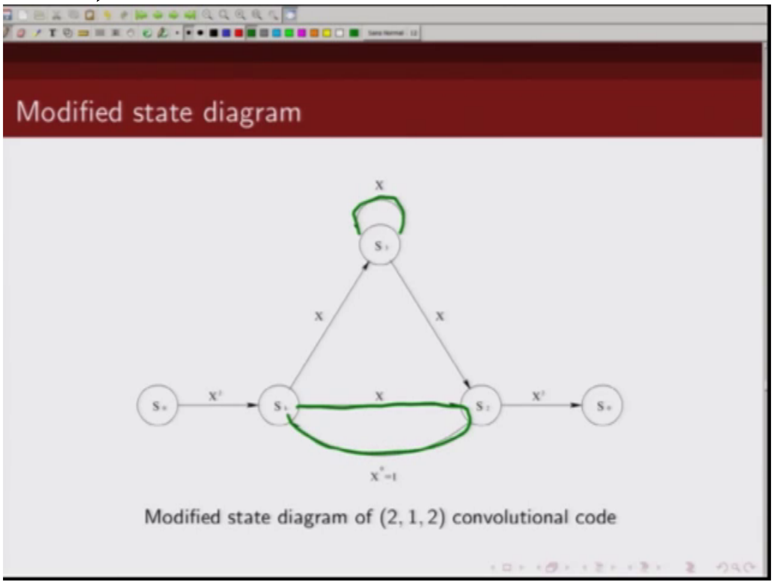

and then there is another loop. So there are 3 loops here. That's what I am denoting

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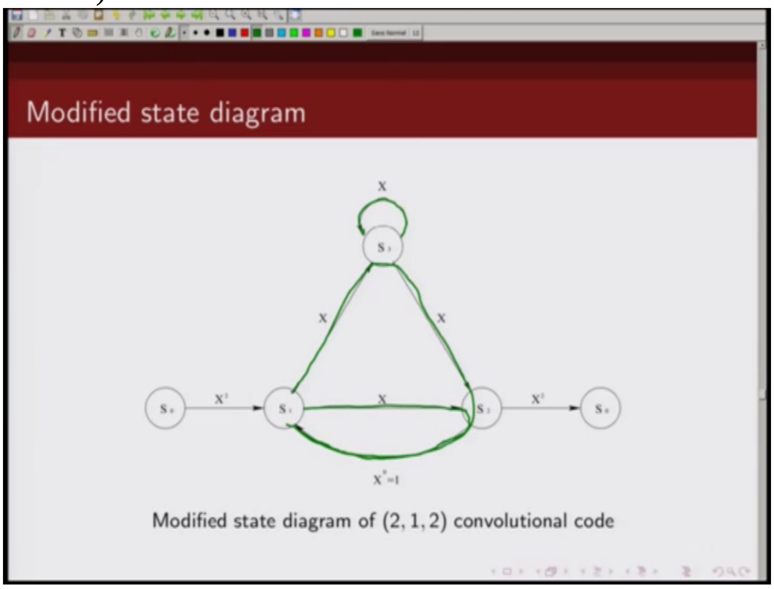

it by S 3, S 3, its gain is x, next one is S 1 S 2 S 1. S 1 S 2 S 1 is this one, Ok and the third loop is given by this. So these are

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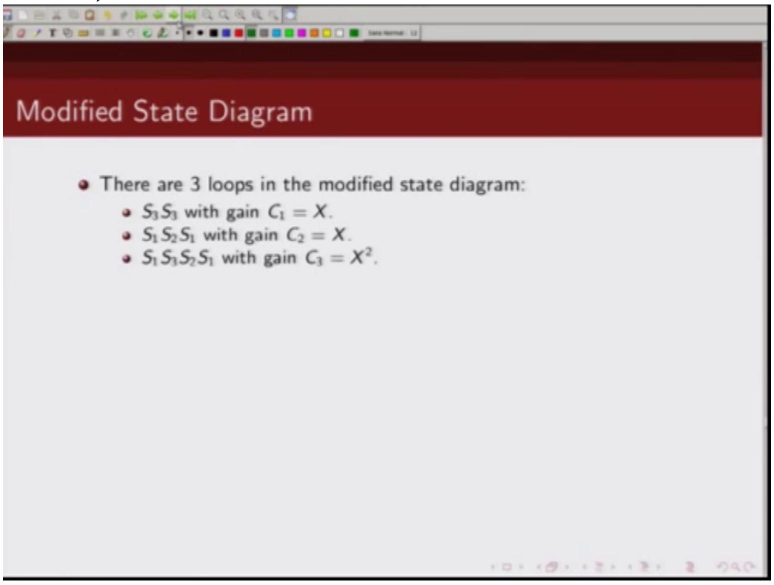

the three loops and corresponding to these 3 loops these are the gains. Next what are the pair of non touching loops? Now only

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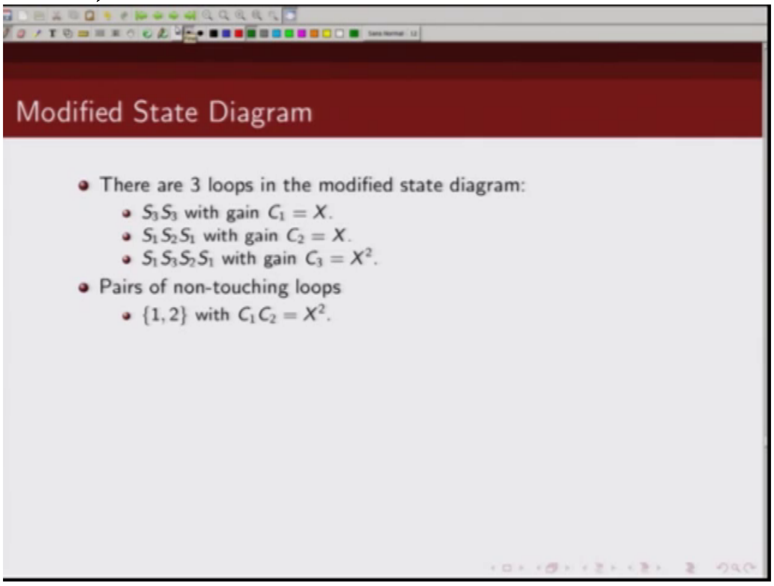

these 2, C 1 and C 2 are non touching loops. You can see, again go back to this example.

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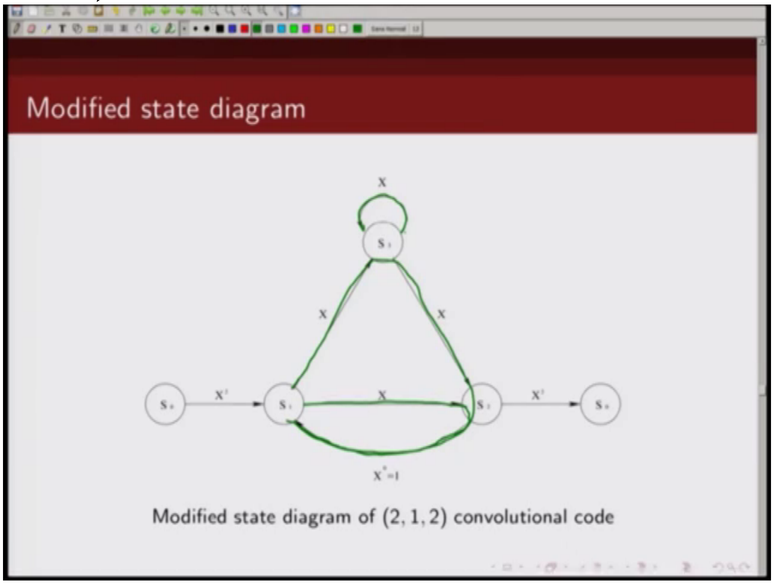

This loop and this loop are non-touching. Why? This loop contains s 3 and this loop contains s 1 and s 2, so they don't have any state common between these 2 loops. So the set of non touching loops is basically this C 1 and C 2 and the gain corresponding to them is basically

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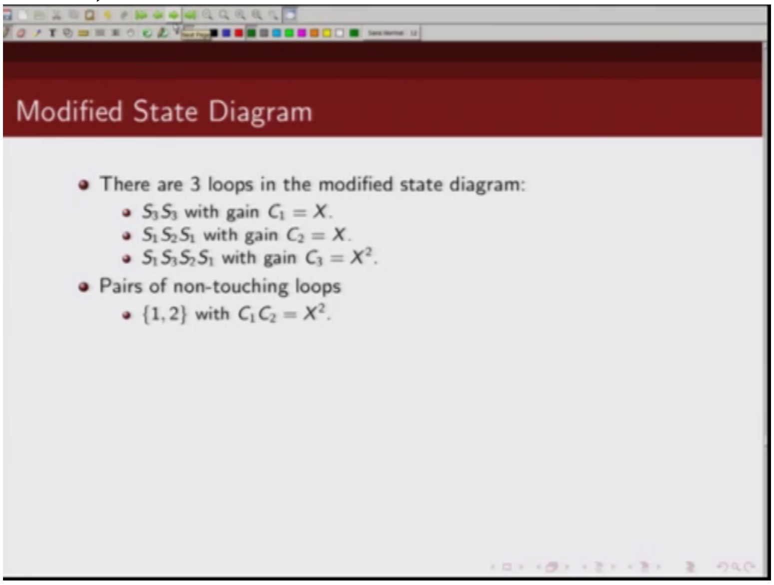

x square. And there is

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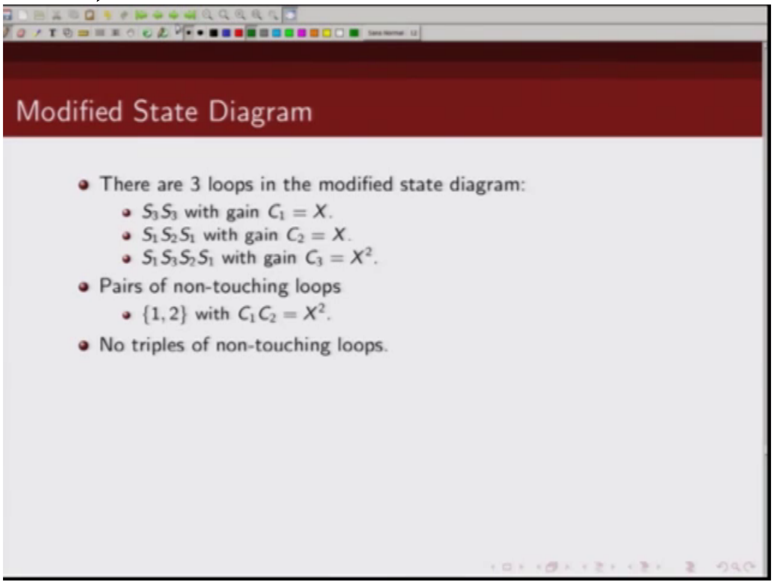

no set of 3 loops which are non touching. So now

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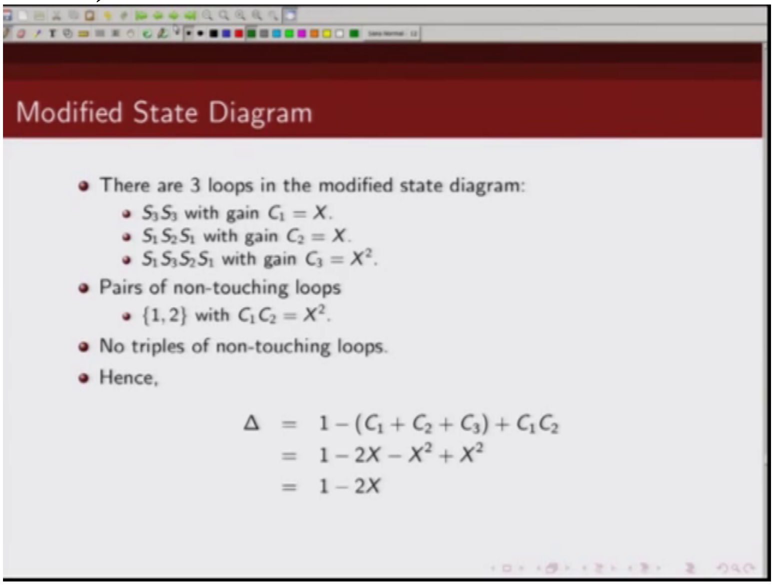

we can then find out the value of delta which is 1 minus summation of these loop gains and plus set of non touching loops so this comes out to be 1 minus 2 x.

Next

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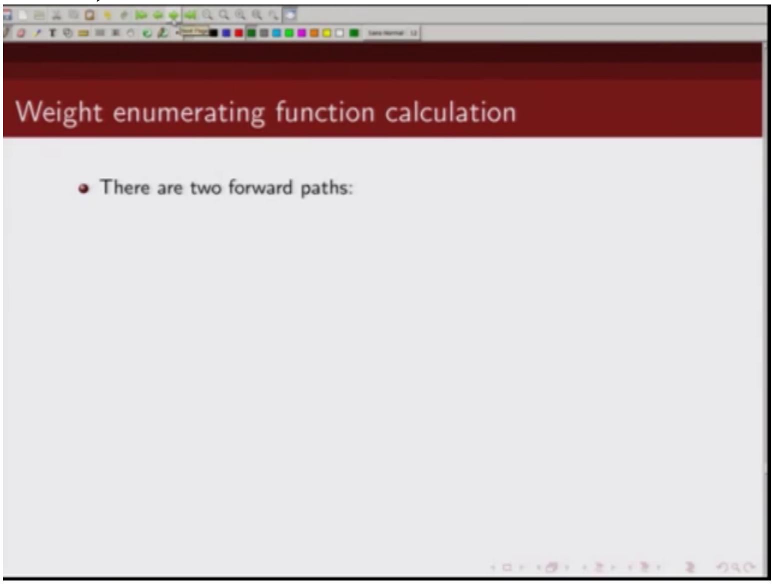

we are going to find out what are the

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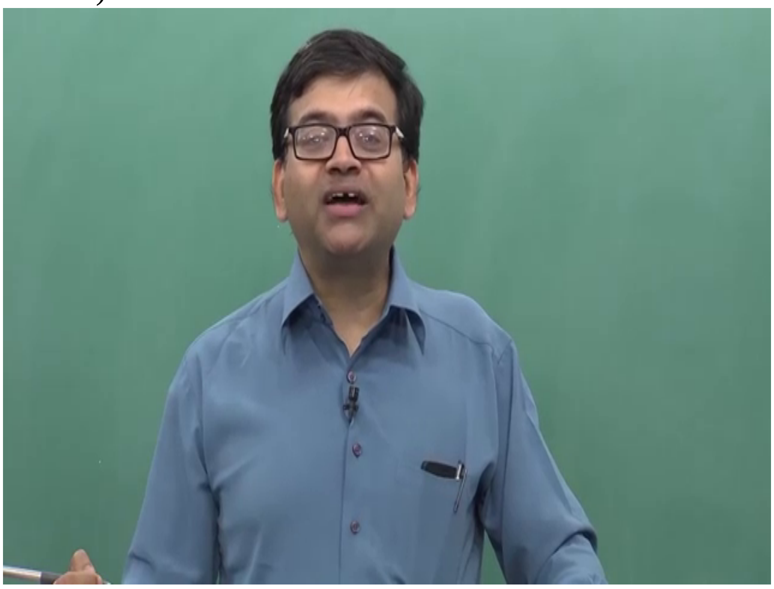

forward paths. So there are 2 forwards paths in

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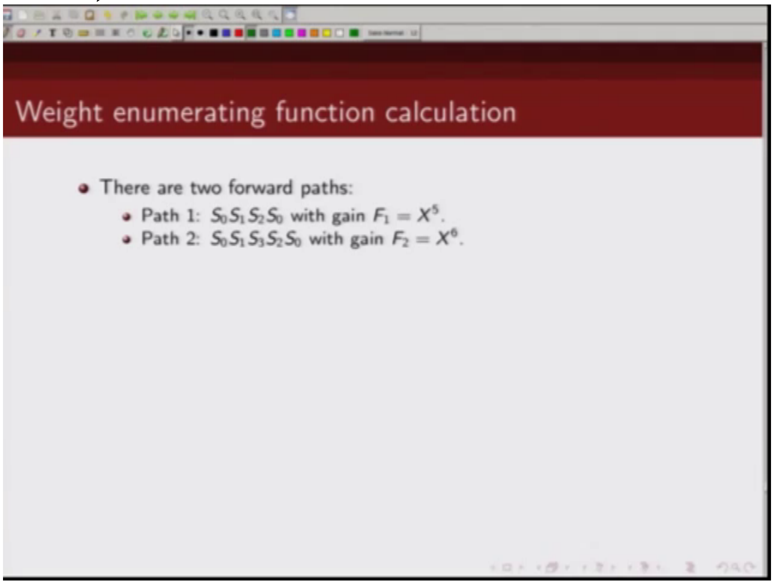

this and we are going to show you. So let's use a different

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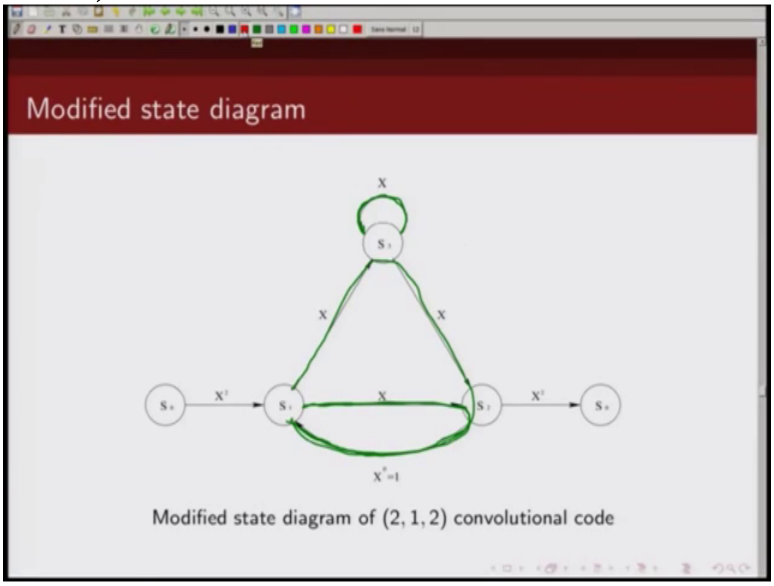

color pen. Let's use a red color pen. Remember what's a forward path? Path from the initial state to the final state without going over any state twice. So one forward path is this, fine. What about another

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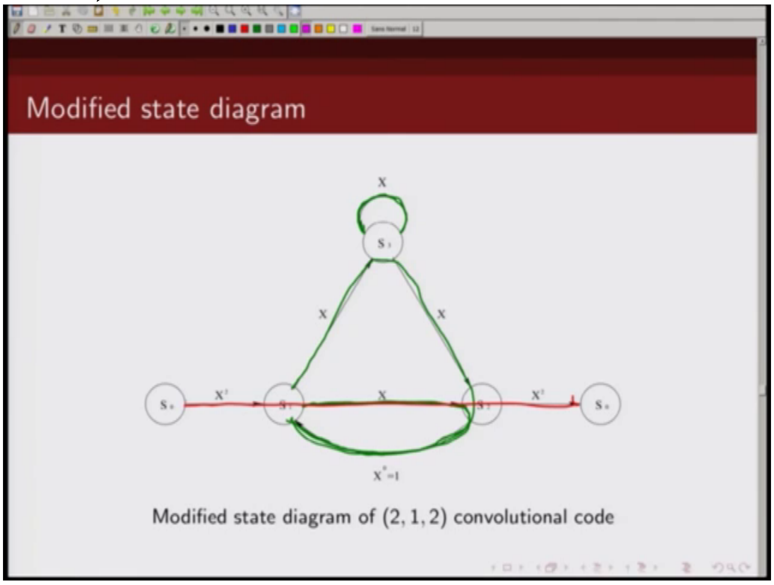

path, another forward path? The another forward path is this.

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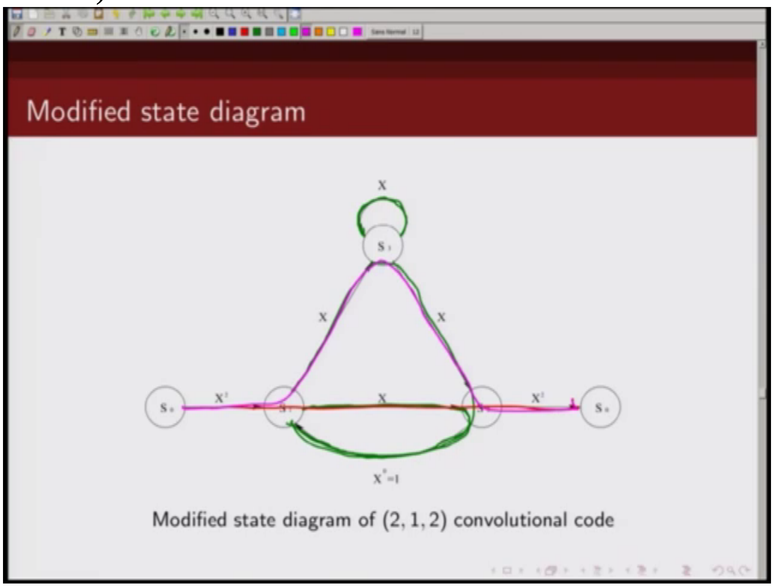

Both the cases you can see I am not going over any state twice. And there are only 2 forward paths in this case. And what are their corresponding path gain? For the one which I marked with red, it is x square x and x square. So this will be x raised to power 5. And this will be x square x x x square. So this will be x raised to power 6. So then

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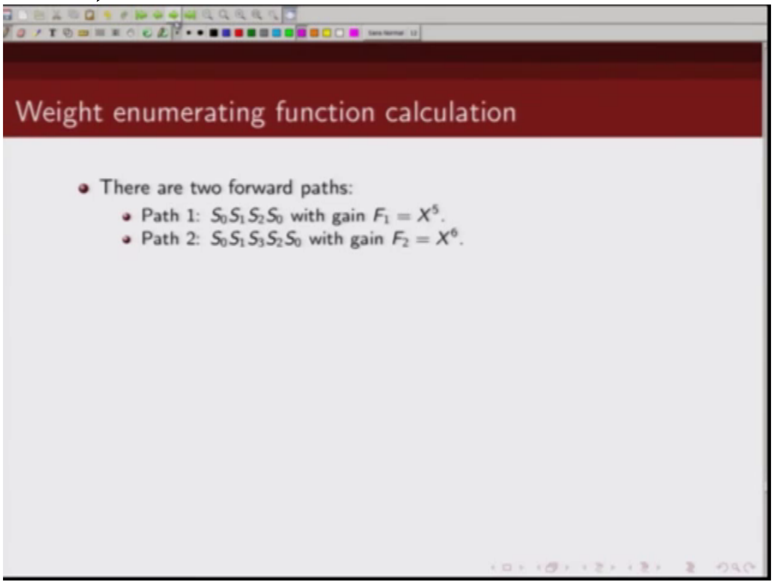

we have 2 forward paths, the one with gain x 5, another one with gain x 6.

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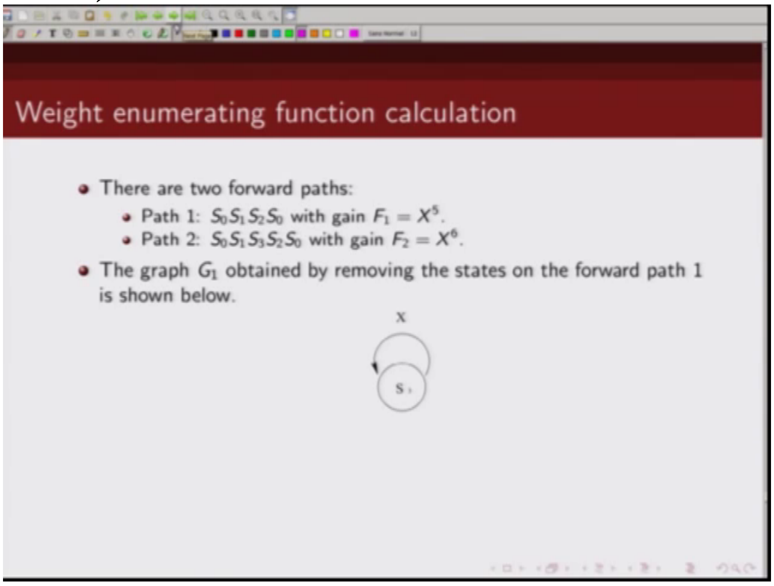

Now what's the next step? We need to remove the forward path and see what is the graph remaining

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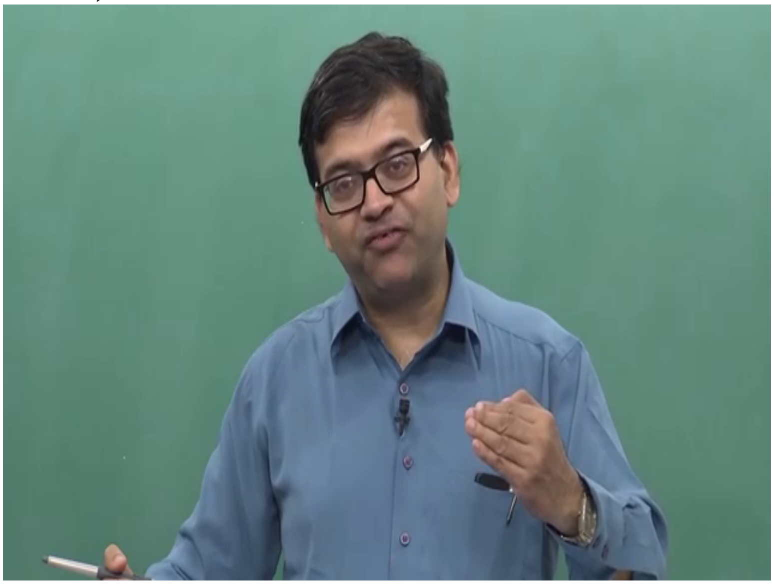

and we need to compute the delta corresponding to that. Now again let's go back

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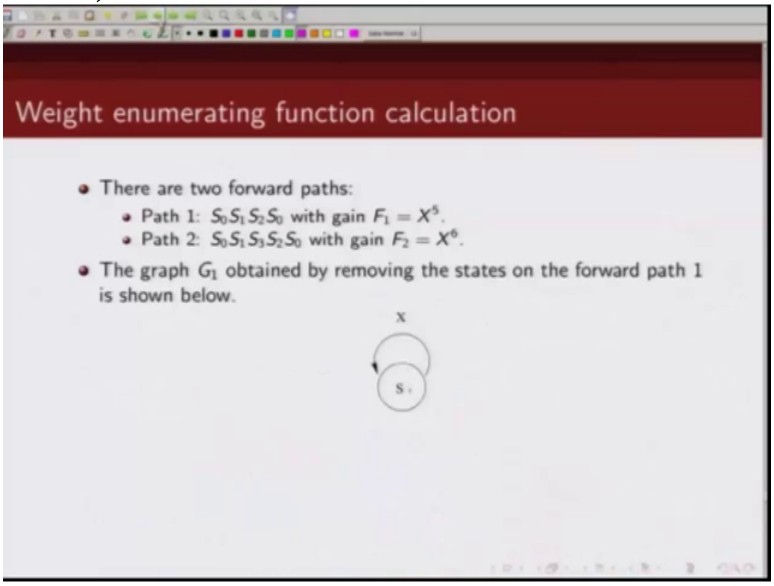

to the same diagram. If I remove this

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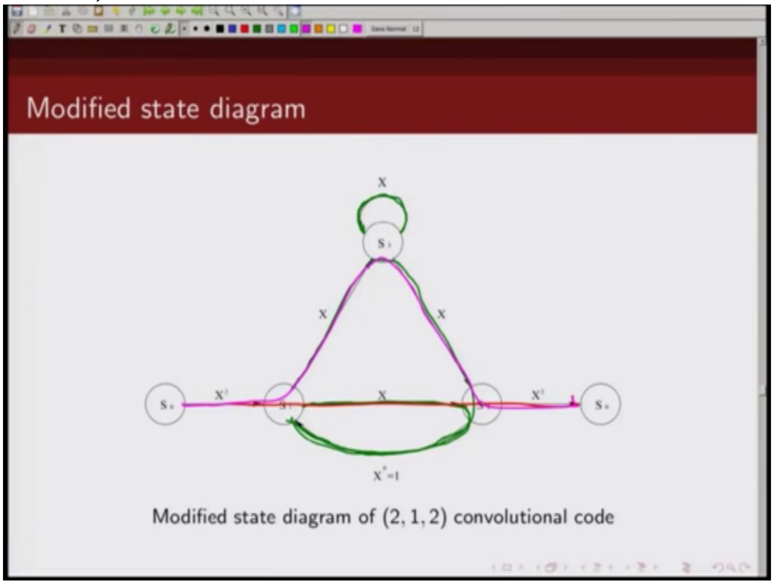

forward path, what is left in the graph? Only this, this node. Only this is remaining. And what if I remove this forward path? If I remove this forward path, everything is gone, nothing left in the graph. So that's what I am saying here. If I remove the

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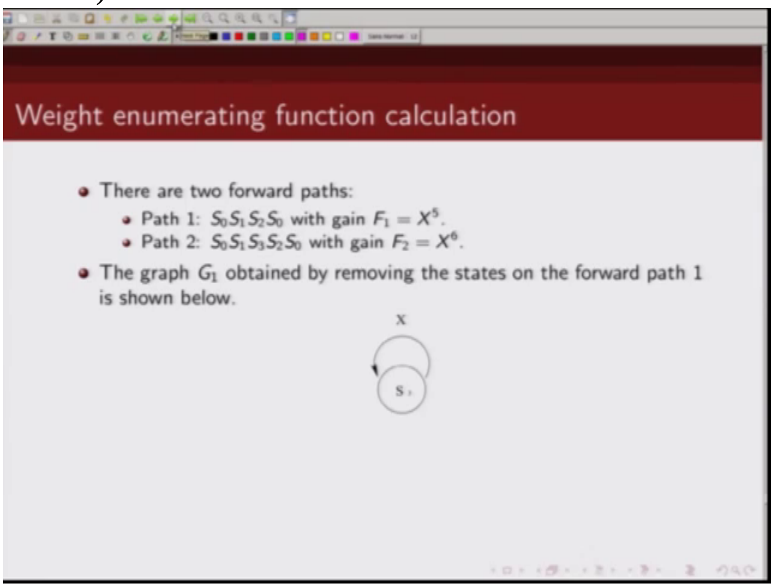

forward path 1, the only graph remaining is this. And the delta

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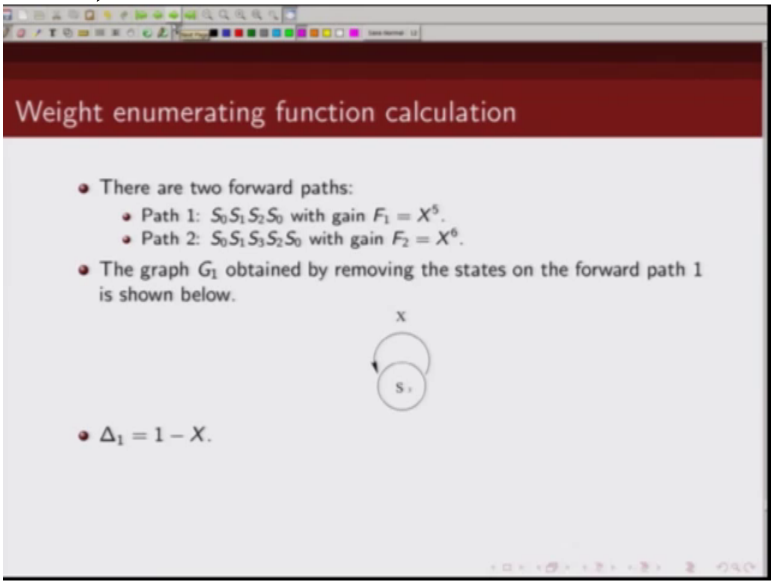

corresponding to this is only one loop with gain x so this is 1 minus x. And for the second case, there is no graph left so delta 2 will be

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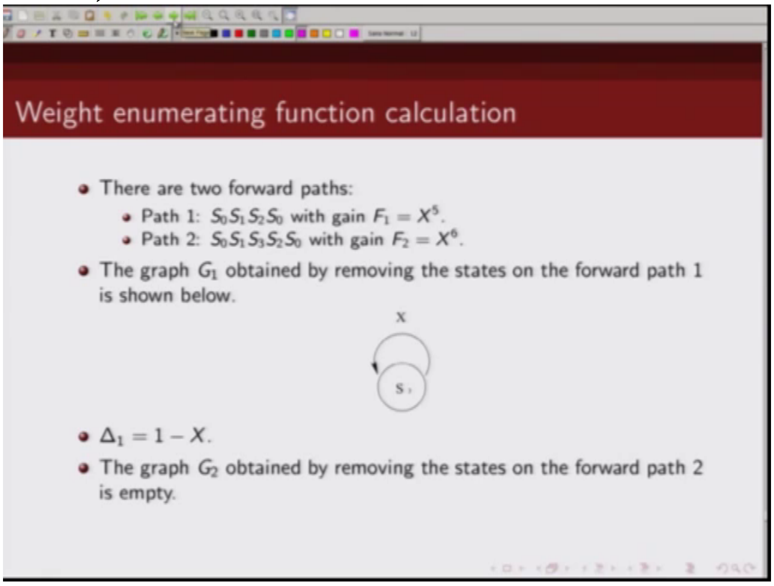

1, Ok. So now I have f 1 delta 1, f 2 delta 2 and I also have the value of delta. So I can then apply Mason's gain formula to

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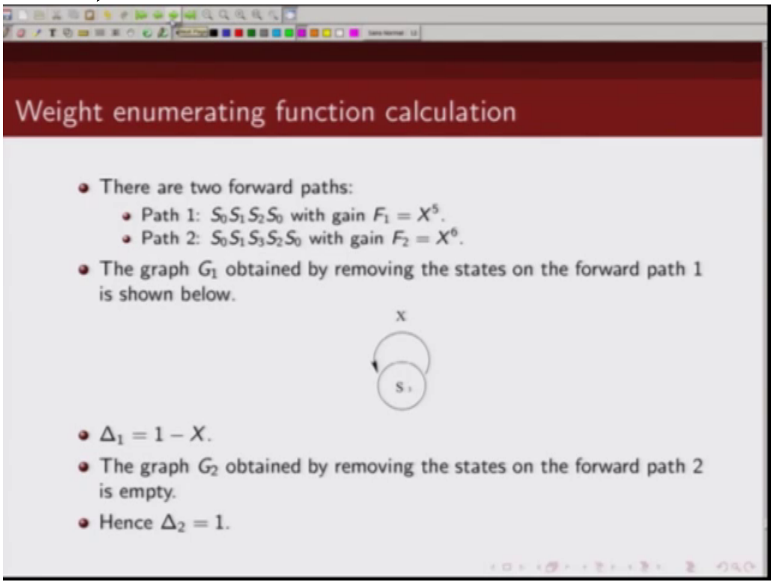

get the weight

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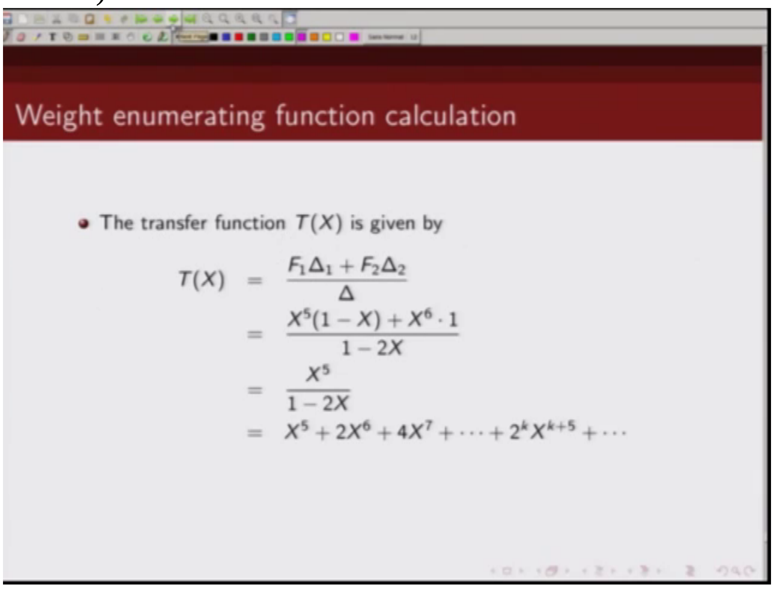

enumerating function. So the weight enumerating function is given by this expression. So I plug in the value of f 1 delta 1, f 2 delta 2 and delta and what I get is this expression which I can write like this. So you can see basically my output consists of 1 codeword of weight 5, 2 codewords of weight 6, 4 codewords of weight 7. So you can see this transfer function is completely enumerating the weight distribution of my convolutional code. The same thing I can do with augmented

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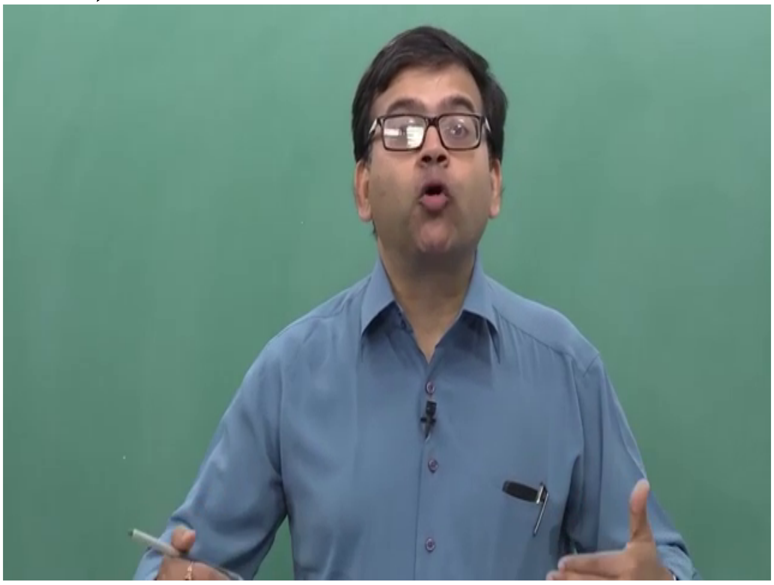

transfer function.

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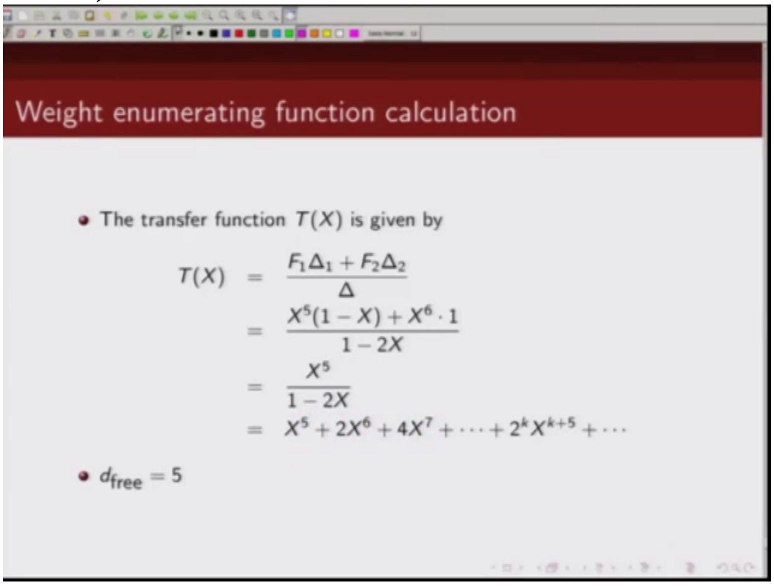

And again because the minimum weight is 5, so free distance of this convolutional code is 5. Now

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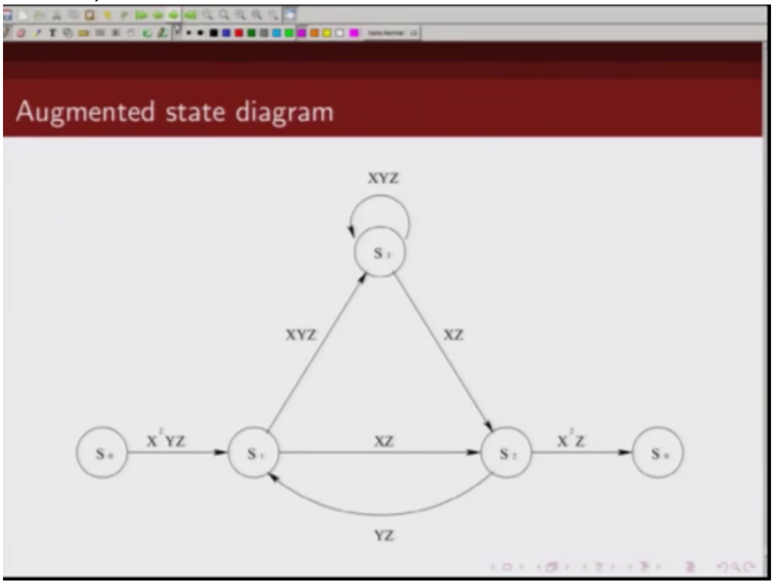

we repeat the same exercise with the augmented state diagram. Now what was augmented state diagram? Each valid branch we added a z to denote this you can reach from one state to another in one step and we also added in each of these branches the weight corresponding

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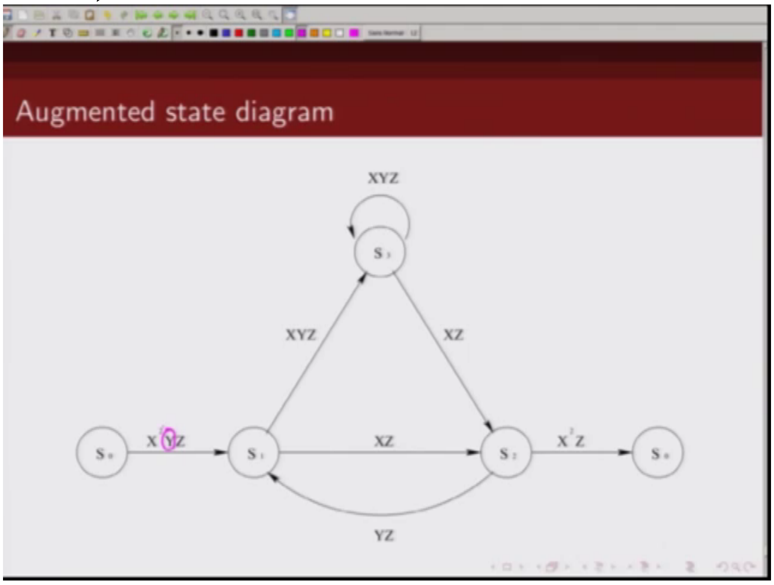

to the information bits. So the information bit weight was 0 so y 0, so that was 1, so you can see in some cases the information sequence weight is 0. So let's just go back to the original state diagram,

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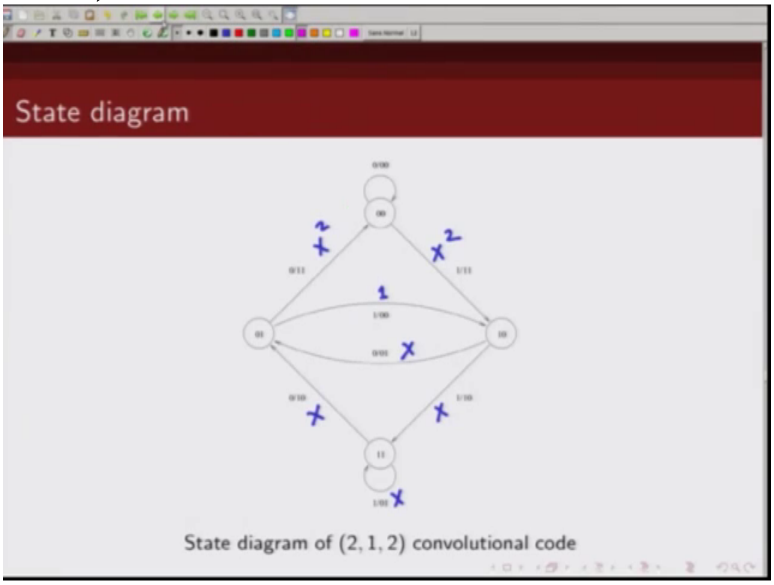

yeah. Let's go back to this. So you can see for this transition from 0 1 to 0 0 what is the weight of the information sequence, that is 0. So y 0 that is basically 1. What about this, the weight of information sequence? Here the input is 1, so this will be y.

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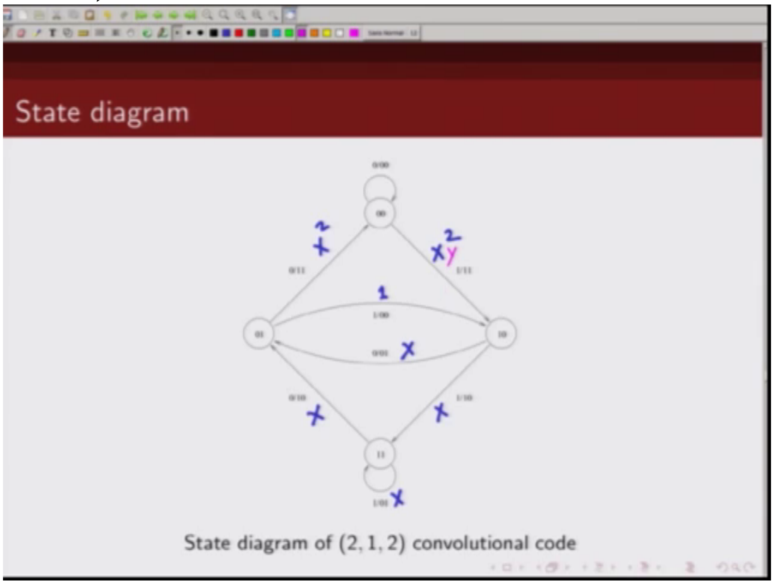

What is the weight of information sequence? That's 1 so it will be y. This weight information

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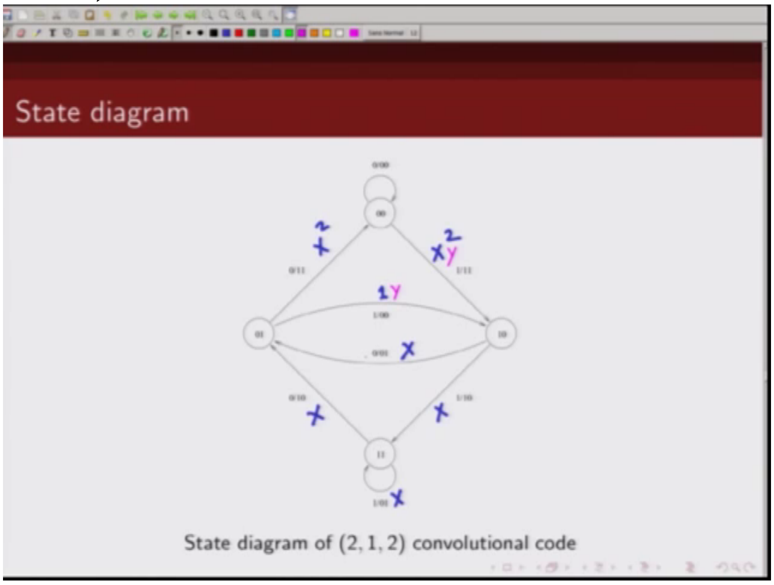

sequence is 0 so y 0 is 1. So wherever you had 1 here you are adding basically y.

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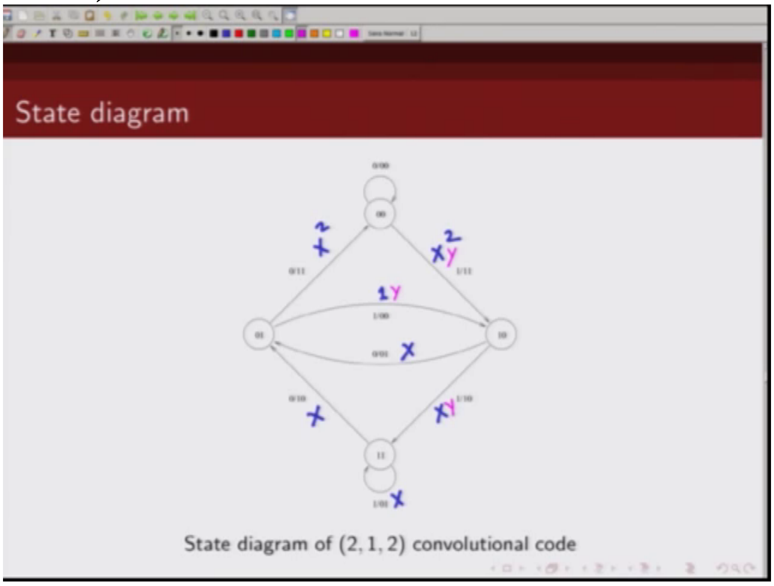

This is 1 and similarly, each of these

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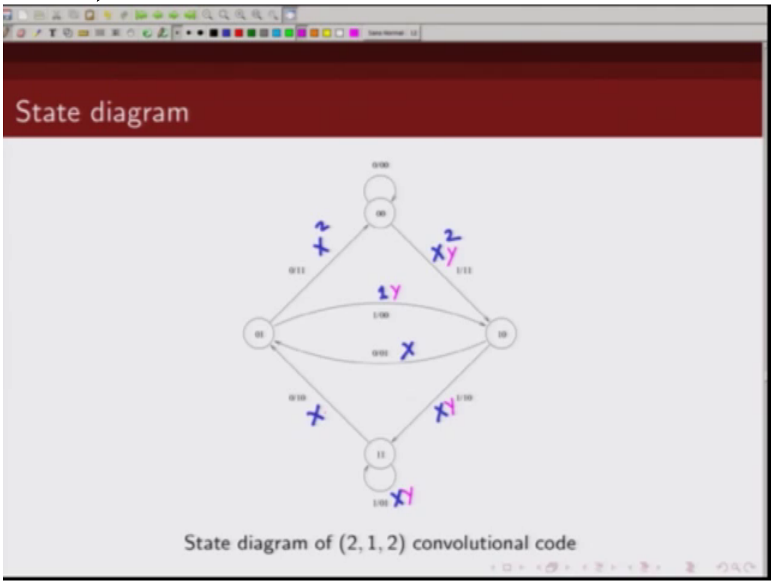

transitions there will be a z added

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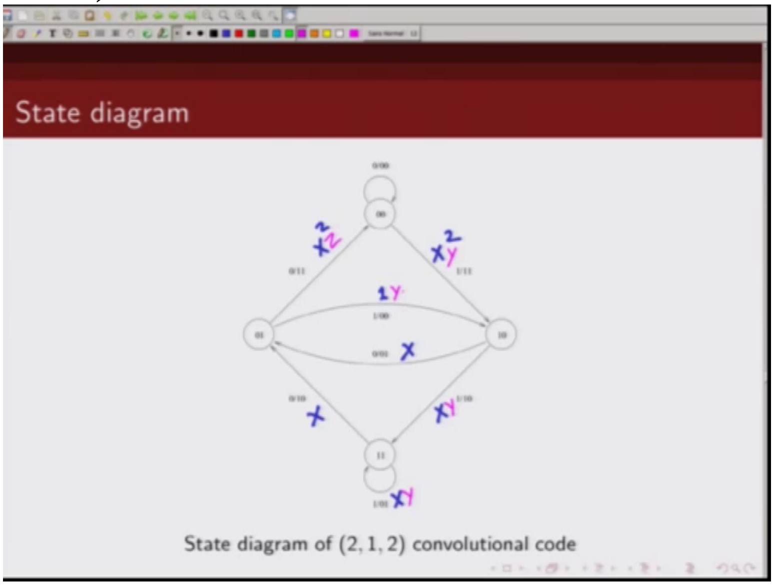

to denote the length

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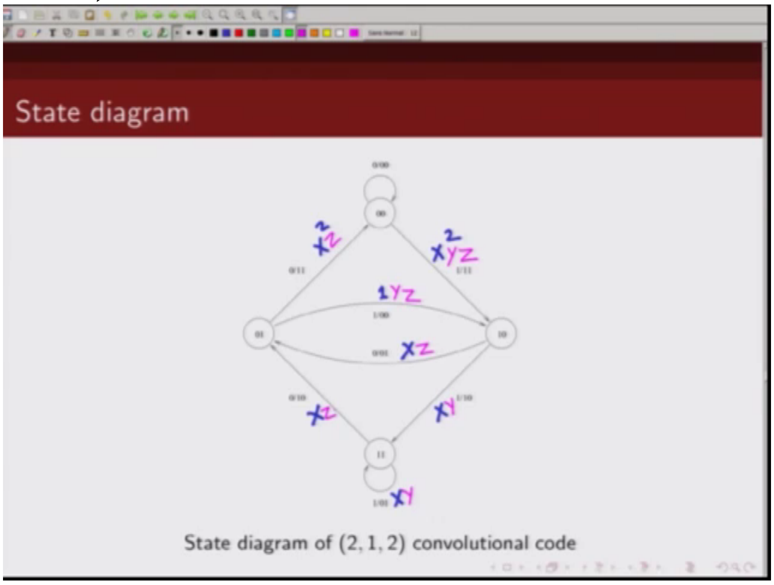

Ok. So that's your augmented state diagram. And that's what; I mean the completed augmented state diagram is what I am showing you here. This

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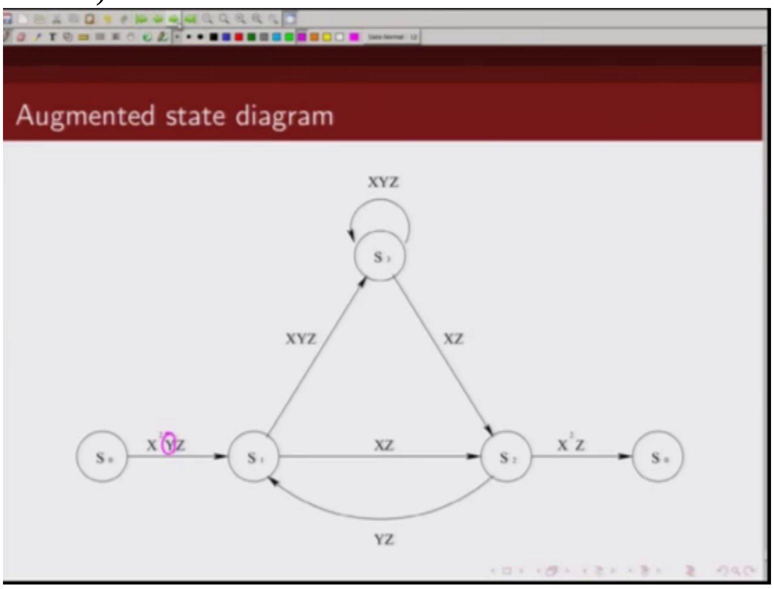

is basically my augmented state diagram where I am not only specifying the coded weight but I am also specifying what input causes that output bit and z to denote the length and I follow the same procedure

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using Mason's gain formula to compute

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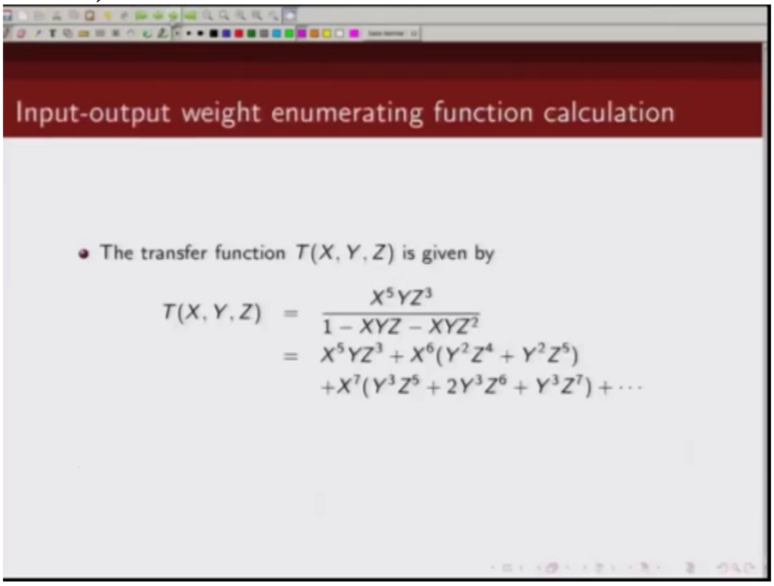

the weight enumerating function. So I get this information. I am skipping the steps. It is exactly the

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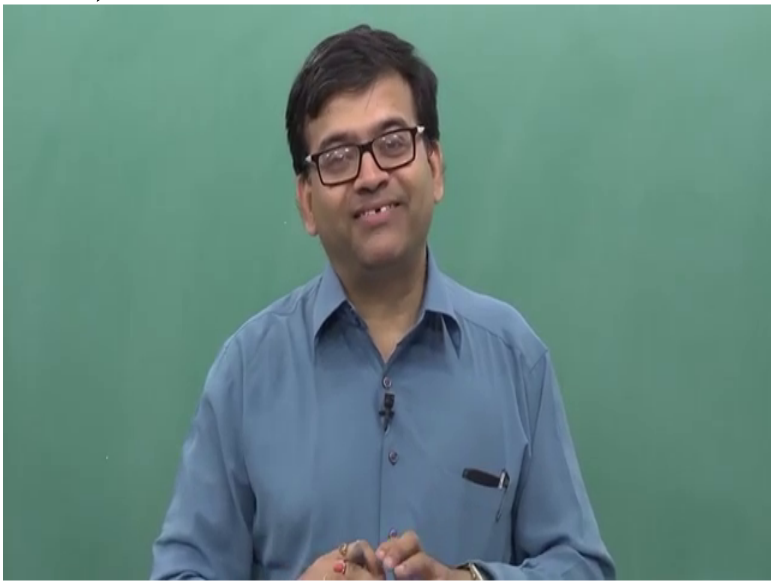

same procedure I just laid out for computing the weight enumerating function and you can see it gives us lot more

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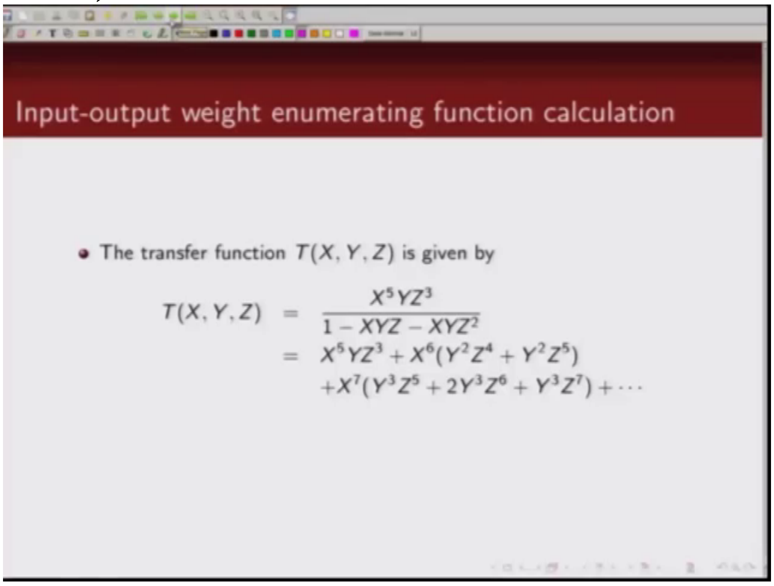

information. The weight enumerating function said we had one codeword of weight 5. Now it says that codeword of weight 5 basically was caused by message information bit 1 and the length of the digression from all zero state before it merged with all zero state was 3. Similarly there we had shown there were 2 codewords of weight 6. This completely specifies what those two codewords was. One which was generated by message bit weight 2 of length 4; this was message bit 2 length 5. So you can see the augmented state diagram, if we use it to generate

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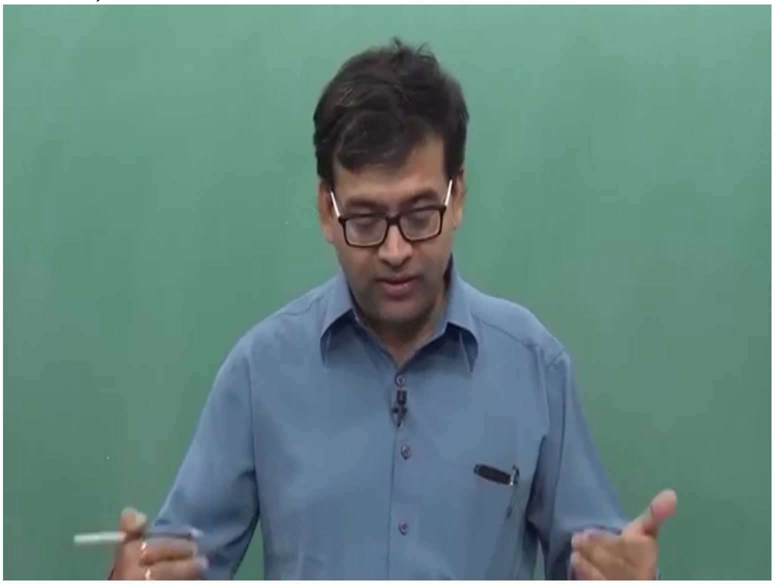

the transfer function it gives us lot more information. So with this I will conclude this lecture.

Thank you.