Principles of Digital Communication
Prof. Abhishek Dixit
Department of Electrical Engineering
Indian Institute of Technology, Delhi

Detection
Lecture – 37
Performance of Non-Coherent Systems

Good morning, welcome to a new lecture in detection. And in this lecture, we will be talking about performance of non-coherent systems ok. So, up to now what we have done is we have understood these coherent systems.

(Refer Slide Time: 00:34)

So, in coherent systems, the receiver has complete knowledge about the frequency and phase of the carrier. And what it does basically it computes the real part of inner product of received signal with the signal from the signal set ok. So, if I assume that there is no noise, so the received signal is exactly same as one of the signal from the signal set, then what it does is, it takes the inner product of the signal with itself, and it takes the real part of this inner product ok.

Inner product of the signal with itself simply gives us the norm square of the signal and norm square is anyway a real quantity. So, this is simply norm square of the signal. So, basically in the coherent receiver, these operations are carried out.

Let us see what happens in non-coherent receiver if these operations are carried out. So, here the received signal might be at certain phase offset then the signal from the signal set ok. So, let us assume that this is the signal from the signal set, this is the received signal, and the received signal is offset by this phase of theta from the signal in the signal set.

When we take the inner product of this, so inner product of u with u e to the power j theta is simply inner product of u with u multiplied by e to the power minus j theta ok, we have seen in the first week itself that this inner product is an Hermitian by linear operation ok. So, we know that this is norm square of u multiplied by e to the power minus j theta. And then if I take the real part of this, I simply get norm square of u multiplied by cos theta and now theta can be anything.

(Refer Slide Time: 02:51)


And if to make our life worse if the theta is 90 degree, then r is 0, and hence everything is lost. And thus performance of your receiver degrades when there is certain phase offset ok. Frequency offsets also results in phase offsets in some sense ok. Hence what we do in this non-coherent receiver, we change our strategy; and instead of taking the real part of inner product of signal with the received signal, what we do is we take the mod of this inner product and now you see that this problem is sorted out.

So, if you take the mod of this thing instead of taking the real part of this thing, what you will receive is simply norm square of signal ok. And thus in absence of noise, this

receiver works in the same way as the coherent receiver, there is no degradation right. We will see now in this lecture what happens in the presence of noise, but overall what we are pointing out is, if you use the same strategy as we used in the coherent receiver, and if there is a certain phase offset, then the real part of inner product of signal with the received signal might give you flat 0. And thus in case of non-coherent receiver, what we do is we take the mod of inner product of the signal with the received signal ok. And so far it seems that it works fine.

(Refer Slide Time: 04:42)
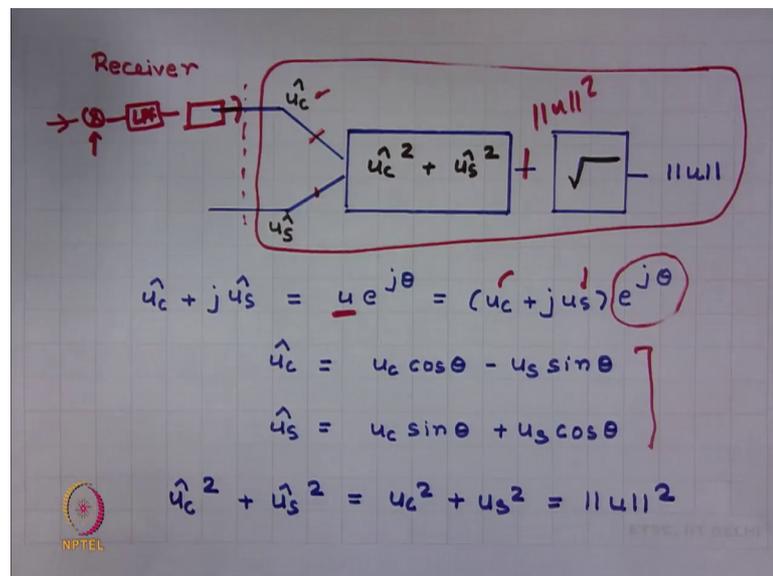
Let us look at the receiver for a non-coherent system. And we are not showing this part, because this part remains exactly as for the coherent receiver, but then you do certain operations. Once you have received the cosine and sin part using the techniques that you used in the coherent receivers, namely you have a multiplier which shifts down the received signal to basement signal, you will have a low pass filter, you will have a matched filter and a sampler ok. Everything remains same in the front end of this receiver ok.

So, what we receive in this non-coherent receiver is u c cap and u s cap, and what is this u c cap and u s cap, these are the cosine and sin parts of the received basement signal in the presence of frequency and phase offset. So, if I assume that my received basement signal will be simply u multiplied by e to the power j theta, and from this I can get the cosine and sine part that I have in here ok. u is simply u c plus j u s, so these are the

cosine and sin paths that we intend to receive, but then this thing in the presence of frequency and phase offset gets multiplied by this factor. And just working out the arithmetic, you can see that this u c cap is u c cos theta minus u s sin theta and u s cap is u c sin theta plus u s cos theta. Actually this we have derived in one of the previous lectures in modulation, and hence I am not repeating it and sort of revising the stuff for you ok.

So, what you do is you square this thing in here, and you square this thing, and you add these two components, and then you take the square root ok. Why are we taking the square root? Anyway, it is a deterministic function and we have seen in detection already by reading this theorem of reversibility which states that any deterministic function does not change or alter probability of error. Thus if you have it or you do not have it does not do anything to probability of error, but it let us just us have nice PDF's which we will see and that is why we are having this square root function ok.

Anyway, so you can see easily that if you square this u c cap and square this u s cap. And you add these two things up, what you simply get is norm square of u and that is what we wanted. Once you take the square root of this norm square of u, you get norm of u, easy ok. So, in non-coherent systems, we add this thing up here, and we can detect again the signals, so far so good.

(Refer Slide Time: 08:25)
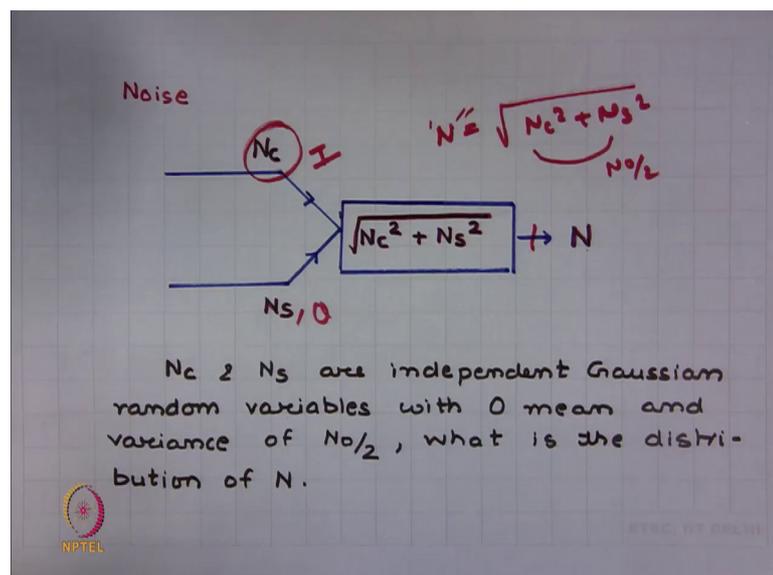
What happens to this noise? So, if we assume that there is a noise depicted by a random variable N c in the cosine part or in the N phase channel. And let us assume that there is a noise depicted by this random variable N s in this Q channel or in the sin part then what happens is when you have this operation of squaring N phase and Q phase component adding them up and then taking the square root of this you also do this with noise. So, at the end of the system you have coronoid random variable N, which is square root of N c square plus N s square, and of course, these random variables are independent of each other, they are Gaussian with zero mean and certain variance, variance we can assume to be N naught by 2. And the first question that we like to ask is what is the distribution of this N and that we will see ok.

(Refer Slide Time: 09:46)

So, let us do one quick exercise from random variables and processes word, and let us say we have two random variables x and y, these random variables are independent of each other. Let us assume that these random variables are Gaussian with zero mean and variance of sigma square. And what we want is, we want to find out the distribution of this R and phi, these random variables are described by these relationships. And if you look at these two equations you know what I am trying to do. I am trying to change my coordinate system from Cartesian coordinate system to polar coordinate system ok.

And I want to understand what is the joint PDF of this random variable R and phi that is what we are up to. And the first thing that we have to start with is writing the joint PDF

of these random variables x and y. And if these random variables are independent, the joint PDF is simply obtained by multiplying the marginal PDF, marginal PDF of this random variable x is this marginal PDF of random variable y is this, they are Gaussian zero mean and variance of sigma square. So, you multiply these two PDF and you get this PDF. This we have also seen when we have discussed proper complex Gaussian random variables. So, we have been able to define the joint PDF of these random variables X and Y, and now we have to think about the PDF of these random variables R and phi.

(Refer Slide Time: 11:52)

Now, what you can see easily is that when you go from one coordinate system to another coordinate system, the probability of finding a point in this coordinate system let us in this space d x d y would be same as the probability of finding the map of the points in this space in the space d r d phi ok. That means, what I am simply saying is if I have a space, if this space gets mapped to a different space in another coordinate system, the probability of finding a point in this space will be same as probability of finding a point in this space. So, probability remain conserved. And this is the probability of finding a point in d r d phi space, this is a probability of finding the point in d x d y space.

And the first thing that you must have learned when you have learned about interchanging the coordinate systems is this relationship between Cartesian coordinates and polar coordinates. You must have seen that d x d y in polar coordinates is r d r d phi

ok. And you must have seen this picture as well. It is very easy picture to remember this. So, if we take an incremental length of d r, and if you take this, this is r d phi, then you know that this space corresponds to space d x d y. And hence r d r d phi is simply d x d y this is all what you need to find out the PDF or joint PDF of R and phi if you know the joint PDF of X and Y.

(Refer Slide Time: 13:59)

So, what we do here is we have already known this. So, we have substituted joint PDF of X and Y, which is this that was in the form of let me look at this again an x square plus y square is simply r square. So, if you see this is simply this ok. Now, I think we do not need it. And what we have done more is we have replaced this d x d y with r d r d phi. And now we have got this expression. From this what we get is the joint PDF of r and phi is simply this function.

If you look at this function for 30 seconds or 1 minute, whatever rely is that this function is not a function of phi, it is just a function of r. And this phi must be uniformly distributed random variable; that means, it should have same distribution for all angles. If that was not the case then the joint PDF must be a function of phi which it is not and thus if you look at this PDF or joint PDF you can realize that phi must be uniformly distributed random variable. And now the next job is to find out the marginal PDF of this random variable r, how do we go from joint PDF to marginal PDF.

(Refer Slide Time: 16:00)

It is a straightforward. You take the joint PDF and you integrate it with respect to one random variable. So, you integrate this with respect to d phi, and that will kill out this phi from this joint PDF. And you get the marginal PDF for r, easy. So, integrating this will let you have the marginal PDF of random variable R, which is this function. Note that this r must be strictly positive, because finally you are having r as square root of x square plus y square which is a positive quantity ok. And what will be the PDF of phi? It is a uniformly distributed random variable between 0 to 2 pi. So, the marginal PDF of phi should be 1 by 2 pi for phi between 0 and 2 pi ok.

If you multiply these two PDFs, you can see that the product of this PDF is simply the joint PDF of these two random variables. And this is a quick proof that these R and phi are independent of each other. So, we have learned lot of stuff here. What we have started with we have started with two random variables X and Y, and we want to derive the probability distribution function of two another random variables R and phi. R is square root of x square plus y square; and phi is tan inverse of y by x. Doing some arithmetic we have derived the joint PDF of R and phi; from that we have got the marginal PDF of these two random variables. And we have also made a quick proof that these random variables are independent.
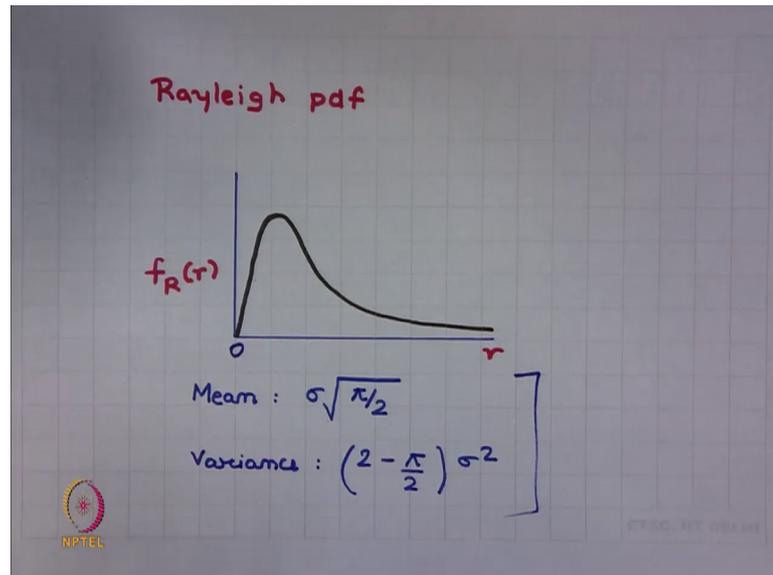
So, it is the time now to look at this equation little harder. And I have got a figure for that. So, here I am plotting the marginal PDF of this random variable R, and this PDF has also got the name because of its applications to various fields. It is used in non-coherent detection; it is also used in wireless communication you model the wireless channel as Rayleigh channel ok.

It is very useful probability density function. It looks like this. It is it is not symmetric PDF the numerical values of r are only positive numbers. So, unlike Gaussian which runs from minus infinity to plus infinity it does not run from minus infinity to plus infinity, but it goes from 0 to plus infinity it is not at all a symmetric PDF.

The mean of this PDF is sigma times square root of pi by 2 and the variance of this PDF is two minus pi by 2 multiplied by sigma square. So, you might have to use this sometimes in your life, and thus it is a good idea to know that these mean and variances exist ok. Now, from this knowledge of Rayleigh PDF, we can start looking into the question of how can I do detection in these non-coherent systems.

(Refer Slide Time: 20:10)
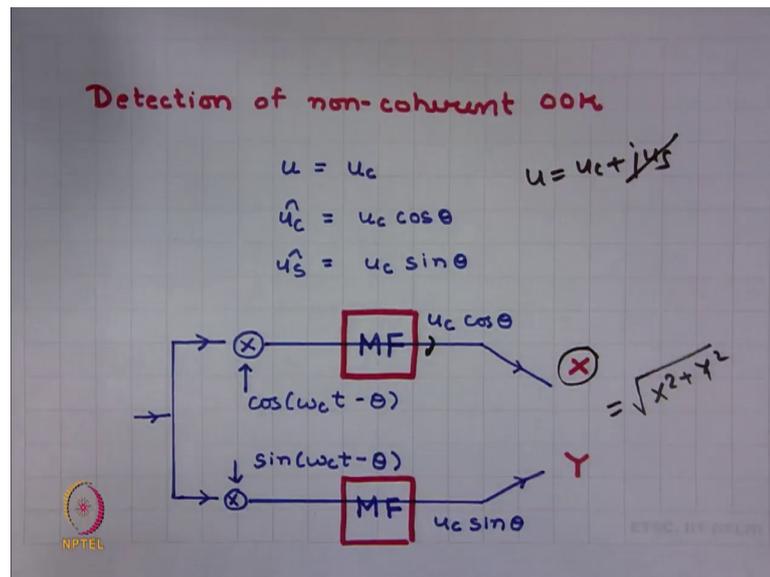
And we start with this OOK system on off keying ok. It is a binary PAM all right. So, now, in OOK we are having just one real number. So, u in general is u c plus j u s. And you do not have this thing in here if you are just using one real dimension. So, u is simply u c. And what is this u c cap, u c cap is u c cos theta, and u s cap is u c sin theta. That means, after this match filter and the sampler, we are going to get u c cos theta in this i channel or x channel a cosine channel, the ways names for the same thing. And I also use them interchangeably to confuse you. And thus it is good idea to get accustomed to all these words. In this y channel or q channel or sin channel what we get is u c sin theta.

And the first question that we like to ask is what is the probability density function of this random variables x and y, of course, there must be some noise hitting this receiver somewhere, and this will be a Gaussian noise as usual. So, what we will believe is that there is a Gaussian noise riding on their signal. And let us try to see what is the PDF of X and Y and from that, we will be investigating what is the PDF of the square root of X square plus Y square that is the big plan.

(Refer Slide Time: 22:06)

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - u\cos\theta)^2}{2\sigma^2}\right)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y - u\sin\theta)^2}{2\sigma^2}\right)$$

$$\int f_{R,\phi}(r,\phi)\, dr\, d\phi = \int f_{X,Y}(x,y)\, dx\, dy$$

$$R = \sqrt{x^2 + y^2}$$

So, what is the PDF of X, x being Gaussian things are simple just now the mean shifts 2 u cos theta. And what is the PDF of Y again it is caution is simple, the mean here is u sin theta. And we assume that the variance of the noise is sigma square. Now, from these X and Ys, what we are interested in finding is R which is the square root of X square plus Y square and thus we can use the same strategy as we used in this lecture while deriving Rayleigh PDF. So, I want to equate the probabilities in the two words in Cartesian coordinate and in polar coordinate. And then I would like to substitute the joint PDF X and Y, substitute the fact that d x d y is r d r d phi. And from that I hope to get this joint PDF of r and phi.

(Refer Slide Time: 23:20)

So, the joint PDF of r and phi by using the same thing that we have just done can be obtained like this. In the previous example, I just had x square plus y square and here because there is a signal also present in these two branches. I have these factors popping up that is the only difference everything else remains same. And it does not take more than high school trigonometry to work out these identities.

So, we have to expand this and you just have to use a square plus b square plus 2 a b identity which even I can remember. So, expanding on this I get x square plus u square cos square theta minus 2 x u cos theta, expanding on this I get y square plus u square sin square theta minus 2 y u sin theta. Then I add these terms, I get x square plus y square, I can combine these two terms into u square, and I get these two terms here ok.

(Refer Slide Time: 24:41)

And then I use the fact that x square plus y square is r square and tan phi is y by x. So, these are the numerical values of the random variables. If random variables are related by this expression, the numerical values should also be related by a similar expression ok. And from this I get x is r cos phi and y is r sin phi from these two basic equations. So, while trying to solve for this, I can substitute x here as r cos phi, this y here as r sin phi. I can substitute r square in place of x square plus y square, so I get r square plus u square minus 2 r cos phi u cos theta minus 2 r sin phi u sin theta. And I can combine this term like this that is it not hard.

(Refer Slide Time: 25:52)

Now I have to substitute this in that argument and divide this by 2 sigma square. And then same as before to get the marginal PDF from this joint PDF, I have to take this joint PDF. And I have to integrate with respect to phi if I have to find out the marginal PDF of random variable r. And what you can do is, you can collect the terms which are independent of this phi. So, I have this, this, two terms which are independent of phi. I can take this out of the integration because they are constants with respect to phi. So, I have this function, and I have this integration to solve.

Solving this integration in the first site looks scary, but actually it does not. And as an engineer you must know how to deal with such functions, because you must have seen them several times. For example, whenever you have cylindrical geometries or polar coordinates, we mostly get to these kinds of functions. You must have seen these functions in wave guides or in optical fibre. So, we know how to solve this if you think little bit, and if you have done a course in electromagnetism where you have seen these wave lights or optical fibres.

This is actually a modified Bessel function of first kind in zero order, simpler. So, I can replace this integration, why this modified Bessel function of first kind in zero order or modified zero order Bessel function of first kind also ever you would like to call that does not matter. And this is simply I naught u r by sigma square. And thus I can get the marginal PDF of this random variable R. So, it is like some maths involved here.

(Refer Slide Time: 28:09)
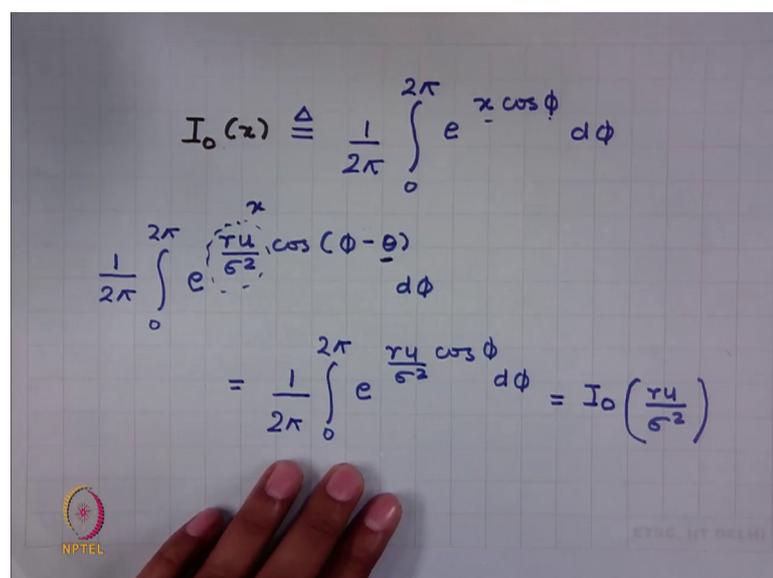
$$I_0(x) \triangleq \frac{1}{2\pi} \int_0^{2\pi} e^{x\cos\phi} \, d\phi$$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\frac{ru}{\sigma^2}\cos(\phi-\theta)} \, d\phi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{ru\cos\phi}{\sigma^2}} \, d\phi = I_0\left(\frac{ru}{\sigma^2}\right)$$

So, let us look at this modified zero order Bessel function of first kind. It is defined as 1 by 2 pi multiplied by the integration of e to the power x cos phi n the limits 0 to 2 pi ok. This is the definition of this modified zero order Bessel function of first kind. The integration that we were dealing with was this and there was also 1 by 2 pi here.

So, we were dealing with this. So, you can easily appreciate that this thing is x ok, but what we have here which is different from this basic definition that we have cos of phi minus theta instead of simply cos phi, but that is not a problem because cos is a periodic function. And if you integrate a periodic function in one period, then the initial phase from where you start does not matter.

This theta simply shifts the phase of this function, but being a periodic function, the initial phase does not matter if you are carrying out the integration over one time period is not it. So, this integration is same as carrying out this integration. And this is simply I naught of u r divided by sigma square ok. So, do not get confused.

(Refer Slide Time: 30:07)

So, what we have got is the marginal PDF of random variable r is this. Does it have a popular name? Yes, it does. It is known as Rician PDF named after the famous engineer rise. So, Rician PDF looks like this. So, it contains this modified zero order Bessel function of first kind ok. We will try to look this Rician PDF again.

But let us first ask some basic questions what is this I naught of 0. If you have to investigate this at 0, e to the power 0 is 1. So, here we just have 1 and this will be 1. So, I naught 0 is 1. And if in this PDF if you have u as 0 what do we get. So, if you substitute u as 0, this term vanishes. So, you simply have e to the power minus r square by 2 sigma square multiplied by I naught 0 because u is 0 and this is 1 right. So, this converts to Rayleigh PDF.

And this was bound to happen because what was the difference between Rician PDF and Rayleigh PDF when we derive this Rayleigh PDF we just assumed that there is a noise there was no signal riding on the noise. And when we have done the derivation for Rician PDF, the thing that changed here is now we have signal plus noise hitting the block, which is carrying out first the square of the terms summing them up and then taking the square root operation. So, the basic difference between the Rician PDF and the Rayleigh PDF is we get Rayleigh PDF when we just assume that this noise hitting us in the i and q channel, and we get this Rician PDF when there is a signal riding along with noise ok, so that is the main difference.

(Refer Slide Time: 32:23)
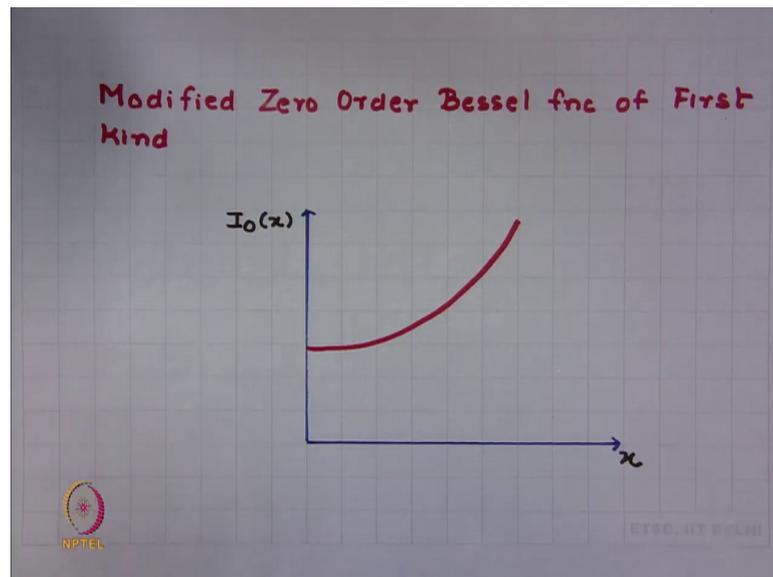
Let us see how this modified 0 order Bessel function of first kind looks like it is a monotonically increasing function of x, so that is nice.

(Refer Slide Time: 32:36)
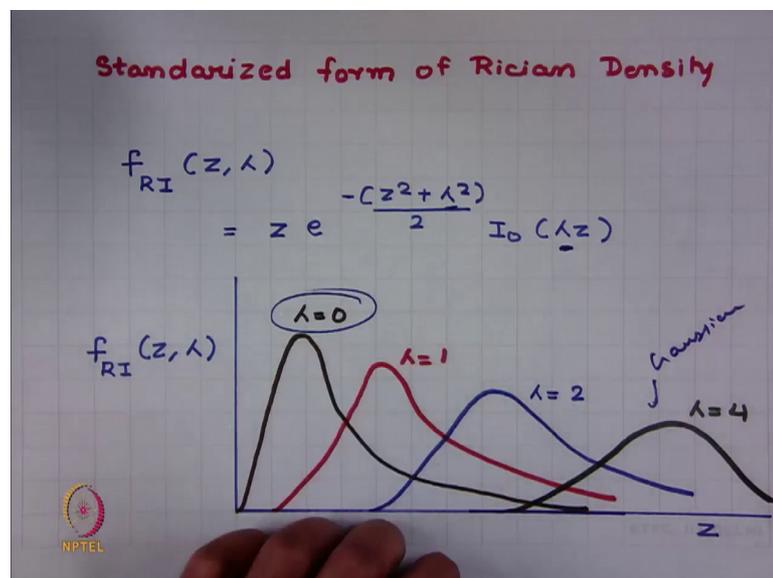
And like we have this normal random variables we also have a standardized form of Rician density functions ok. The standardized form of Rician density function is this. So, it is very similar to the density function that we have versus a standardized Rician density function. What you can see is, if you change this lambda in here, so you can think about this lambda as u that we have in our density function. And you can think

about this z as r that we had in a density function, and variance is assumed to be 1 in this case ok.

So, what you can see is if you change this lambda that means, it is akin to increasing the signal power that is riding along with the noise, you get to different Rician density functions. So, when there is no signal power, then you get a Rayleigh PDF. As you increase the signal power, lambda increases and you get other density functions. If your lambda is pretty large, then the Rician density function begins to become like a Gaussian density function. So, it is like Gaussian density function; it will never become a Gaussian density function, because Gaussian density function runs from minus infinity to plus infinity. And this Rician density function runs only over positive arguments.

So, in no sense it will be a Gaussian density function, but it will become like Gaussian density function. So, then the tail of this Gaussian density function does not matter, then you can very well approximate it as a Gaussian function ok, and that happens when a signal power is large. We have a mathematical kind of proof for this.

(Refer Slide Time: 34:43)

So, this modified Bessel function of the first kind can be approximated like this. Do not ask me for the proof of this; this is well established relationship which I am just using. And if my modified Bessel function can be approximated like this for large u or signal power, then the Rician density function that I had before you can substitute this approximation of this Bessel function in this density function ok.

So, you are just substituting this thing in place of this in here. What will I get? So, I will get this term which is here, then I have got this scaling factor in here. So, I am replacing this term with these two newly added term. So, one is this term and one is this factor. And then I can multiply this with two, and I can try to combine this with this.

(Refer Slide Time: 36:08)

And my result in density function begins to look like this. And then again using this identity of a minus b whole square, I can reduce this to this. So, if I look at r around u, this factor becomes 1; and this thing can be approximated as this thing. So, it begins to behave like a Gaussian density function in approximation. And that is one nice thing that we have in here. So, we can assume that r E b N os is large, when my E b N o is large instead of thinking in terms of Rician density function, I can think in terms of Gaussian density function, and I can do quick calculations, so that is the strategy that we have in here.

(Refer Slide Time: 36:58)

So, after completing this derivation of Rician PDF and Rayleigh PDF, we are good to go and derive the performance of non-coherent O k system. And what we are assuming is that when we have this hypothesis 1, we are transmitting a signal u; and when we have this hypothesis 0, we transmit 0. So, the PDF when 1 is transmitted, that means, you have the noise riding with the signal would be a Rician PDF. And this will be the expression of that PDF we have derived this. And the probability density when you are transmitting 0 will be a Rayleigh probability density, because u is 0 here ok.

So, what we have in here is we have different kinds of likelihoods when 1 is transmitted and when 0 is transmitted. Now, first thing that you have to identify is, where is the threshold of detection ok, when will receiver will decide for 1 and when will it decide for 0. So, to think about that we need to find out the threshold and threshold is the place where these two likelihoods are same.

(Refer Slide Time: 38:36)

Let me show that. So, we have to find out this threshold a naught which you can simply find by comparing the likelihood when hypothesis 1 is transmitted where the likelihood when hypothesis 0 is transmitted. And can I make use of some simplifications here. So, this term will simply cancel out with ok, and this will also cancel out. So, what you have is e to the power minus u square by 2 sigma square times this modified Bessel function must be 1. It is little bit tedious to compute a naught from here.

(Refer Slide Time: 39:22)

But you can show that approximately a naught is given by this thing. Now, we use the properties that we have learned while covering this matched filter that if I use a matched filter which I will because it is the most optimum filter to use. My u is simply E s and my sigma square is N naught by 2 times E s and this we have seen even in the last lecture. So, from this the square of u by sigma is E s square divided by N naught E s by 2.

So, I am simply substituting these numbers here. And from here you get that this is 4 E b by N naught. Remember in OOK systems E b is E s by 2 because for 1 bit I am transmitting E s energy; and for the another bit I am transmitting 0 energy. Thus on an average the energy is symbol is E s by 2 and bit energy is also the average symbol energy ok substituting in place of this the value that we have derived for u by sigma square.

(Refer Slide Time: 40:40)

$$a_0 = E_b \sqrt{1 + \frac{2}{(E_b/N_0)}}$$

Thrushold depends upon E_b/N_0 and this is a serious problem

For $\frac{E_b}{N_0} \gg 1$

$$a_0 \simeq E_b$$

The a naught that we get is this ok. And now the problem is in this system and in an OOK system, whether it is a coherent OOK or non-coherent OOK, this is the problem that you have to live within OOK system that threshold is a function of E b by N naught. For example, if I would have used an antipodal system or an antipodal signalling scheme where does the threshold lie? If the signals are of equal priors, then the threshold lies always at 0, there is no problem. But here in case of OOK system, threshold depends upon E b N o ok.
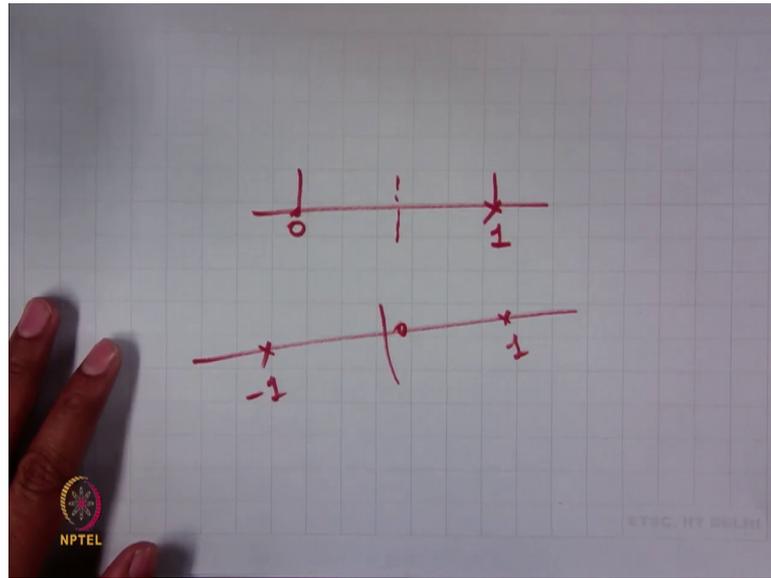
So, for example, if you have an OOK system, so here is your signal 1, here is your signal 0. Your threshold will lie in between these two signal points. And thus it depends upon with what energy you are transmitting 1. And because the energy that has a role in the receiver performance is the energy which receiver is actually receiving, and thus it depends upon several factors like how far the receiver is from the transmitter, what kind of channel is in between the transmitter and receiver, what fraction of the transmitted energy is reaching the receiver, what powers the transmitter is transmitting and on several factors.

Whereas, if I would have had used an antipodal scheme, my threshold always sits at 0 in spite of the channel conditions, and that is why it is very useful, because threshold detection in antipodal schemes is does trivial its 0 always. Whereas, the threshold detection becomes a serious problem in case of uni polar signalling mechanisms like on off keying ok, so that is why practically the receiver becomes complicated for OOK systems.

(Refer Slide Time: 43:02)

$$a_0 = E_b \sqrt{1 + \frac{2}{(E_b/N_0)}}$$

Threshold depends upon E_b/N_0 and this is a serious problem

For $\frac{E_b}{N_0} >> 1$

$$a_0 \simeq E_b$$

So, threshold depends upon E b N o and this is a serious problem spatially in fading channels. And what we will like to do is we want to assume this E b N o is pretty large than 1 so that I can approximate the threshold to be just E b. So, we do set of approximations because we want to arrive at a simple formula for probability of symbol error rate.

(Refer Slide Time: 43:26)

$$P_{e|H=0} = \int_{E_b}^{\infty} \frac{e^{-r^2/2\sigma^2} \, r}{\sigma^2} \, dr$$

$$= e^{-E_b^2/2\sigma^2}$$

$$= e^{-E_b^2/(2\frac{N_0}{2}E_s)}$$

$$= e^{-(E_b^2/(N_0 \, 2E_b))}$$

$$= e^{-E_b/2N_0}$$

Now, we ask the question what is the probability of error given that hypothesis is 0.

(Refer Slide Time: 43:39)
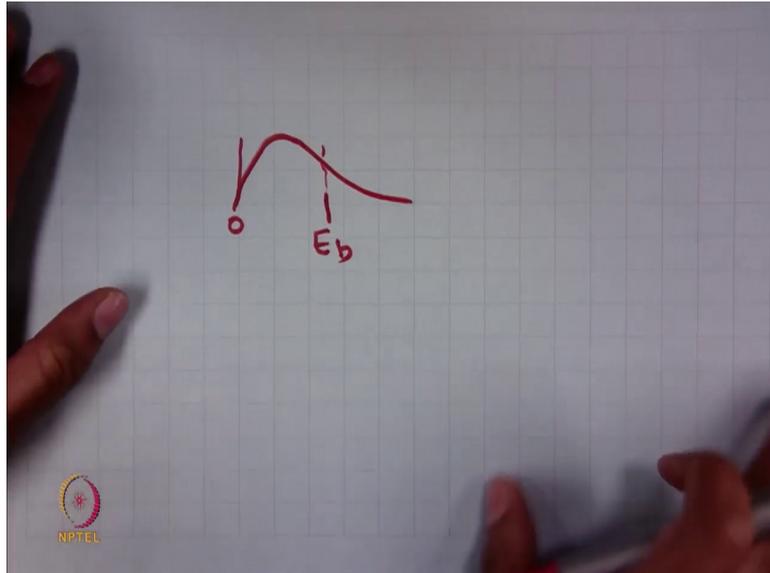
So, when we have transmitted hypothesis 0 that means we are not transmitting anything. So, ideally I should get a 0, but because of this noise acting in here I might get a certain number, a certain numerical value. And if this received number exceeds the threshold of E b, I create an error because then my receiver will decide for 1 ok. So, you can think it like this that the perpendicular bisector sits at E b.

So, if there are receive numerical value is more than E b, then my detector will go for hypothesis 1 which will be an incorrect choice. And thus to find out the probability of error given that we have transmitted to hypothesis 0 can be simply obtained by evaluating this PDF's integration in the limit E b to infinity.

And one thing that you must be really comfortable with is integrating this function as very simple. There is a trick here. If you have to integrate this, what you have to do is simply replace this r with E b that is the final answer that you will get. Mostly in this non-coherent systems and in wireless communication, you end up with these kinds of integration ok. And here the trick that we use is simply replace r with E b, and you get this function where r is replaced with E b. For example, this integration is simply this.

Now, putting for the sigma square sigma square is N naught by 2 times E s and thus this is simply E b square by N naught times 2 E b E s is 2 E b. So, one E b cancels with one E b. So, I get the probability of error given hypothesis 0 is transmitted as e to the power minus E b by 2 N naught. So, it is straightforward, not that complicated ok.

So, now we have to find the probability of error given that we have transmitted hypothesis one, and this can be simply obtained by integrating this PDF from minus infinity to E b. So, now you notice that we have Gaussian PDF instead of Rician PDF. And we have done this because it makes our life simple. And we have seen that this will be the case when E b N o is pretty large.

Then why not assume E b N o to be pretty large, because that will simplify the life of an receiver and it will also simplify our life ok. So, we have just replaced the Rician PDF with Gaussian PDF. And then we have to integrate this between minus infinity to E b. Then if we can get the probability of error given hypothesis one is transmitted, and this is Q of root of E b by N naught ok.

Now, we make use of another approximation we will do it later we have not done it yet let us first see what is the probability of error. So, this is the probability of error given hypothesis 0 is transmitted; this is the probability of error given hypothesis 1 is transmitted. You add these two probability of errors and divide by 2 to take the average. And now we are using the approximation that Q of x is 1 by root 2 pi x e to the power minus x square by 2 why are we using this approximation instead of e to the power minus x square by 2 approximation which we have used several times before.

This approximation has to be used when you are finding the slope at which BER curves decreases right. So, when you are looking to think about numbers how this BER

decreases with E b N o. When you are trying to find out the exact value of the expression, then this is a better approximation to use. So, using this approximation for Q of x, we can write this as this. And now you can see that if your E b N o is pretty large, then this term will have no contribution, it will be negligible.

(Refer Slide Time: 48:29)
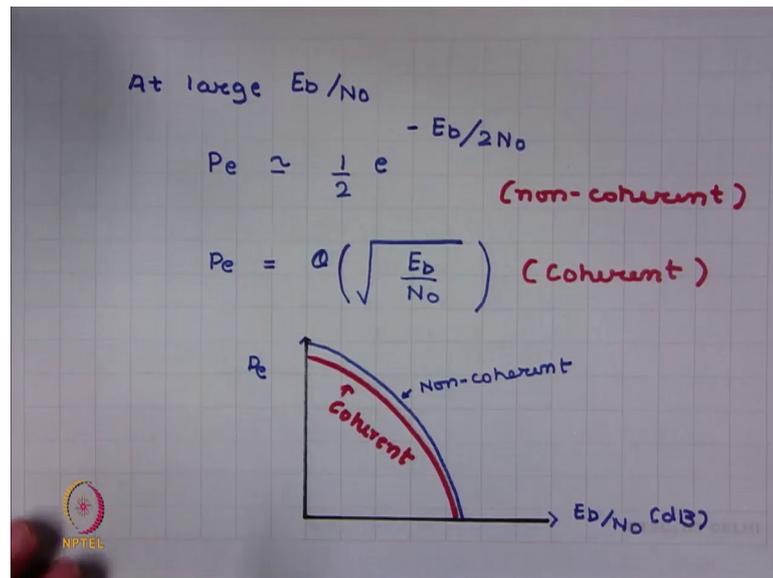
And thus my probability of error simply trims down to this value. So, this is the probability of error in non-coherent OOK systems. And we have seen in one of the lecture that probability of error in coherent systems is this. And there is not much to take between non-coherent and coherent OOK system probability of error. They look exactly similar specially at large E b N o's ok.

(Refer Slide Time: 49:00)
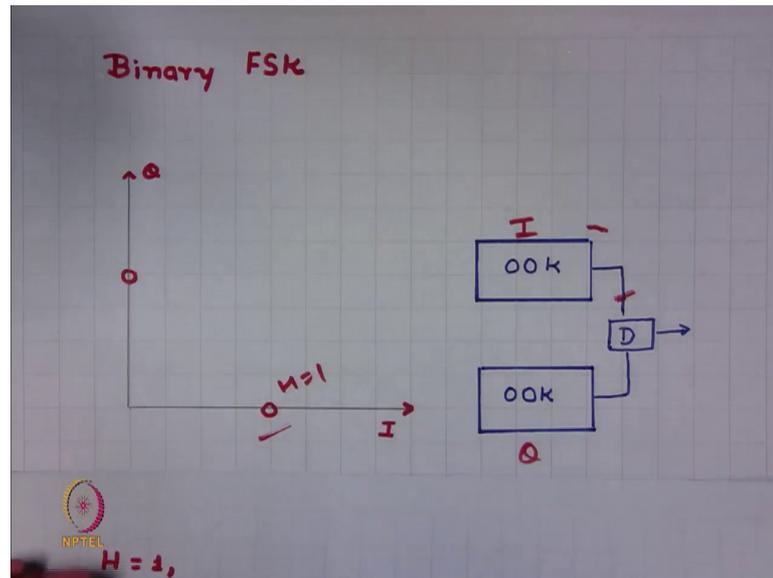
Now, let us quickly derive the probability of error in the case of binary frequency shift keying. So, in binary FSK, you have two real dimensions to worry about. You can think of it as the two OOK systems non-coherent one; one in I channel, one in Q channel. And then you have a decision device which decides based on the fact whether the output of this OOK system is larger or whether the output of this OOK system is larger.

For example, if this OOK system is in the I channel, and this OOK system is in the Q channel. And if here the output is larger, then the detector knows that this symbol must have been transmitted or at least it guesses so. So, now let us assume that hypothesis 1 is transmitted, and let us assume that this is the signal corresponding to hypothesis 1, that means, we are assuming that we have a higher output in the I channel.

(Refer Slide Time: 50:03)

$$H = 1,$$

$$f_{RI}(r_1) = \frac{r_1}{\sigma^2} e^{-(r_1^2 + u^2)/2\sigma^2} J_0\left(\frac{r_1 u}{\sigma^2}\right)$$

$$f_{RQ}(r_0) = \frac{r_0}{\sigma^2} e^{-r_0^2/2\sigma^2}$$

$$P_{e|H=1} = P(r_0 > r_1)$$

This is the likelihood of the random variable available at the output of the I channel if hypothesis 1 is transmitted. This is the probability density function of the random variable at the output of the Q channel if hypothesis 1 is transmitted. Namely here we will have a Rician PDF because the signal writes on noise and here we simply have Rayleigh PDF because there is no signal in the Q channel.

Now, when will there be an error, there will be an error. If the numerical value of this random variable exceeds the numerical value of this random variable, because then the detector will decide that hypothesis 0 is transmitted because the output of the Q channel is larger than the output of the I channel that is so simple.

There will be an error if hypothesis 1 is transmitted if the numerical value of the random variable corresponding to the Q channel exceeds the numerical value of the random variable corresponding to the I channel ok. Then what we have to do is we have to simply write down the joint PDF and we have to carry out this integration. So, r 1 can take any value between 0 to infinity. And if r naught takes a value between r 1 and infinity, of course there is an error. Remember that these two random variables are independent of each other and thus the joint PDF can be written as the product of the marginal PDFs which we do in here.

So, I have taken the PDF of this random variable r i and I have taken the PDF of this random variable r q, I have multiplied these two things and now I want to compute this integration ok. And using the trick that I have introduced that when you want to integrate this function, the output is simply this thing, and you have to replace this r naught with r 1 and you are done ok, so that is it what we have.

(Refer Slide Time: 52:23)

We have this function which remains exactly like this; we have this function which remains like this. And now when you were thinking about integrating this this function will be replaced. So, this function with d r naught will simply be replaced by e to the power minus r 1 square by 2 sigma square so, that is what we do in here we have this function ok.

Now, we make some trivial changes in variables. We assume that x is root 2 r 1 alpha is u by root 2. And then first we identify which terms in here are independent of r 1. We have this term independent of r 1, we shift this out of this integration, and then we get this function remaining to be integrated. And if you look at for 10 seconds, you would be surprised that this is a very simple integration, because this is formula of Rician PDF.

And if you integrate any PDF between minus infinity to plus infinity, you get 1, but for Rician and Rayleigh PDF because they can only take positive arguments. If you integrate this from 0 to infinity this should be 1, because it is a probability density function. And

thus we easily get rid of this integration and we simply get 1 ok. So, my probability of error given hypothesis 1 is transmitted is this.
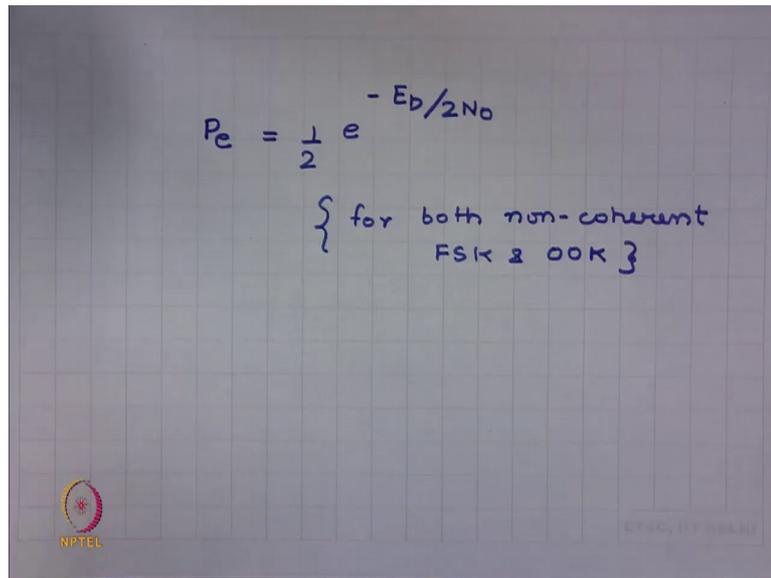
(Refer Slide Time: 54:17)

And now I replace the values of u at sigma, so that I get my answer in terms of E b N o. So, u square by 4 sigma square is 1 by 4 E s square, because u is E s sigma square we know that it is N naught by 2 E s. So, we get E s by 2 N naught. And E s is E b in case of frequency shift keying because every signal has E s energy average energy is also E s and there is one bit per signal, so E s is simply E b. So, probability of error given hypothesis 1 is transmitted is this ok, because u square by 4 sigma square is E b N o by 2.

And this corresponds to overall probability of error because we have seen this thing several times, because all signals are of equal energy they are at same distances from each other. There is a probability of this signals hit will be uniform, and thus the average probability of error is simply the probability of error when one hypothesis is transmitted ok. I will not belabour it any longer all right.
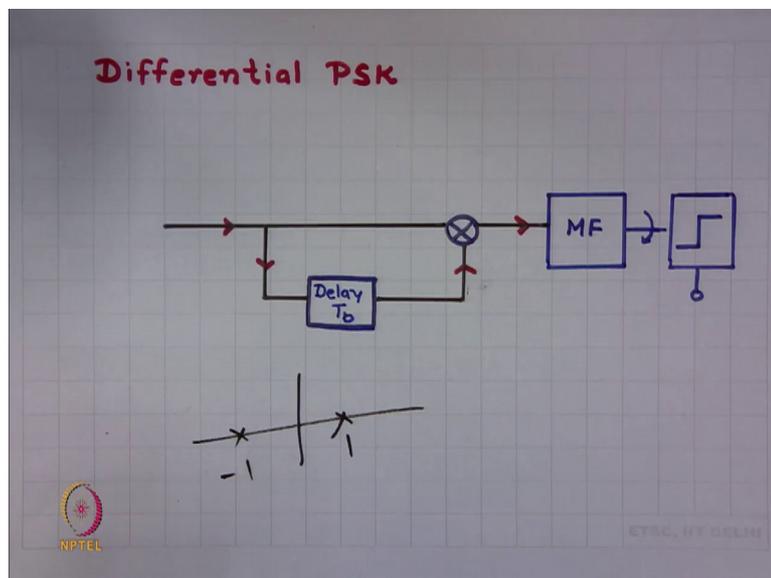
(Refer Slide Time: 55:45)

So, probability of error in case of both non-coherent FSK and OOK systems turn out to be same. And that was also the case in coherent FSK and OOK systems, the probability of error was same.

(Refer Slide Time: 56:06)

Now let us look at the last stuff one of the last, two more things to go. And one thing is thinking about this differential PSK systems, because they also set in this non-coherent systems. Well, when you are talking about the phase you can get confused that for example, if my phase is 180 degree far and if I have to receive this one in minus 1 and if

there is an squaring operation how would I be able to distinguish between these two signals.

So, differential PSK when we say that it is an non-coherent system, I simply mean that this does not require a carrier because it uses the previous bit as the carrier. So, you know in differential phase shift keying one of the pulses is delayed. And this pulse delayed pulse acts as a carrier for the incoming pulse because in differential phase shifting we are rather worrying about finding out what is the difference in the phase between the present symbol and the past symbol.

And so we use the previous pulse as the carrier for the incoming pulse ok, and that is why you do not use any carrier, but you draw your own carrier from the signal. So, that is the point that is why we call this as belonging to family of non-coherent systems.

(Refer Slide Time: 57:48)

This idea also we introduced before that DPSK is actually an example of orthogonal modulation scheme. So, it should not be surprising that it should have the same error performance as we got for FSK, but with one small difference. So, in these orthogonal signals two signals that we have in DPSK. So, refer to lecture on differential modulation scheme if you have forgotten this that we have these two orthogonal signals in DPSK. One let us say its u, u, one is u minus u; these signals are orthogonal.

And now what is the energy of each signal energy of each signal is two times energy of u and what is the energy of u energy of u is same as the energy of bit. Thus in DPSK energy per signal is two times energy per bit and that is why it gets a 3 dB improvement compared to FSK. In FSK, E s was simply E b here E s is 2 times E b, because in one signal you are having two bits and that is why this probability of error improves by a factor of 3 dB. So, you do not have this 2 that you had in FSK system that gets killed because of this fact simple.

(Refer Slide Time: 59:19)

Now, the last point of this lecture how can I think about this non-coherent multi level frequency shift keying why we are just talking about this multi level f s cannot about other modulation schemes, because practically this is the most used modulation scheme ok. Coherent FSK systems had certain problem because we have to track frequency and phase. And thus when we want to use a large number of carriers it is to live with a non-coherent system if it does not give a serious degradation performance.

Deriving the probability of error in this case is little bit tedious, but conceptually it is same. So, idea is exactly what we used in the case of deriving the probability of error in case of coherent FSK. So, remember we had this bunch of random variables. And we said that probability of being correct is when the numerical value of the random variable corresponding to the first hypothesis is maximum, then you are detecting the correct

hypothesis. So, the probability of being correct is that the probability of the numerical value of the first random variable should be largest and we say that thing in here.

So, if the numerical value of the first random variable is r one the numerical values of all other random variable should be smaller than r 1 that is it that is it conceptually. And then you write what is the probability of the first random variable picking up on a value around r 1 which is this one. So, use the PDF of this random variable r 1 multiply this with d r 1 to find out what is the probability of this random variable sticking on a value around r 1, and then you find out what is the probability of all these random variables taking value less than R 1.

So, you pick on the probability density function of one random variable which is Rayleigh. If we want to find out what is the probability of this random variable taking a value less than r 1, you integrate this PDF from 0 to R 1, and because we have m minus 1 terms in here we multiply this with m minus 1 times, and we integrate the whole thing. You can easily do this in mat lab and can arrive at the expression of probability of being correct. This someone has done for us and solved it.

(Refer Slide Time: 61:59)

So, probability of being correct you can prove turns out to be this. Of course, it is a very cumbersome formula which I will not advise you to remember. But one thing and one message that you can draw from here is if you work this out, you will see that the probability of error in non-coherent multi level frequency shift keying systems is very

close to that of multilevel coherent frequency shift keying systems. And this practically is very useful, because now you do not have to track the phase and frequency of m carriers if m is pretty large ok.

So, with this, we have completed the discussion on non-coherent systems and we will move to some other interesting stuff in next lecture.

Thank you.