Principles of Communication- Part I
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Module No 4
Lecture 21
Frequency Domain Description of Hilbert Transform - Fourier Spectrum of the Hilbert Transformer

Hello welcome to another (mono) online course, seen the previous module is we have looked at SSB that is single sideband generation using frequency discrimination, in this module let us start looking at a different technique for SSB generation that is generation of single sideband modulated signals using phase shifting technique or phase shifting method, okay.

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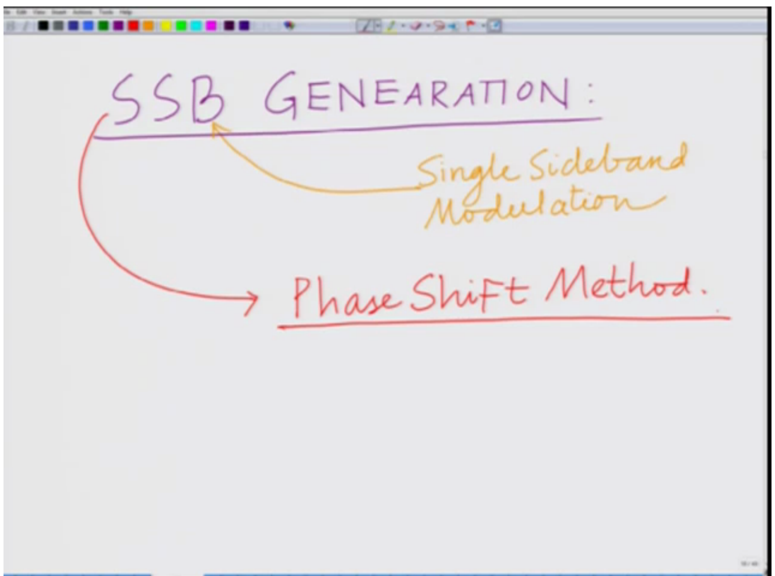

So this is the phase shifting method so what we want to look at the want to look at SSB generation we have already looked at SSB generation using frequency shifting SSB generation and well, let me remind you once again SSB stands for single sideband SSB stands for single sideband modulation and we are going to look at SSB generation using employing the phase shift so what we want to look at is the phase shift method for the generation of single sideband modulated signal.

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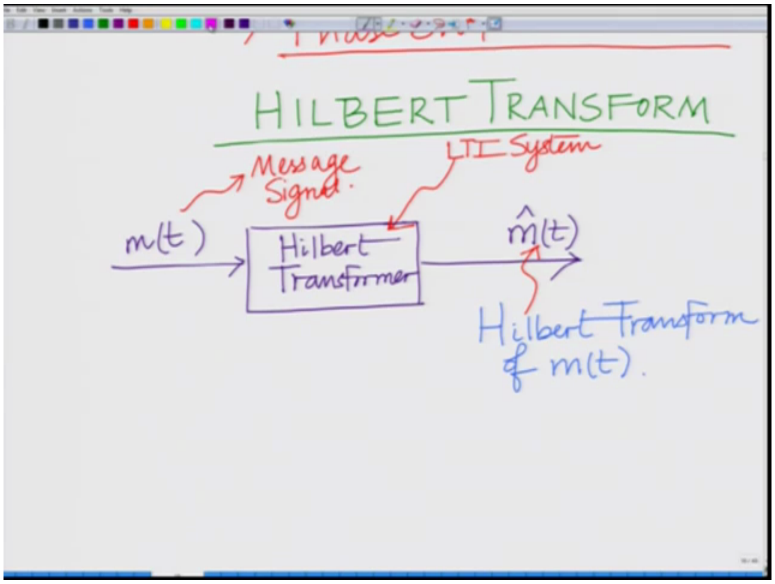

And to understand this better we have to start with the Hilbert transform, so we want to (sta) understand the phase shift is method, let us start by first learning about another important transform technique that is the Hilbert unlike the Fourier transform the Hilbert transform is a time domain transform that is it transforms one signal that is Hilbert transform is a time domain transform in the sense it transforms one time domain signal into another time domain signal.

So the Hilbert transform of a signal is another signal, alright. So what is a Hilbert transform? The Hilbert transform is basically let us say let me describe it at a high level m(t) passed through this system which is termed as the Hilbert transform or the Hilbert transformer passed through a Hilbert transformer gives your signal m hat of t and m hat of t so this is the your message this is the message signal or the original signal passed through a Hilbert transformer which is a LTi system this is an LTi system.

We are going to see how to characterize this later passed through an LTi system and this is m hat of t which is the Hilbert Hilbert transform of mt, okay. So m hat of t Hilbert transform of m(t) m hat of t the Hilbert transform of m(t) and this Hilbert transform remember we said the Hilbert transformer is an LTi system that is it is a linear time invariant system that is characterized by an impulse response, alright.

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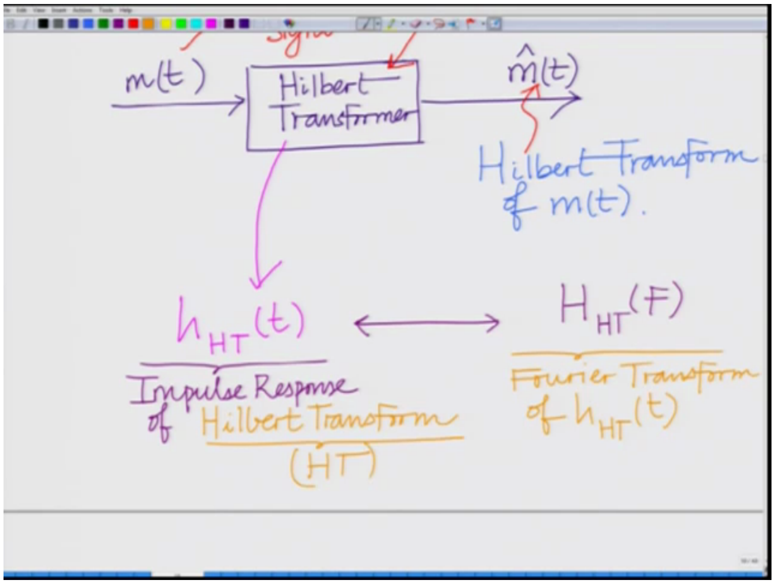

Every linear time invariant system is characterized by an impulse response, let us termed this response as h ht of t this is your impulse response of the Hilbert transform or the Hilbert transformer of the Hilbert transform I am going to abbreviate this Hilbert transform as HT. Now let the Fourier transform of this Hilbert transform, so please pay attention because there are several quantities here, let us denote that by H Ht of F that is the Fourier transform of the impulse response HT.

Fourier transform of the impulse response H the impulse response HT hHT of the HhT t of the Hilbert transform let me write this is your Hilbert transform which we are abbreviating using the acronym HT. So what we are saying is we need to understand this Hilbert transform can a Hilbert transformer can be characterized as an LTi system, alright. And with impulse response H subscript HT of t, alright. Where t denotes the time and with corresponding Fourier transform of hHt t given as H capital H HT of F that is a spectrum of the Hilbert transform, okay.

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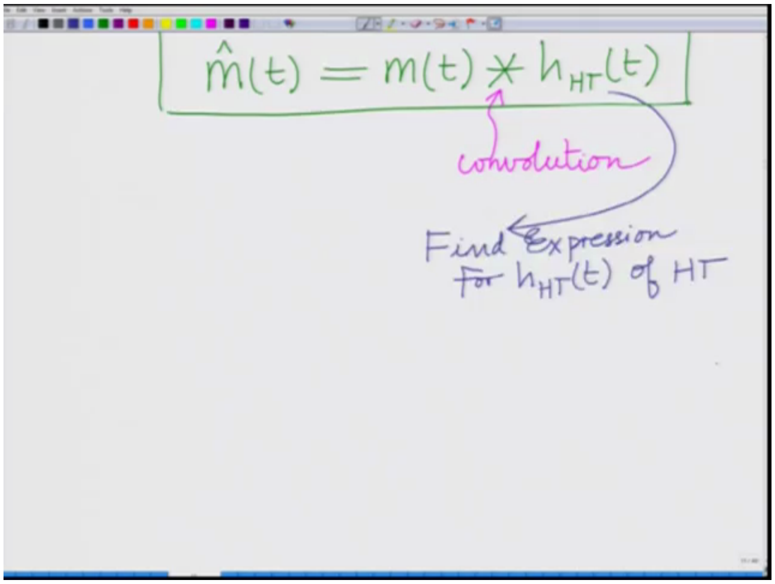

Now it is easier to characterize that and therefore what we have is before we proceed further the Hilbert transform m hat of t of m(t) is nothing but the (con) output of m(t) to the Hilbert transformer which is basically the convolution of so this is the relation convolution of the original signal m(t) with the impulse response of the Hilbert transform, so this is your convolution this is the convolution this is your Hilbert transform of m(t), m(t) is the message signal h HT is the impulse response of the Hilbert transform. So m hat of t is basically the convolution of m(t) with the impulse response of the Hilbert transformer that is hHT of t.

Now what we want to do is to characterize the (hil) Hilbert transform we want to find an expression for the impulse response for the impulse response of hH hat of t of the Hilbert transform, however the Hilbert transform is more readily expressed in the frequency domain it's easier to understand its characterization in the that is it is easier to characterize the spectrum it has an elegant characterization the spectrum of the Hilbert transform that is hHT of F, however it is easier to characterize hHT of F.

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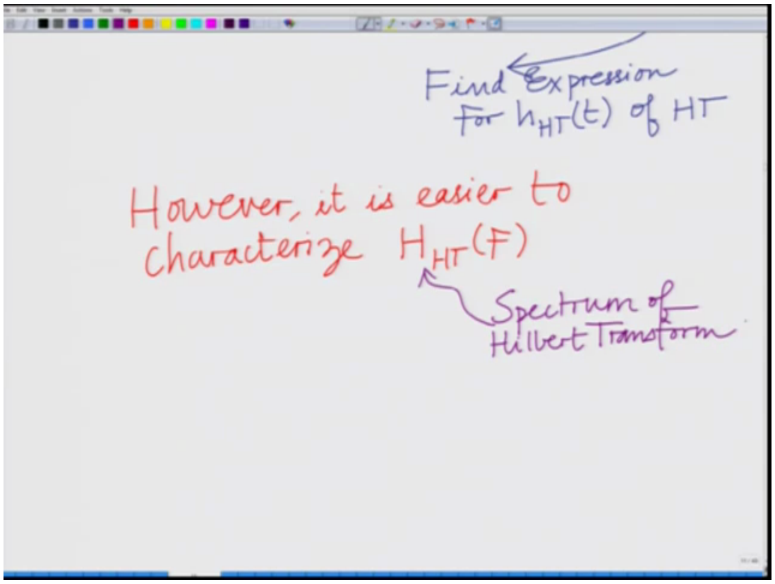

So we will start with that however that is a spectrum of the Hilbert transform you can also call this as is spectrum of the Hilbert transform.

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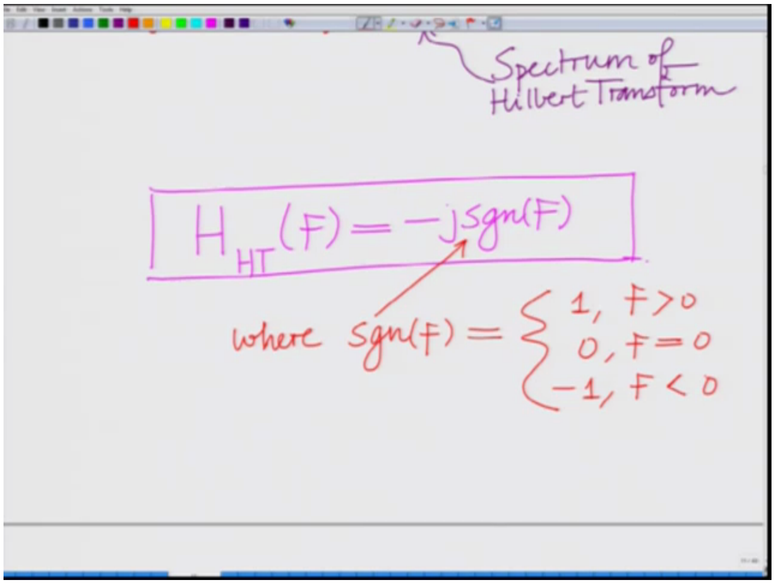

And indeed this spectrum of the Hilbert transform it has an elegant representation that is given as hHT of F is minus j sgn of F I am going to explain what this sgn function is shortly where this sgn function this is sgn of F is literally the sgn function which is one for all positive F that is F greater than 0. 0 or F equal to 0 minus one for F less than 0, so the sgn function sgn of F is basically the sgn of a number, so it is 1 for every positive F that is when F is greater than 0 minus 1 for negative F that is when F is less than 0 and at F equal to 0 it is equal to 0 and the Hilbert transform hHT of F the spectrum of the Hilbert transform rather is basically minus j sgn F.

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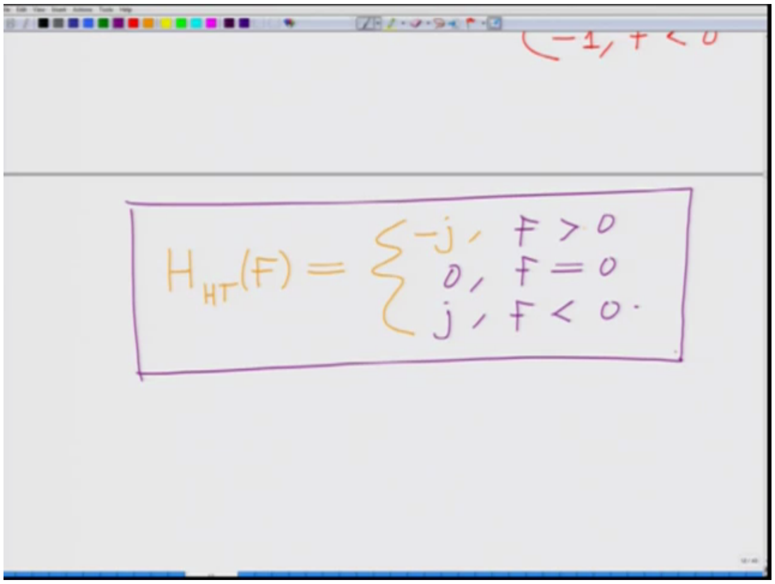

Which means now using the definition of sgn F above the sgn function of F hHT of F equals minus j for F greater than 0 minus j for F greater than 0. 0 or F equal to 0 and j or F less than 0 this is an equivalent expression of the spectrum of the Hilbert transform and therefore now if you look at the sgn function the sgn function looks something like this, so let us first draw the sgn function.

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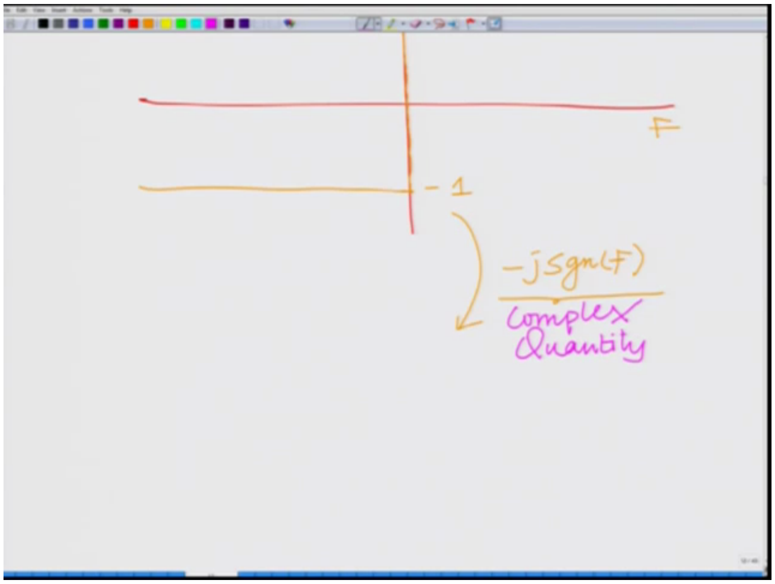

Sgn function is 1 for F greater than 0, so this is in the frequency domain this is one for F greater than 0 at 0 it is equal to 0 and for F less than 0 it is minus 1. So this is basically your sgn function so this is your is your sgn function this is sgn of F, okay. Now therefore now if I multiply this by minus j that is to obtain minus j sgn of F. Now look at this, this is a complex quantity this is a complex quantity so I cannot readily plot it this is a complex quantity, so minus j sgn F is a complex quantity it is minus j for F greater than 0 j for F less than 0 and at F equal to 0 it is 0 however I plot the magnitude and phase response of this.

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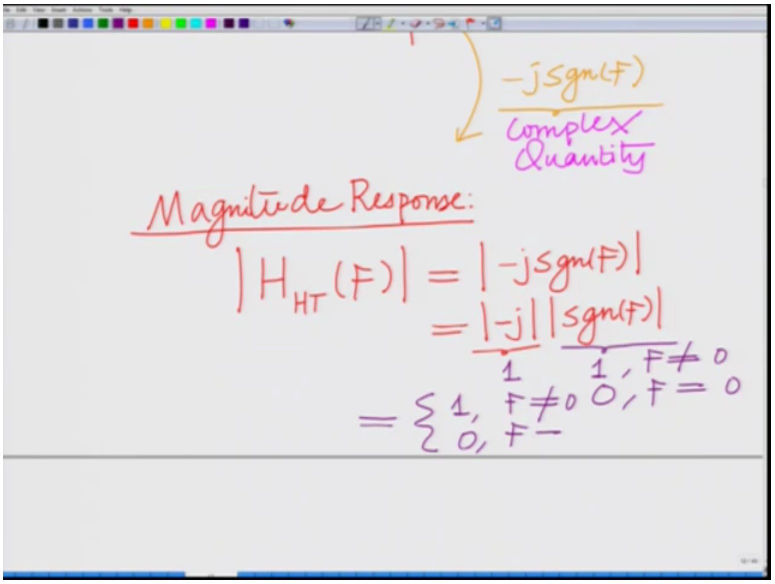

So the magnitude of this the magnitude of the magnitude response HHT of F equals magnitude you can see this is the magnitude minus j sgn F which is basically the magnitude of minus A j times the magnitude of sgn F. Now magnitude of minus j is basically one, magnitude of sgn F is one except F equal to 0 where 1 for F not equal to 0 and magnitude of sgn F equal magnitude of sgn F equal to 0 for F equal to 0, so the magnitude response HH TransForce F is 1 for F not equal to 0. 0 for F equal to 0 and therefore if I plot the magnitude response that is if I plot the magnitude, so since it is a complex quantity I am plotting the magnitude and phase response of the spectrum of the Hilbert transform the magnitude a magnitude response is 1 for all frequencies except at 0 ignoring that (fo) except for that one point 0 where it is 0 it is one it is unity.

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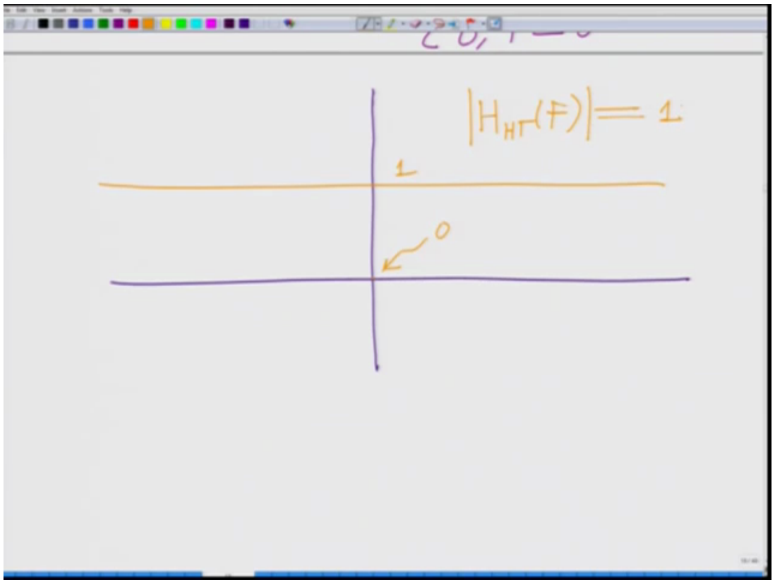

So the magnitude response like this except at 0 where it is equal to 0 except for this single point the magnitude of this magnitude of H the magnitude response of HHT of F is 1 and the phase response.

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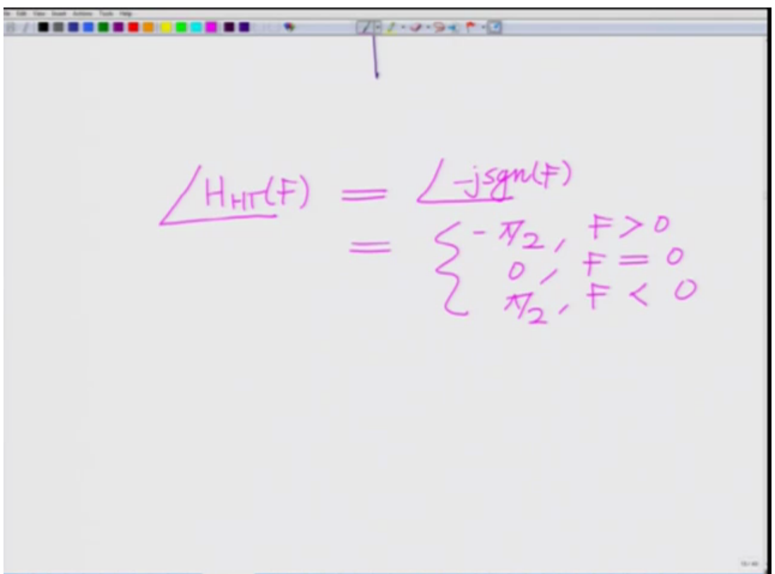

Now if you look at the phase response of the Hilbert transform that is the phase of minus j sgn F which is minus j for F greater than 0 which means phase of pi by 2 or F greater than 0. 0 for F equal to 0 which means phase of 0 for F equal to 0 j for F less than 0 which implies pi by 2 for F less than 0.

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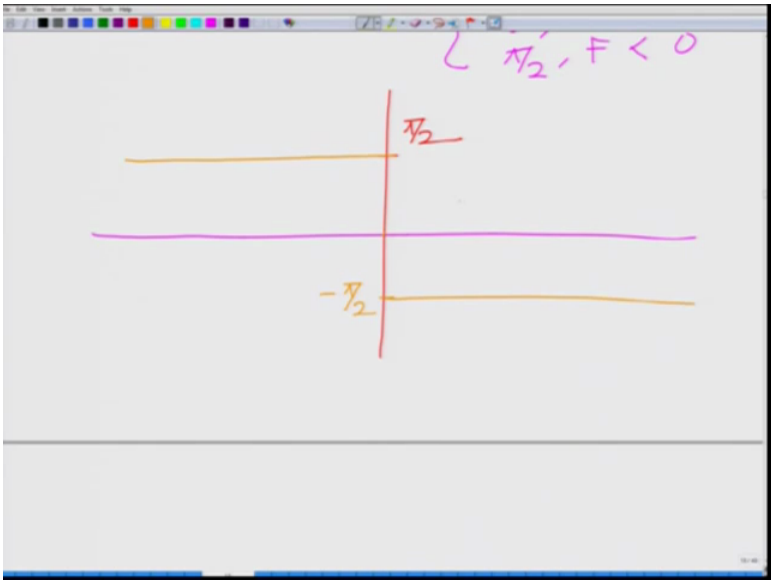

And therefore the phase response if you can look at the phase response of the Hilbert transform the phase response of the Hilbert transform that looks at as follows. For F greater than 0 it is equal to pi by 2 for F less than 0, F greater than 0 it is minus pi for F less than 0 it is pi by 2 at F equal to 0 it is 0 so this is basically your phase response, so basically if you can look at this you can see that the gain is unity except at F equal to 0 and all positive frequencies are shifted by minus pi by 2 phase all negative frequencies are shifted by the phase pi by 2 therefore this is a phase shifter.

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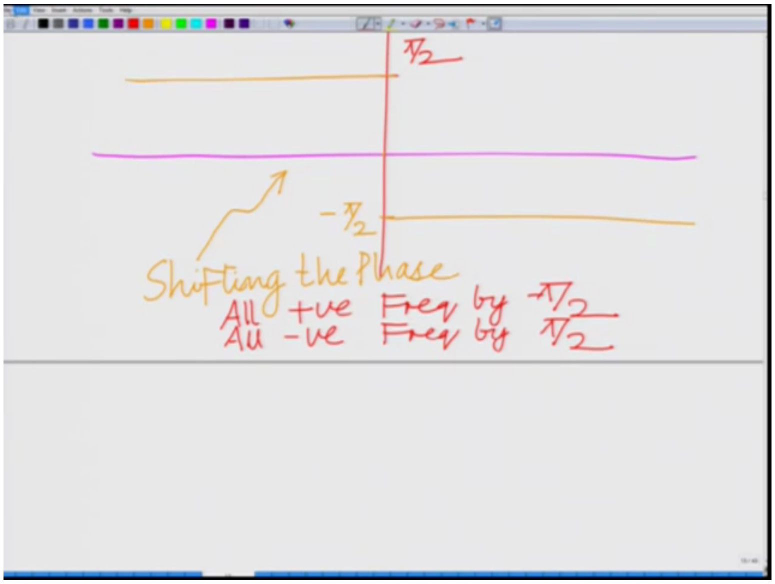

Basically this is what this is doing is this is shifting the Hilbert transform if you can see it is shifting the phase all positive frequencies by minus pi by 2, all negative frequencies by negative frequency components by pi by 2. So this is basically a phase shifter, alright. So Hilbert transform it has a gain of unity and the phase for all the positive frequencies phase shift is minus pi by 2 for all the negative frequencies the phase shift is pi by 2, okay.

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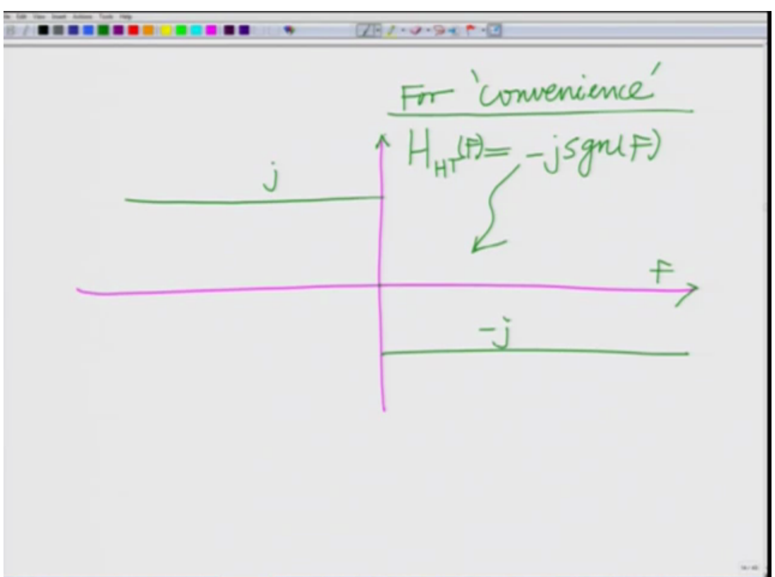

So now simply although I cannot represent this complex quantity in terms of a two-dimensional plot I am going to simply represent this for convenience as I am going to draw this although strictly speaking this does not make sense I am going to simply draw this as j I am simply going to represent this minus j sgn F by minus j and j, so this is basically minus j sgn F although this is complex quantity. So strictly speaking this is not correct but for the sake of convenience they can simply represent this as minus j sgn F with 0 that is equal to HHT of F that is minus j for all F less than equal to 0 j for F greater than equal to 0 and this is basically remember for for note for convenience, alright.

Simply although it does not make this plot does not make much sense is a simply for our convenience we are drawing this as j, I am sorry j for F less than 0 minus j for F greater than 0, alright. This is simply for the sake of our convenience, okay. Now what we are going to do is we have characterized the magnitude we have characterized the (hil) the Hilbert transform in the frequency domain that is the spectrum of the Hilbert transformer which is an LTi system, alright. We have characterized the spectrum of the Hilbert transform; let us now from the spectrum try to construct its impulse response.

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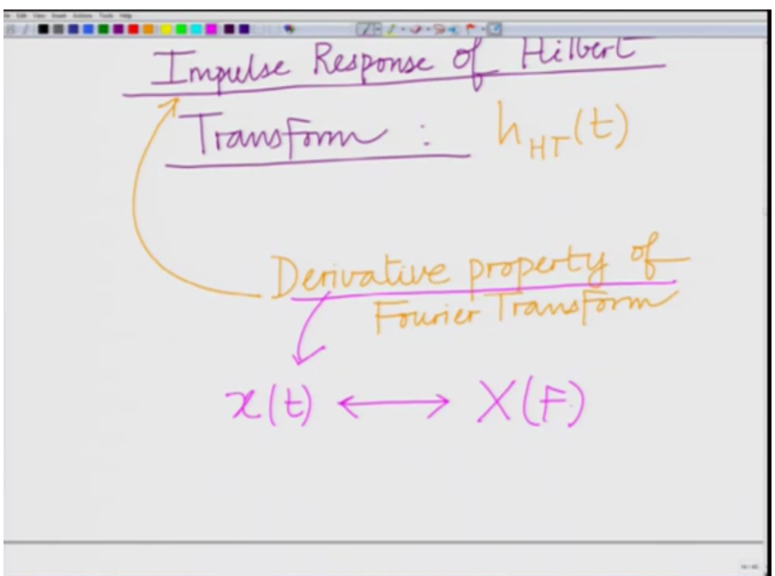

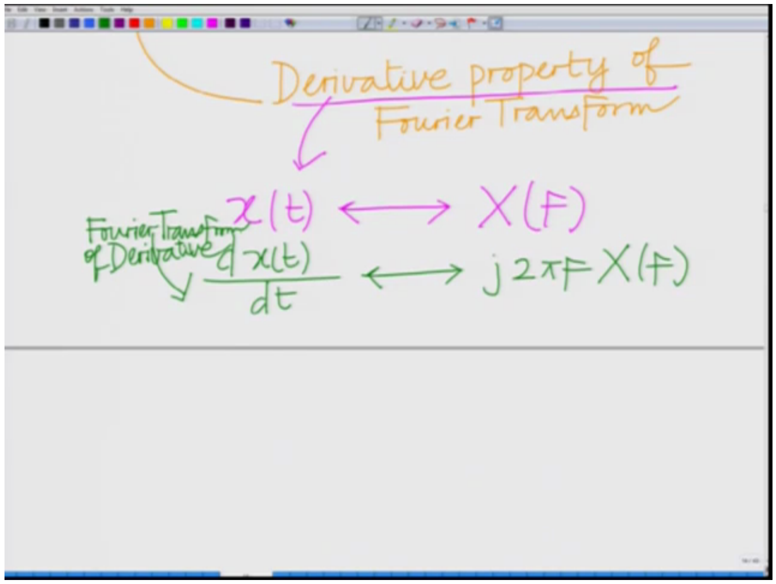

So now what we want to do is want to construct the impulse response of the Hilbert transform and to characterize the impulse response of this Hilbert transform, correct? This impulse remember this impulse response of the Hilbert transform that is denoted by the Hilbert transform impulse response of the Hilbert transform which is denoted by h HT of t this is a impulse response of the Hilbert transform, now towards this we will use the we will use the derivative property of the Fourier transform to characterize this impulse response we will use the derivative property of the Fourier transform.

The derivative now the property of the derivative property of the Fourier transform is as follows, its derivative property is as follows if consider as signal x(t) which has Fourier transform given by X(F) that is x(t) is a signal whose Fourier transform is given by X(F) than the Fourier transform of the signal d x(t) that is the derivative of x(t) is given by j 2 pi of x 2 pi of F times X(F), so this is the Fourier transform of the derivative of x. So what we have is that if we have a signal x(t) which has Fourier transform given by X(F) than the Fourier transform of its derivative that is d x(t) by dt is given by j 2 pi F times capital F that is j 2 pi F times the Fourier transform of x(t) that is a Fourier transform of d x(t) by dt, alright. So we will use this property in our next module to derive the impulse response of the Hilbert transform, alright.

So in this module we have seen we have introduced the Hilbert transform as an LTi system that is we have passed a signal m(t) through the Hilbert transform it gives the Hilbert transform that is m hat t of m(t) we characterized it in the frequency domain the spectrum of the Hilbert transform as minus j sgn F therefore it is minus j for F greater than 0 j for F less than 0 and 0 at F equal to 0. We have seen that it is a magnitude response of unity and phase shift of minus pi by 2 for F greater than 0 and pi by 2 for F less than 0, alright. And now we are trying to construct the impulse response of the Hilbert transform from the frequency response, thank you.