Principles of Digital Communication
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Lecture – 24
Modulation Spectral Description of Sources using Markov Chains & Cyclostationary Random Processes

Good morning welcome to the next lecture on Modulation. So, in the last lecture what we did we investigated the power spectral density of various line coding schemes. And, in this lecture again we would be doing that what we would use another tool to investigate power spectral density and this tool is known as the Markov chain.

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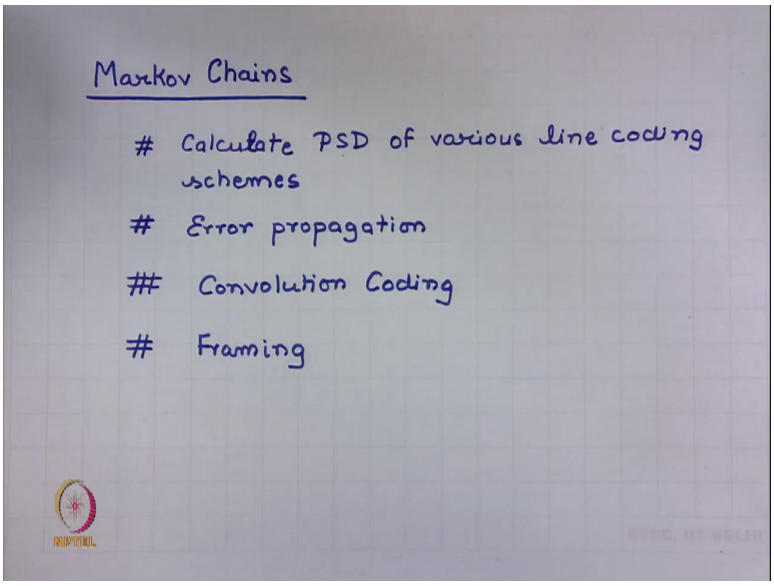

So, today’s lecture is about Markov chains and towards the end of lecture we would also study about what is known as Cyclostationary Random Processes ok. So, Markov chains are very useful because we can investigate power spectral density of various line coding schemes and also it is used in calculation of error propagation, it is very useful in the analysis of convolution codes and framing structure. So, this is typically the advantage of Markov chains; sadly in literature it has not been given its due importance particularly in the context of digital communications.

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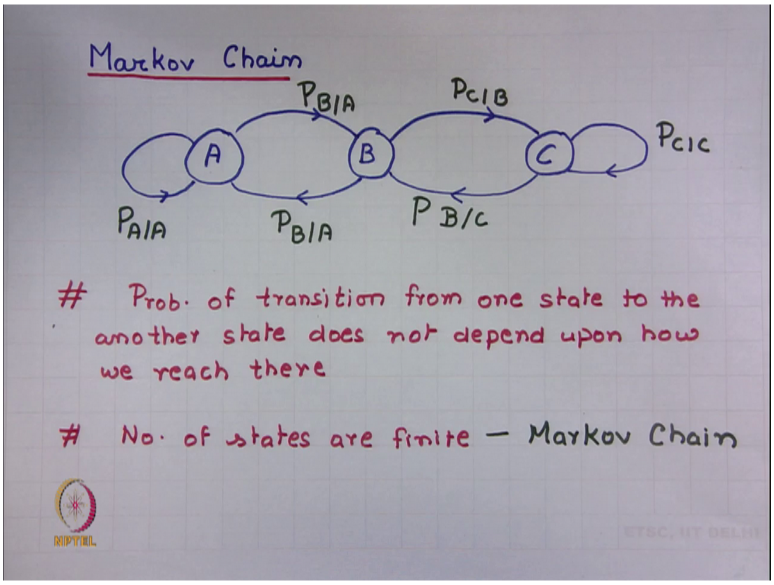

So, let us introduce Markov chain, I believe that you must have had some introduction of Markov chains in the courses on probability and so on so forth. So, we will not like to dig deep in this Markov chain. I will give a kind of very superficial introduction to Markov chains. So, let us first talk about the Markov process, a Markov process consists of various states. So, as you can see in this picture we have three states A, B and C. So, one thing is states and the second thing is the probability of transition. So, probability of going to state B from state A; so if you are in the state A with what probability you can move to state B.

So, this is known as probability of transitions. So, if you have some states and if you have these probability of transitions you are good to go and we call this is Markov process. Particularly, if this probability of transition let us say the probability with which you go to state C, if you are in a state B does not depend upon how you have reached B ok. So, it does not have any memory whatsoever, the probability of going to C from state B is independent of how you have reached B and this is particularly a Markov process ok.

So, Markov process we understood it is consisting of some states and we also need to know some probability of transitions. And, the probability of going to one state from another state does not depend upon how you have reached that it state ok. What is Markov chain? So, this was Markov process, a Markov chain is a Markov process where the number of states are finite or countable in general. So, if the number of states are countable we call this as Markov chain ok.

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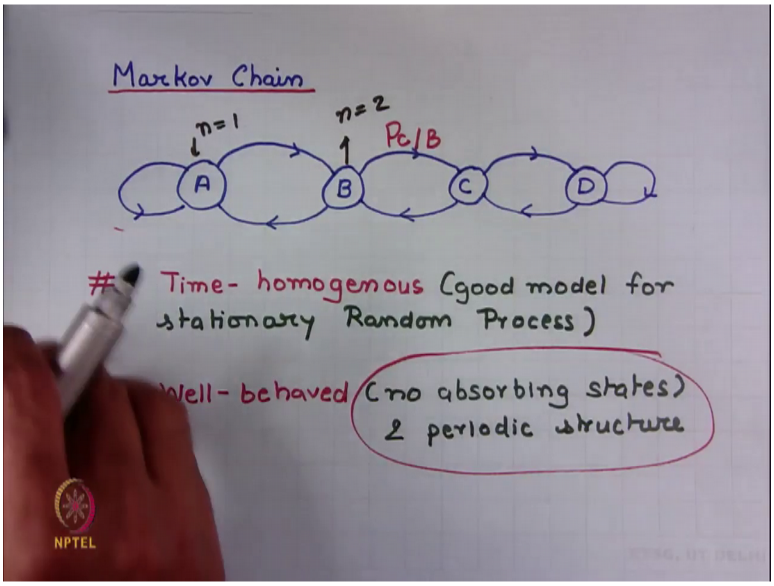

So, I have a Markov chain for you. So, it is a Markov chain because, number of states are only 4 finite. What we also assume is this Markov chain is time homogeneous, meaning that the probability that you go to state C from state B does not depend upon the time at which you are looking this Markov chain; so it is independent of time. So, these probabilities are kind of constant with respect to time and because, we want to model a stationary random process time homogeneous Markov chains are really a good model.

Also what we usually assume is these Markov chains have well behaved; that means, there are no absorbing states and periodic structures. This is slightly mathematical you will not be talking about this a lot, but what we assume that are Markov chain is good. It does not have absorbing states meaning that there is no particular state in which it gets a stuck right. So, it really moves across all states and also there is some probability that it can stay in that state ok. So, we have those kind of Markov chains; in short it should be well behaved.

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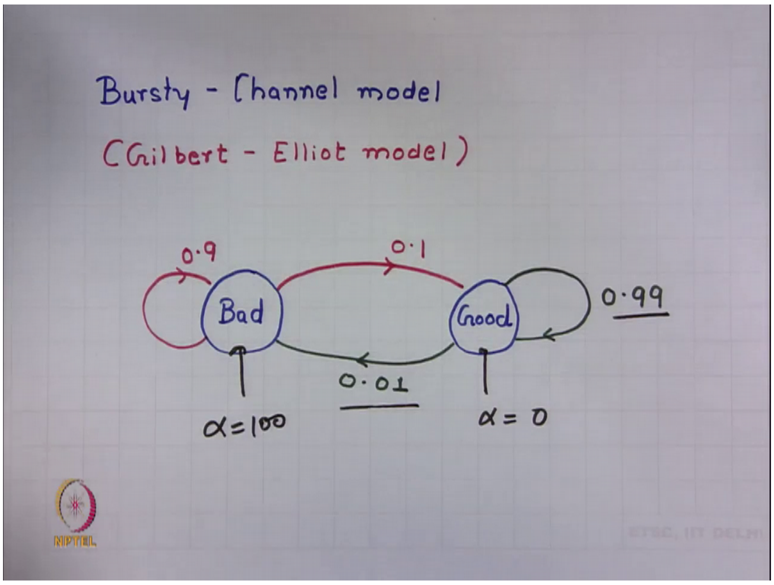

It is best to study Markov chains through examples and this is an example that we use to discuss Markov chain, we call this as Gilbert Elliott model and this model emulates bursty channel. So, sometimes in research you want to create a bursty channel and or a bursty traffic and this is a good way to do this, it is very simple.

So, we have said that there are two states is a good state and a bad state, two states good and a bad and good means that the channel is in good condition. That means, what possible it can mean that it has 0 attenuation and if it is in bad condition it has some very high attenuation ok. So, one way to understand good and bad channels can be based on the attenuation that the channel offers; attenuation means the loss that your signal will experience when it moves through the channel.

So, if the channel is in good state it continues to be in good state with a very high probability ok, with a probability of 0.99. It can also go to a bad state with a small probability, here we say that the probability is 0.01 and you should see now that this probability should sum up to 1. Meaning that if it is in good state it would continue to be in good state or it would move to bad state, this probability is 1 ok.

Similarly, if the channel is in bad state it continues to remain in bad state with a probability of 0.9. And, if this channel is in bad state it can move to a good state with a probability of 0.1. And again these probabilities should sum up to 1, meaning that if the channel is in bad state it remains in a bad state with point nine probability and it goes to a good state with 0.1 probability.

So, very simple Markov chain model to emulate a bursty channel ok. What I have not said in Markov chain is, that we also assume that this Markov chain is a discrete time Markov chain; meaning that if I am in this state as let us say time instance 1 I can go to this state only at time instance 2. So, I am observing this Markov chain at discrete instances of time. So, we want to investigate power spectral density and power spectral density can be obtained as I have said by taking the Fourier transform of autocorrelation function.

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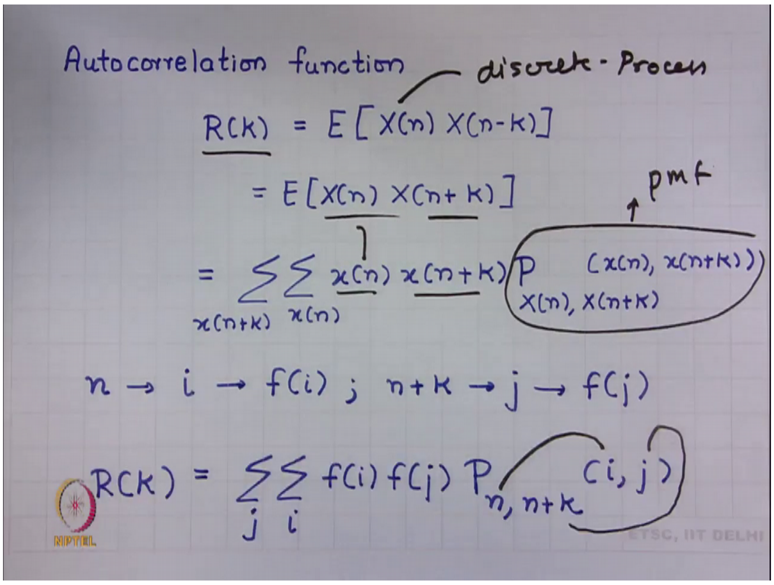

Now, let us revise what is a autocorrelation function what discrete process. So, X we assume as a discrete process and so, you take the sample of the process.

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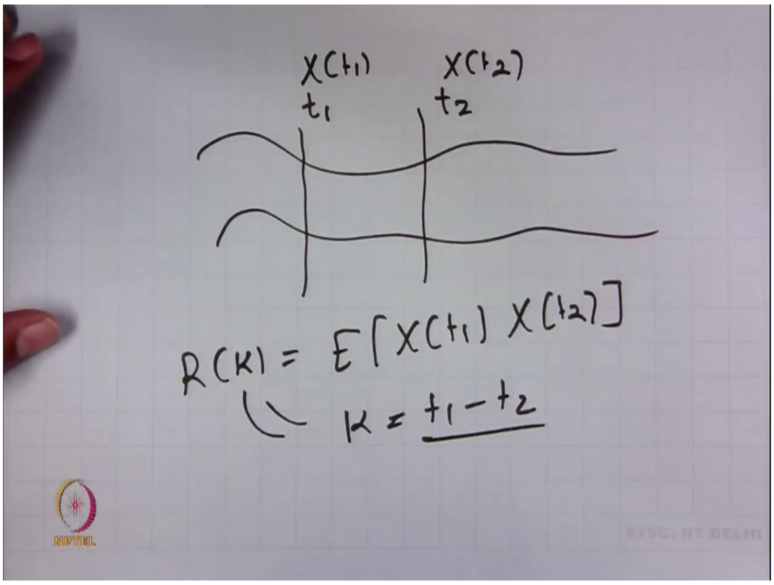

So, if I have a process in continuous time domain, it is easy to the continuous time process. How we took autocorrelation function? We could sample this at let us say time instance t 1, I sample this as time instance t 2, I collect two random variables X t 1 and X t 2 and autocorrelation function is defined as taking the expected value of these two time instances.

And, if this processes is stationary the autocorrelation function would depend just on the timing difference. So, k in this case would be t 1 minus t 2 ok. So, this is a simple idea this is how we took the autocorrelation function in the case of continuous time random processes. Similarly, we define the autocorrelation function for a discrete time random process.

Here you sample the process at time instance n and at time instance n minus k and you take the expected value. And this autocorrelation function being an even function whether you define it like this or like this it does not matter, it is one and the same thing; I will like to use this definition it is more convenient. So, when you sample you get this random variable, this random variable takes in a value x of n, this random variable takes in a value x n plus k.

So, these are the random variables, I said when you are taking the expectation you need to have the values. These are the values of the random variables and then we define a probability mass function. This is the probability mass function, all these things we have done I am just revising this. So, if you have forgotten about this refer to the lectures on random processes where, we have done all these things.

So, what we are saying is we are collecting two random variables X n and X n plus k and we are asking the question what is the joint probability mass function? This is a probability mass function because we are in the discrete domain it is not a probability density function. So, meaning that I am asking what is the probability with which X n takes in a value numerical value x n and X n plus k takes in a numerical value x n plus k at same time ok.

And, then we sum this up for all values of x n and x n plus k, this is how we took the autocorrelation function. In this case for Markov chain the idea is very similar only the notations have changed. So, let me introduce the notations.

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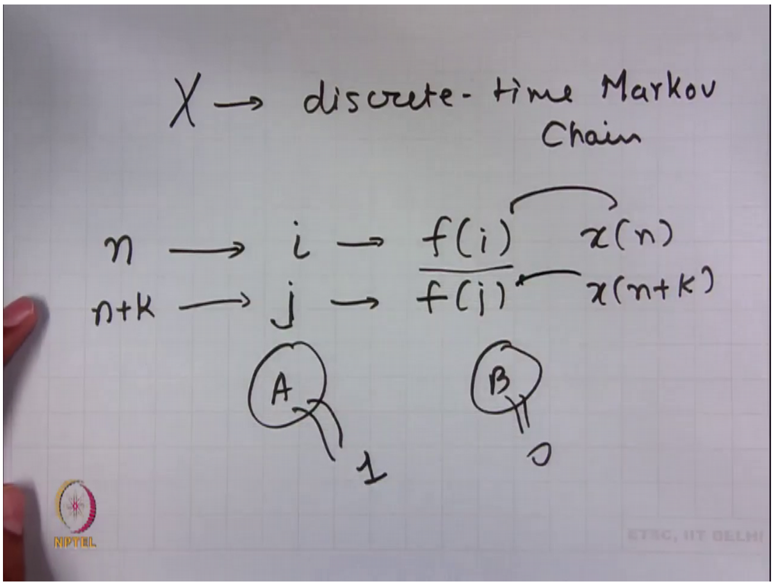

So, what we would do is we would have a discrete time Markov chain and at time instance n, when I look at this Markov chain let us assume that I get into state i. What I see is that I am in a state i and when I am in the state i that state i has a value numerical value f of i. So, for example, if I have two states A and B to this state A I can assign a numerical value 1 and to this state I can assign a numerical value 0.

So, this f of i is the numerical value of i and so, let me repeat. So, I am looking this Markov chain and the n th time instance and I see that it is in the state i and if it is in the state i it takes a numerical value f of i. So, x of n is in fact, f of i I look down this at n plus k, I assume that this is in the state j and when it is in the state j it takes in a numerical value f of j. So, x n plus k is nothing, but f of j right. So, so simple ideas. So, what I am saying is x of n is f of i meaning that I look down this Markov chain at time instance n, it is in the state i.

When it is in the state i it takes in numerical value f of i, when I look down this as n plus k I find that it is in the state j. When it is in a state j it takes in a numerical value f of j and this pmf is replaced by this notation which means that I am looking down this Markov chain at time instance n and n plus k. What is the probability that when I look down this at time instance n and n plus k, I find this to be in a state i and a state j. So, at n I find it in a state i and in n plus k I find it in the state j. So, what is this probability and then I need to sum it over for all values of i and j. So, this is a new notation idea is exactly same, there is nothing different just a new notation because it is more convenient here ok.

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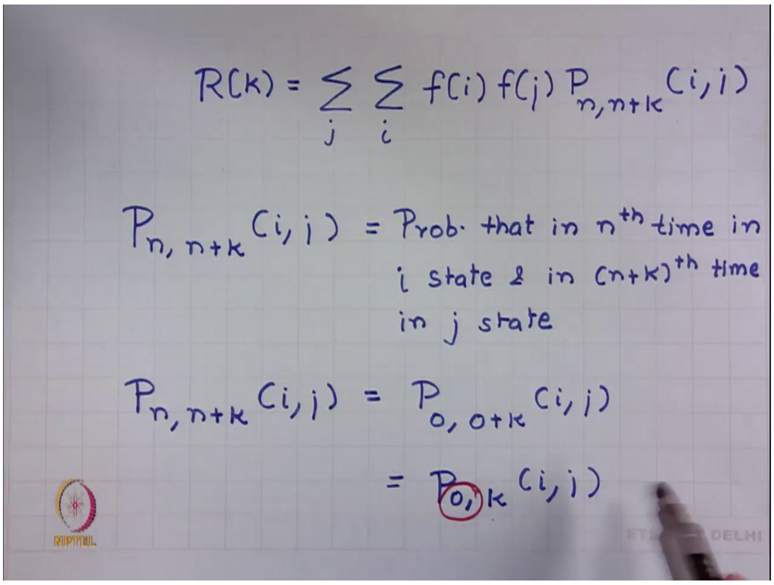

So, let me summarize the autocorrelation function if you have to find for a Markov chain you need to run this summation for all values of i and j. Let me remind what is this notation meaning, it tells me that probability that in nth time I am in a state i and in n plus k th time I am in state j ok. I can simplify this because, if it is a stationary process it does not depend upon the absolute value of time instance. So, this probability should also be the same as this probability where, n I can conveniently choose to be 0. This should be just a function of the difference in the timing instance rather than on the absolute values of time ok.

So, I can replace this with 0 and 0 plus k right, same as we did for stationary random process and 0 plus k is then simply k. So, again as we said for a stationary process this is kind of redundant information, I can also compactly represent this thing by this.

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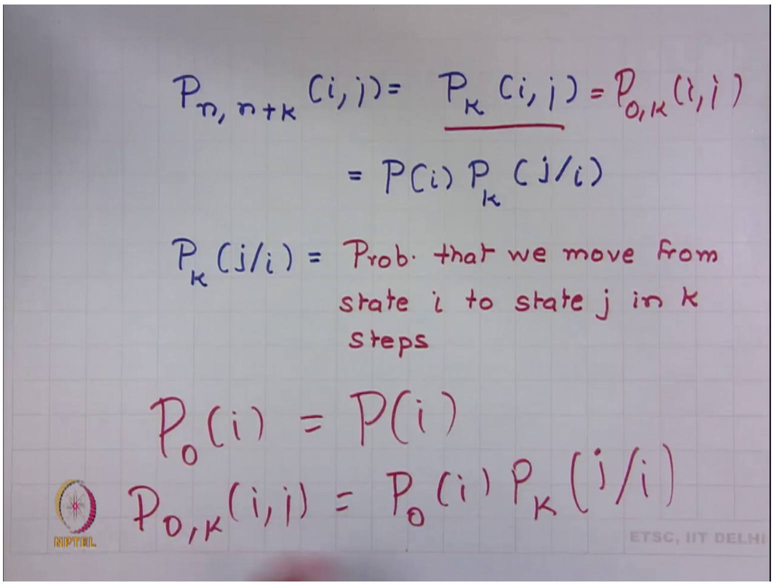

So, I have removed 0 and this should be interpreted in a mind you should interpret this as this. And, what it means is probability that if you look down this Markov chain at time instance 0, you are in state i and if you look down this Markov chain at time instance k you are in a state j ok; interpretation of this is this notation we use this.

Now, what is this probability that you at time instance 0 you are in state i and at time instance k you are in state j. Now, probability that at time instance 0 you are in state i is same as probability that you are in state i, because the probabilities should not depend upon at what time instance you are looking the process because, it is a stationary process. So, this P 0 i the probability that you are in state i at 0 times things should just be P of i.

So, what we were finding is probabilities 0 k i j, this I can write probability that you are in i th state at time instance 0 and then at kth state you are in state j given that you were in state i. So, this we can compute in form of conditional probabilities as we have already explained in the random processes. Again I repeat the probability that you are in a state i at 0th time and in state j and k th time is just the probability that in state i in 0 th time. And, probability that you move to state j from state i in k th time units and what I am saying is probability that you are in the state i at 0th time instance is nothing, but probability of i. So, I can replace this with this thing ok. What is the meaning of this? It says that probability that we move from state i to state j in k steps because, time here is discrete ok. So finally, we can simplify that expression.

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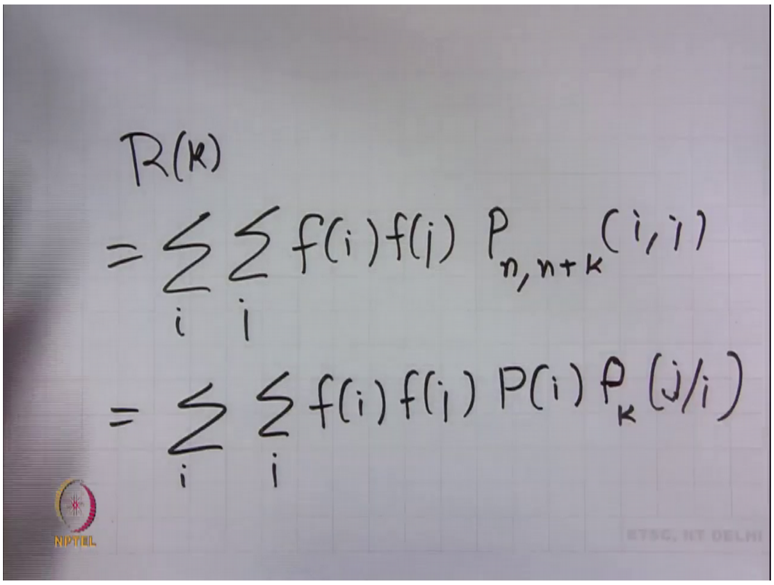

So, we said R of k a summation i summation j f of i f of j P n n plus k i j can be replaced by ok. So, instead of this we can use this.

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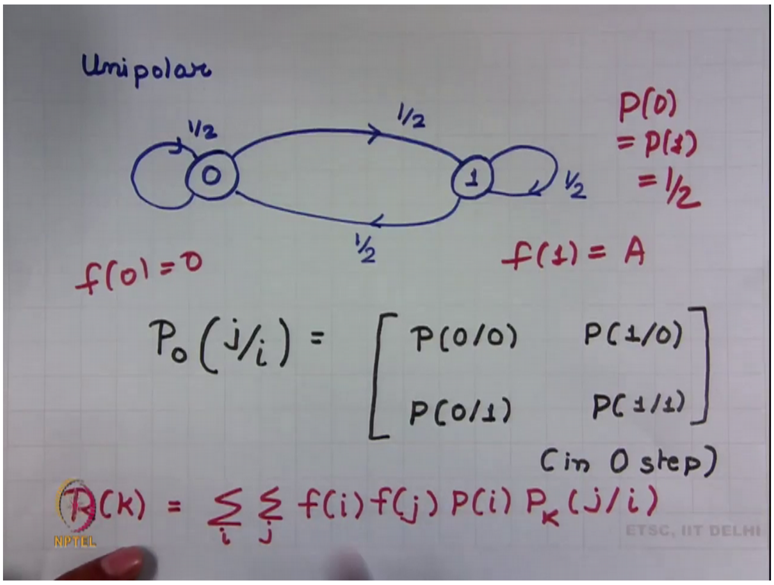

So, let us try to understand this Markov chain through some examples and we will do 2 examples, the first example we will start with unipolar signaling mechanism.

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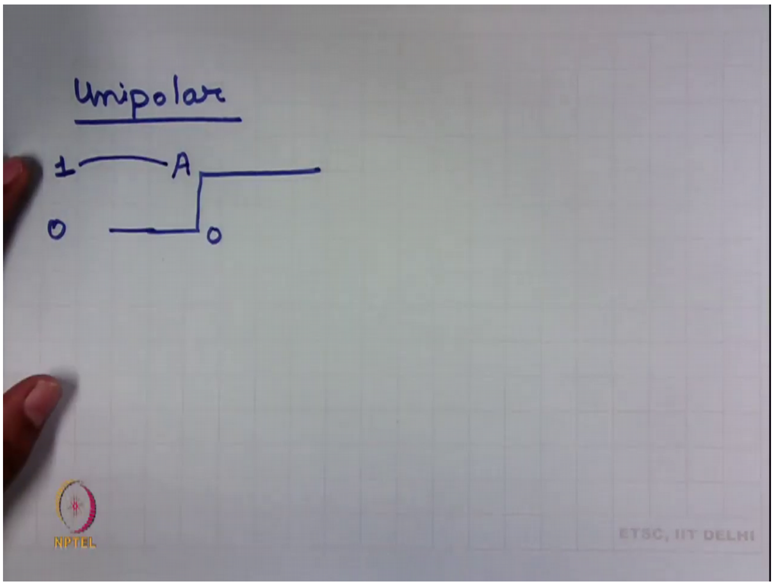

So, let me revise what we set for unipolar mechanism is that we have two states. So, we have 0 state and 1 state, 0s map to let us say 0 voltages and 1 is map to a voltage A. So, unipolar we have two states ok. So, now let us look at this Markov chain, here also we see that we have two states we have 0 th state and 1 th state and the value are in the 0 th state. So, in this state f of 0 the value of the 0 th state would be 0 and the value of the 1 th state would be A and also what we see is that because there are the two states and 0 is equiprobable as 1. So, probability of 0 th state is same as probability of 1 state and this will be half.

So, remember what we are trying to find is this autocorrelation function and in autocorrelation function all these things are known to us, the only thing that is unknown is finding out this matrix ok. So, what does this represent is probability of going from i th state to j th state in k steps alright. So, let us first find out what is this matrix when the number of steps is 0; that means, I want to go from i th state to j th state in 0 step. So, there are two states; so, I can go from 0 th state to 0 th state, I can go from 0 th to 1 state, I can go from 1 th state to 0 th state and 1 state to 1 state. And, these represent the probabilities of these transitions and I want to do this in 0 step ok; without taking any step I want to change stages.

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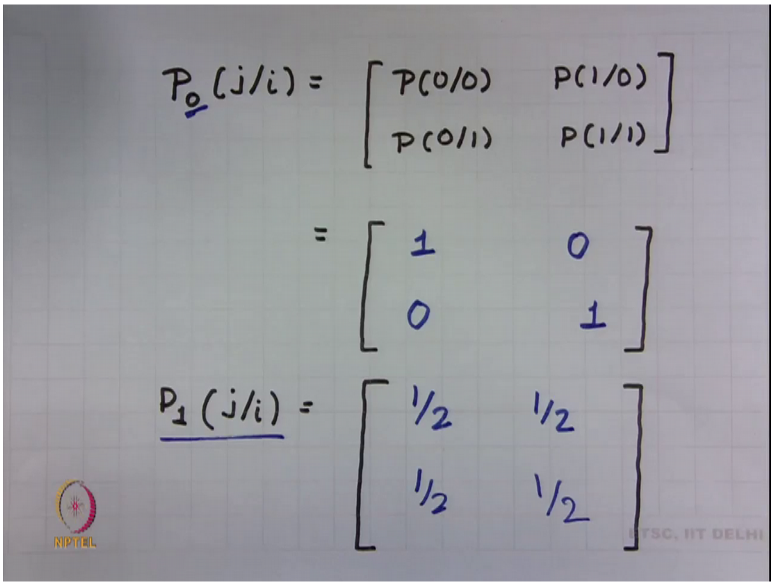

So, let us try to find out what will be these transition probabilities. So, if I am interested in going from 0 th state to 0 th state and I cannot move; what is the probability for that? The probability is 1, because I am not taking any step I will remain in the same state. Similarly, the probability of going from 1 state to 1 state if I am not moving at all is 1. What is the probability of going from 0 th state to 1 th state? If I am not moving that probability is 0. Similarly, this probability will also be 0 hence, P 0 j given i will always be a diagonal matrix. What is this P 1 j given i? So, I now want to move from i th state to j th state and I can take 1 step.

So, what is this probability let us look at the chain. So, let us say if I want to go from 0 th state to 0 th state and I can move 1 time this probability is half right. So, if I am in this state I want to remain this state and I have just 1 step to move, I can get into this state with the probability of half. So, this is half similarly, from 0 th state to 1 th state you can go with a probability of half if you have 1 step to take and similarly you can fill in by using the same logic and this will also be half and half; so that is done.

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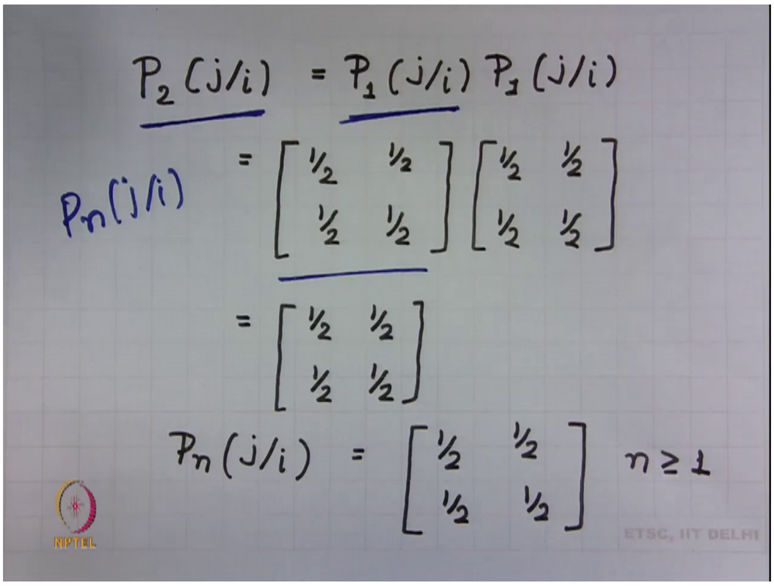

Now, let us try to do this for P 2 j given i and you can work this out yourself that computing this will be easy once you have obtained this. Because, this matrix can be simply obtained by multiplying this matrix with this matrix; if you wanted to find P 3 j given i you need to multiply this again with P 1 j given i and that is why we love this right. Because, once you have obtained this calculating this P n j given i is pretty straightforward, it is just matrix multiplication.

So, to find P 2 j given i you just have to multiply this matrix that we obtained with another such matrix and what you end up with is again the same matrix. So, here we have just done the matrix multiplication and you can prove it to yourself that P n j given i will again be a similar matrix, if n is greater than or equals to 1; so we have obtained these transition probability matrix.

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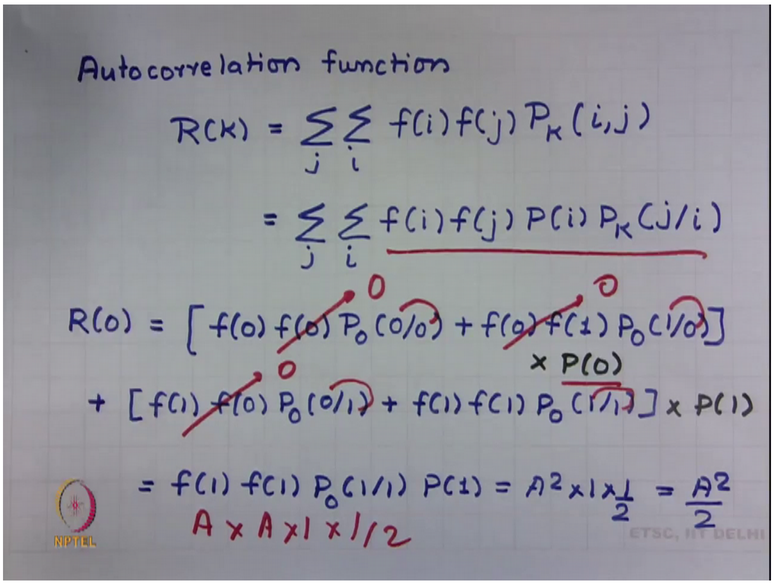

And now the job is to find out the autocorrelation function. So, let us work with this expression, we have already said that this is same as this. So, there are two states so, I have 0 and 0 and let us assume that I am first in the state 0 so, P 0. So, I am in state 0 now, if I am in state 0 I can move from 0 to 0 or I can move from 0 to 1 ok.

So, this thing and then I have to multiply this with the values when I am in state 0 and state 1; similarly I can be in state 1 and if I am in state 1 I can move from state 1 to 0 or state 1 to 1. So, this term right now what is the value of state 0? The value of state 0 is 0 hence, this term would be flag 0; we do not have even to think about this because f of 0 is 0. So, whatever else I have it will not make any influence. So, this is flag 0, this is flag 0, this is flag 0 because f of 0 is 0.

So, what I remain weight is f of 1 into f of 1 P 0 1 given 1 into P of 1. So, f of 1 we know is A the value when I am in state 1 is A multiplied by A P 0 1 given 1. So, f I want to move from state 1 to state 1 and I cannot move at all that probability will be 1. And, P of 1 the probability with which I stay in state 1 is half because, both the states are equiprobable and thus I get R of 0 as A square by 2. And, this is the similar result that we obtained before right and now I am working out this result with the help of Markov chains.

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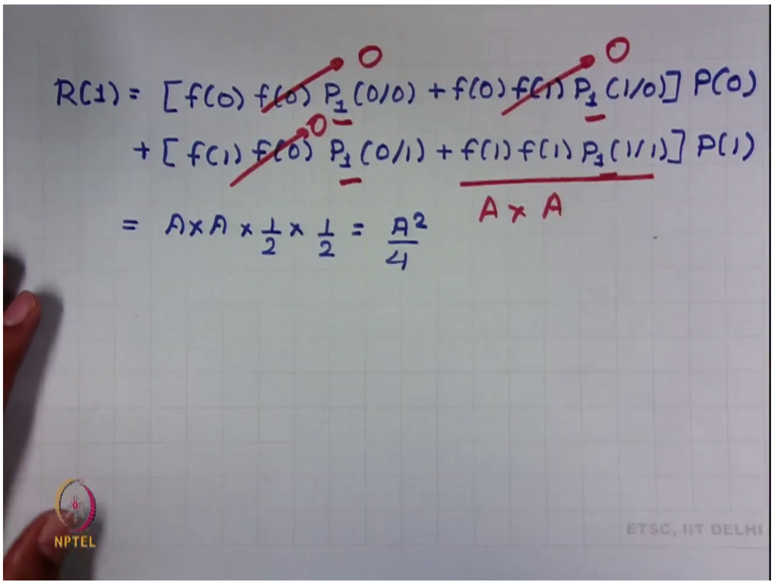

Similarly, let us try to find R of 1. So, what is the difference between R of 0 and R of 1 not much. So, instead of these P 0 0 given 0 now we have P 1. So, this has changed everything else remain same and same as before this term will have no contribution, this term will have no contribution, this term will have no contribution. What is the contribution?

So, f of 1 is A f of 1 is A P 1 1 given 1, let us check this out we have already obtained P 1 1 given 1 this is half; so half times P of 1 which is half. So, I end up with A square by 4 and this is exactly what we derived before as well here I am just using Markov chain. So, why I am teaching this Markov chain is because the beautiful thing that happens here is, once you have obtained this transition matrix you can calculate all transition probabilities involving n steps.

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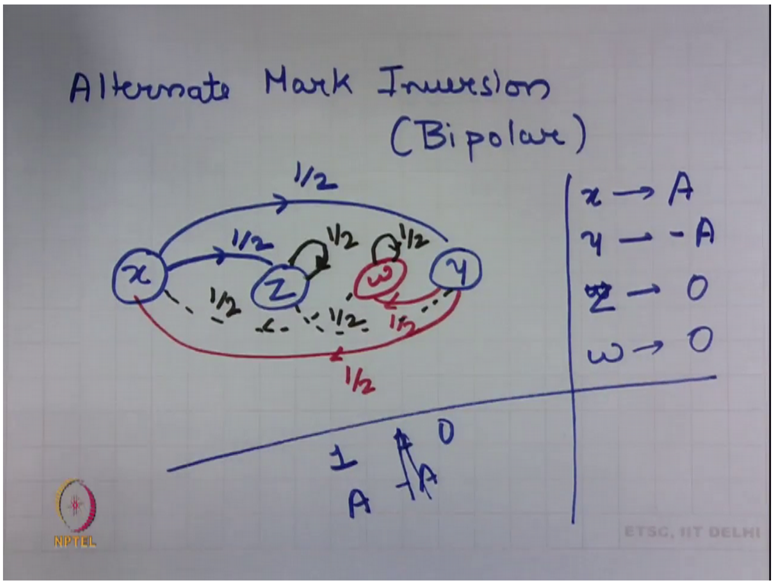

To make you more confident let me work out a second example with alternate mark inversion scheme. I will not do polar, polar is very similar to unipolar you can work this out, this is more interesting by polar. So, let me assume that I have four states here and I will explain why I am assuming four states.

So, let me assume I have a state x and the value when I am in state x is A, let me assume that I have state y and the value when I am in state y is minus A, let me assume the value of state z is 0 let me assume that the value of state w is 0. So, let me assume that I have these four states. So, remember in the case of bipolar signaling mechanism the main idea was that if you have received 1 and if you have chosen the voltage level of f to represent 1; if you get 1 again then the voltage level that you have to choose should be negative of what you have chosen before.

So, you can now not choose A, but you can only choose minus A and that is the main difference between alternate mark inversion and unipolar signaling skip. So, let me assume that I am state x; if I am in a state x; that means, I have received 1 and I have chosen the voltage level A, now my next bit can also be 1 with probability half. So, with probability half I can go to state y ok, if I have received 1. So, I can go to the state y or what could have happened is instead of this one I could have received 0 again with probability half. So, with probability half I can go to state z. Similarly, if I am in a state y with probability half I can go to state x or with probability half I can go to state w. Now, suppose I am in a state z, what can happen? I can get a 0, if I get a 0 I can live in the same state, I can live in the same state with probability half.

If I get 1 I cannot go to x, but I can only go to state y with probability half. Similarly, if I am in a state w I can remain in state w with probability half and if I receive 1 I can go to state x and that is why I need four states instead of three state. Because, state z and state w are different because, if I am in a state z and I get 1 I can only move to state y because, I have reached to this state z from state x; that means, I must have received 1 and 0. So, if I get a next one I can only move to state y and that is why these state z and state w are different. So, this is the Markov chain representation of alternate mark inversion scheme; once you have sorted this out finding autocorrelation function will be easy.

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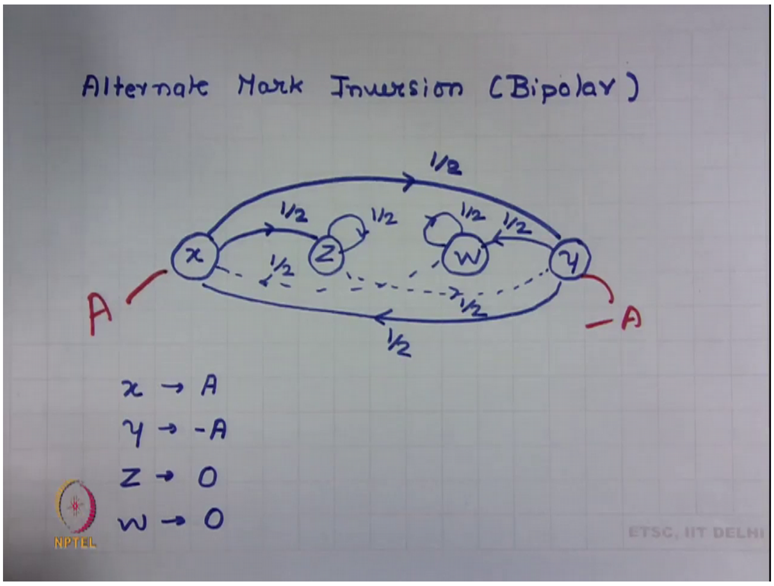

So, I have also drawn a neater diagram for this alternate mark inversion scheme. So, four states this has a value A, this has a value minus A, these two states has value 0.

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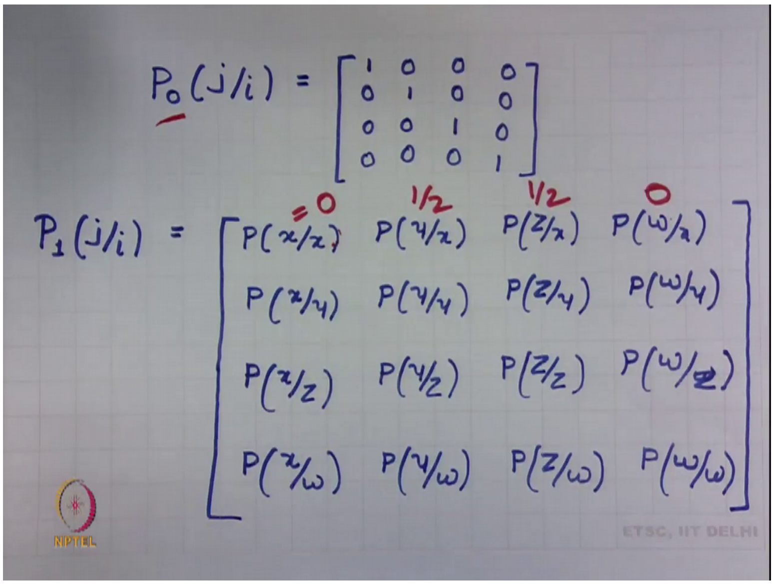

And I leave it to you to sort this out yourself only I will provide some hints. So, P 0 j given I will be a diagonal matrix same as before nothing changes there, if you have no steps to move you can only remain in the same state. P 1 j given i by definition is was the probability that you can move from state x to state x, y is there w and so on so forth in 1 step and by looking at this Markov chain you can fill this up let us try one. So, let us say I am interested in finding what is the probability that I can go from state x to state x and I have just 1 step to make.

So, state x to state x, but that is not possible in 1 step because in 1 step I can either move to state z or state y I cannot come back to state x. So, this probability is 0. What is the probability that you can go from state x to state y in 1 step state x to state y and that probability is half. What is the probability x to z in 1 step, this probability is half. What is the probability x to w in 1 step, x to w no it is not possible right. I can take two steps to reach w, what in 1 step either I can go to state z or to state y so, this is also 0.

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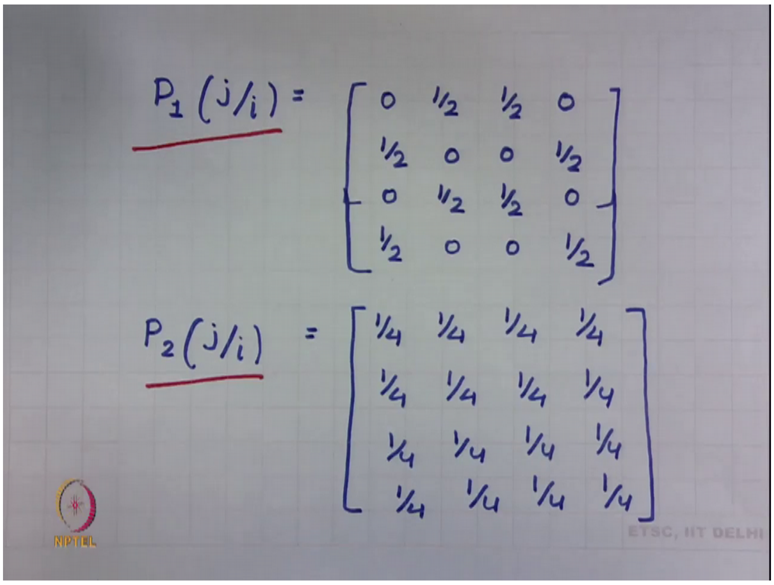

You can work this out yourself and complete P 1 j given i matrix, you can obtain these numbers. If you have got this then rest is easy you can obtain P 2 j given i which is obtained by multiplying this with itself and you get such a matrix do it, get yourself familiar.

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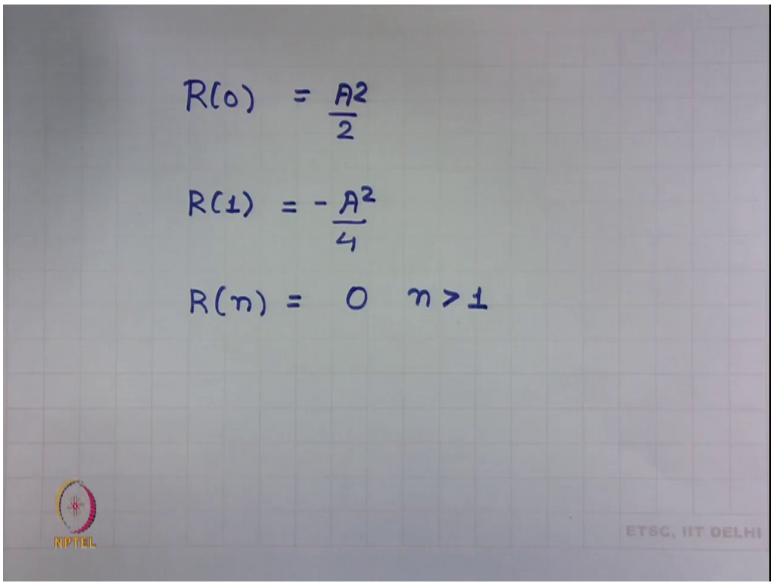

And then also find out the values of R 0 R 1 using the same approach as we have done in the case of unipolar signaling scheme. Doing it yourself will give you more confidence than me doing it for you. I am just tabulating the results that R of 0 will be A square by 2, R of 1 will be minus A square by 4, R of n will be 0 for n greater than 1 with this more or less we have completed the analysis of Markov chain. So, I remind you this Markov chain is pretty useful if you are thinking about proposing, let us say new line coding scheme or if you want to analyze the performance of a very complicated line coding schemes and you cannot calculate these R 0s R 1 and R 2 by hand.

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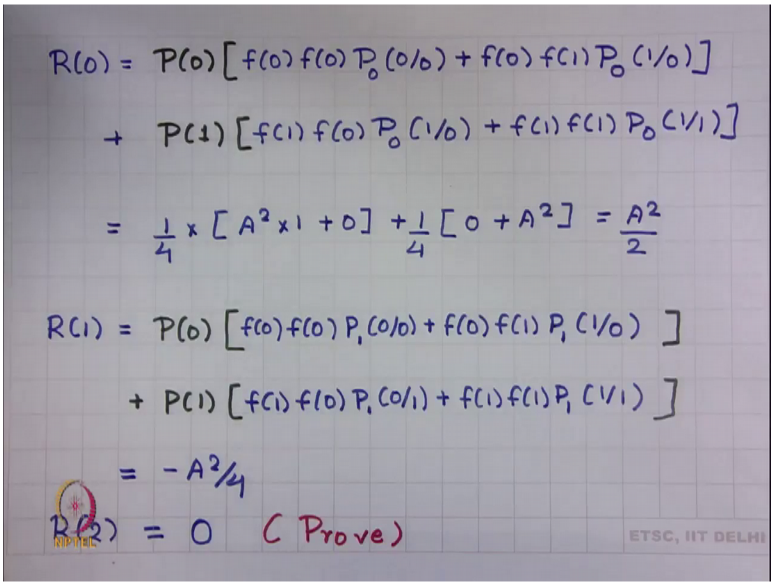

So, in computers you can find these autocorrelation functions by writing a MATLAB program ok. So, this would be pretty useful particularly in the context of research; let us move to what is known as non-linear modulation schemes.

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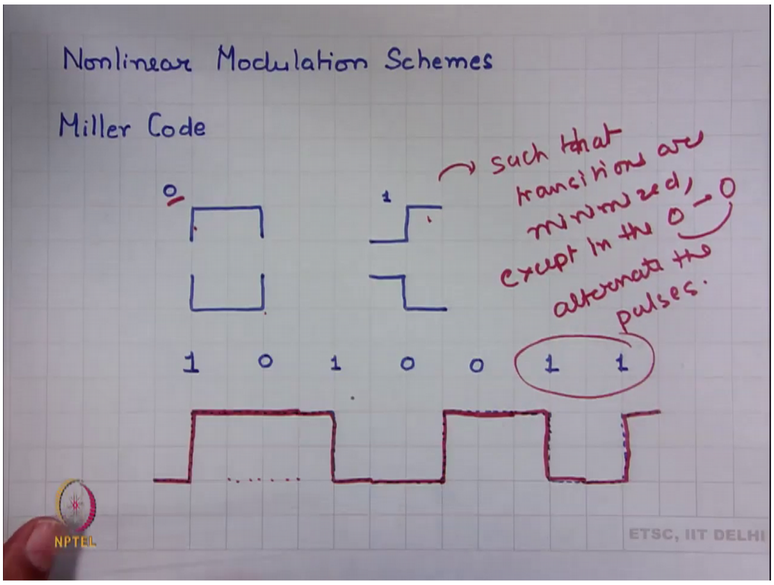

So, till now we have been finding or investigating about the linear modulation schemes which use the same pulse shapes for all sequences; here what we are saying is we use different pulse shapes. So, for 0 we use either this or this pulse shape and for 1 I use this pulse shape and this pulse shape is different from this pulse shape. And, thus it is an example of a non-linear modulation scheme.

And how do I choose this pulse shapes? The pulse shapes are chosen such that transitions are minimized. So, whether to choose this or this depends upon what minimizes these transitions except in the case of 0 followed by 0, where you alternate the pulse shapes. So, if you have chosen this next for next 0 you need to choose this if they are coming consecutively.

So, let us see with an example. So, 1 let us start with choosing this now, for 0 I could have either choose in this or this, but this 1 would minimize my transitions. If I would have chosen this then this would have been bad, because then I have a transition. Then for 1 I choose this instead of this because, it minimizes the transitions, for 0 I choose this again. Now, because this 0 is followed by 0 now if I have chosen this then it should come with a pulse shape this and this is done. So, that you do not lose synchronization. So, long trains of 0 does not reduces the transparency of your line coding scheme. Next 1 so, I have this for next 1 I have this, as you can see long trains of 1's would not impact the transparency because, you have transitions ok. So, this is a example of a Miller code; it is a very simple and very interesting Miller code.

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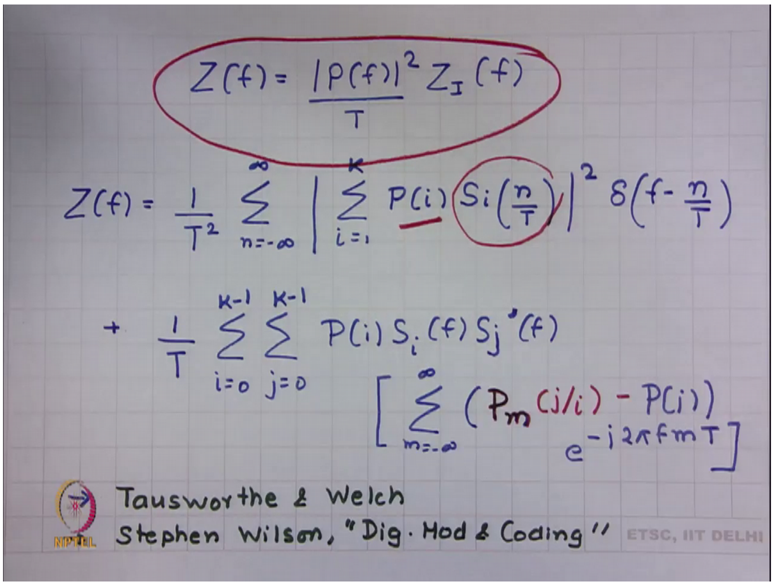

If you have non-linear modulation schemes then are formula could old formula to investigate power spectral density, cannot be used anymore. You need to use a very complicated formula for calculating the power spectral density which depends upon the probability that you take a pulse shape i. If you have let us say 2 pulse shapes this gives the probability with which you choose one of them, this is the spectrum of a pulse shape. This is in pulse function, this I have already introduced. And what is this?

This is interesting these are the transition probabilities that we have already analyzed using Markov chain. So, we will not derive this I am just bringing this out before you; so, that you know that such a formula already exists. So, if you have a need this just refer to this paper by Tausworthe and Welch for the proof and understanding of this formulation. Or, you can refer to the book by Stephen Wilson on Digital Modulation Coding where, again you find the proof and explanation for this result ok. Almost done with power spectral density except this last point where I talk about this cyclostationary random processes.

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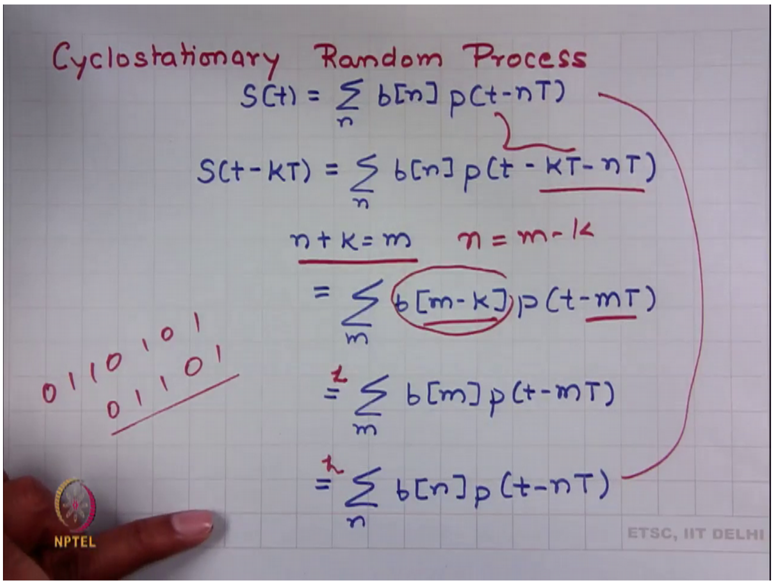

So, what are these and why we are discussing them in the context of power spectral density let us see. So, let us assume that we have a waveform S of t which is given by this. So, again we are confining ourselves to linear modulation scheme and let me ask a question that if I delay this waveform by k T units, T is similar to this T what happens let us see. So, if I delay this with k T, the t gets changed to t minus k T minus n T then I do a change in variable, I substitute n plus k as m. So, n becomes m minus k here, I have m minus k and this becomes p t minus m T, then I have to run this for m. Now, if you see this let us concentrate on this, now if you have a long terms of 1's and 0's and if you delay the sequence you delay the sequence.

So, if I have long terms of 1's and 0's and I let us say delay the sequence; now statistically this sequence should be same as the sequence right because, we are anyway assuming the sequence to be stationary. So, if I delay the sequence by any time units there would be no statistical difference between my sequences ok. So, for this b m or b n I said we always assume stationary because, this is a good model to start with. Now so, this thing is same as this thing is from the point of view of statistics. So, we say that they have the same statistical properties and I can also replace again this m with n because I want to use n instead of m. So, this is statistically same as the thing with which we started.

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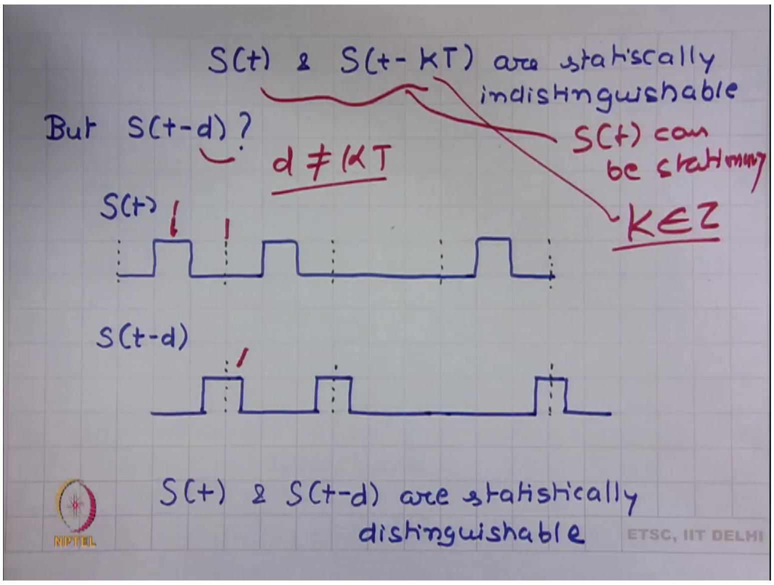

So, what we can say is S t and S t minus k T are statistically indistinguishable, there is no way to differentiate between S t and S t minus k T. And, hence we are pointing out that this probably S t can be a stationary process. We have not yet confirmed this because, we say that as about the definition of a stationary processes you take a process you delay try some time. If the statistical properties do not change for all values of time then we say that that process is stationary. Here we have said that this statistical properties do not change, if I delete t with k times T, where k is some integer ok, k belongs to a set of integer.

Now, if k does not belong to set of integers then this k T is not integer multiple of t, let us assume that I have some d which is not k times T. So, I have taken a process S t, I have delete in my d units such that instead of peak falling here the peak falls in here I can choose any d right. Now, you see that the statistical properties of these two waveforms are completely different. Why? Because, here the peak was falling in the middle of this and here we always would have got 0, but in this case now at these time instances I am getting some value ok. So, this S d and S t minus d are statistically distinguishable.

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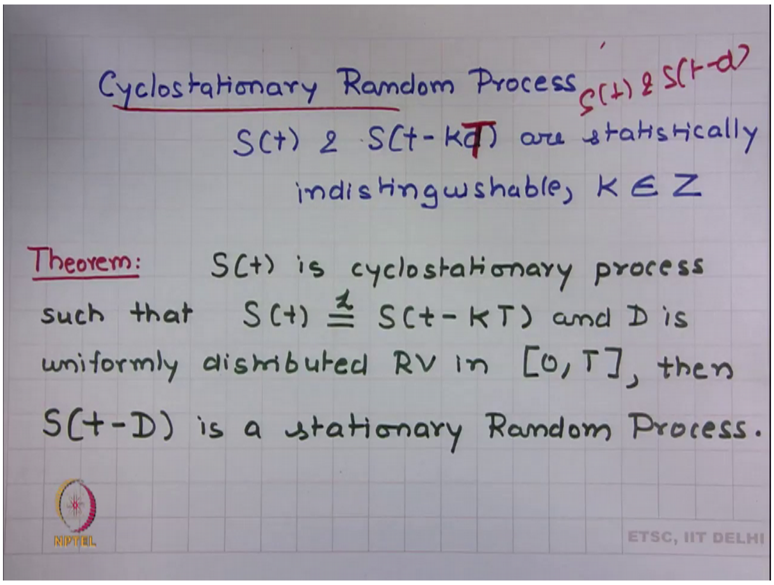

So, that is something interesting and that we say as cyclostationary random process, if S t and S t minus sorry this has to be T; if S t and S t minus k T are statistically indistinguishable where, k belonging to set of integer then we call that process as a cyclostationary random process. So, this is the definition of a cyclostationary random process; remember that S t would not be statistically indistinguishable for S t minus d where, d is not k T. It is only statistically indistinguishable if you delay this by k T times.

So, when S t in general is not stationary we would have cordons do some kind of problems, if we wanted to investigate the power spectral density of that process. So, if we wanted to investigate power spectral density of this waveform from the context of ensemble averaging because, for ensemble averaging what you do first is you find autocorrelation function.

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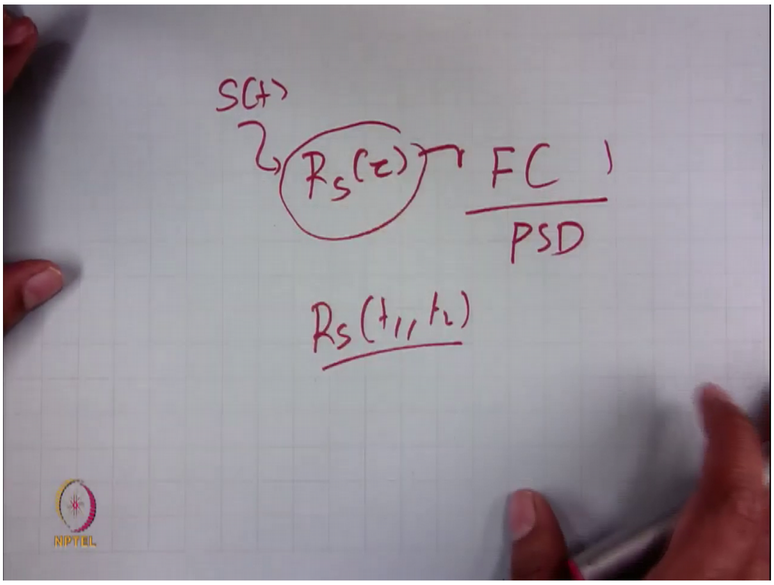

So, let me remind the steps. So, if I have a waveform first we find its autocorrelation function and then we find its Fourier transform right. So, this gives us power spectral density. Now, you get a neat autocorrelation function, need means an autocorrelation function which has only one independent variable, if these processes is stationary. And then you can easily take its Fourier transform and can investigate S power spectral density.

But, if S t is not is stationary to start with this autocorrelation function would not be a function of one independent variable, but it would be a function of two independent variables, if S t is not a stationary process. And, then you cannot take its Fourier transform in investigate power spectral density. So, this would not have worked this ensemble averaging or statistically averaged autocorrelation function could not be derived simply for this process because, it is not a stationary process. So, what do we do with this?

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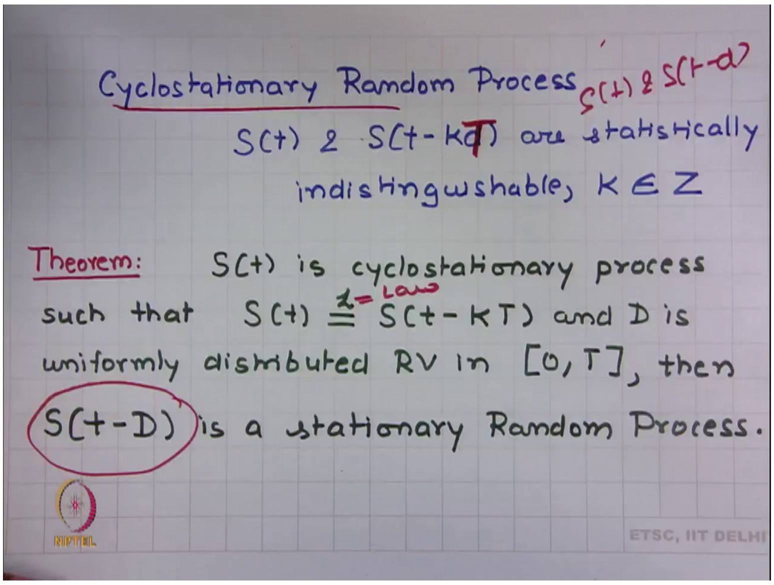

So, there is a theorem which we can make use of that if S t is a cyclostationary process such that S t has the same statistics as S t minus k T, this stands for law has the same law or as the same statistics as S t minus k T and D is a uniformly distributed random variable in 0 to T. Then S t minus D is a stationary random process this is interesting; that means, I can take a process S t which is a cyclostationary process. I can define another process S t minus D and this S t minus D would be a stationary random process; that means, I can stationarized a cyclostationary process by adding to this process a uniformly distributed random variable.

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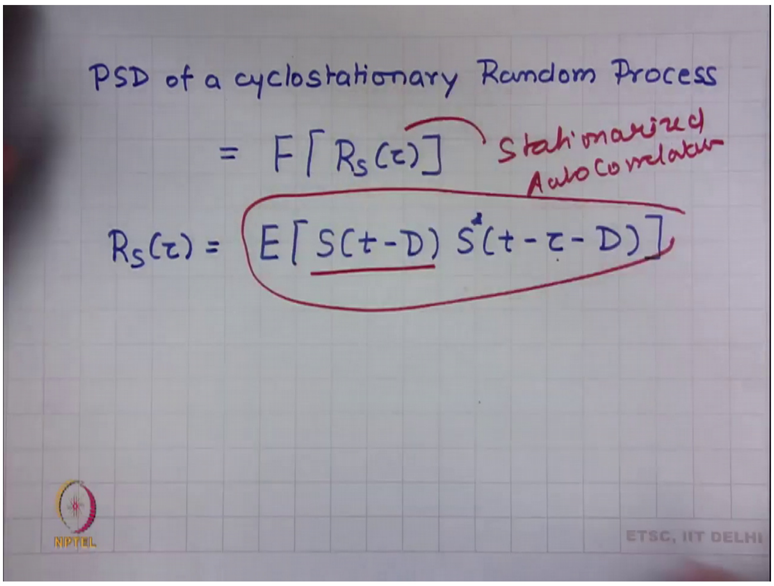

So, this looks useful and thus I can find the power spectral density of a cyclostationary random process by taking the Fourier transform of this autocorrelation function. This is a stationarized it is not a normal for the stationarized autocorrelation function and this stationarized autocorrelation function is thought as by finding the expected value of S t minus D multiplying to the conjugate of S t minus tau minus D.

Now, because S t minus D is a stationary process when you investigate this autocorrelation function would be function of only one independent variable and you can easily take its Fourier transform and can find the power spectral density ok. So, I have stated the general results and how we will look at the proof of these things, let us look at these equations.

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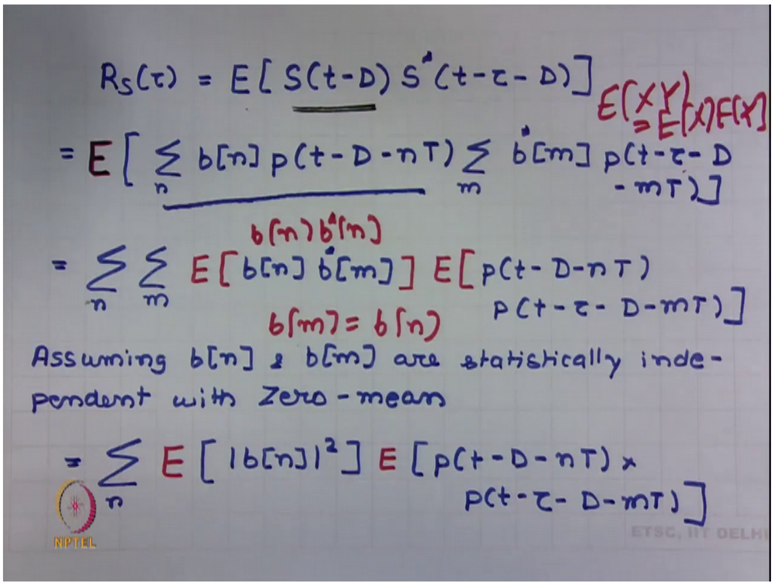

So, what we are trying to find out is R s tau which is calculated by taking the expected value of stationarized random process. And, this random process I stationarized by adding this random variable D.

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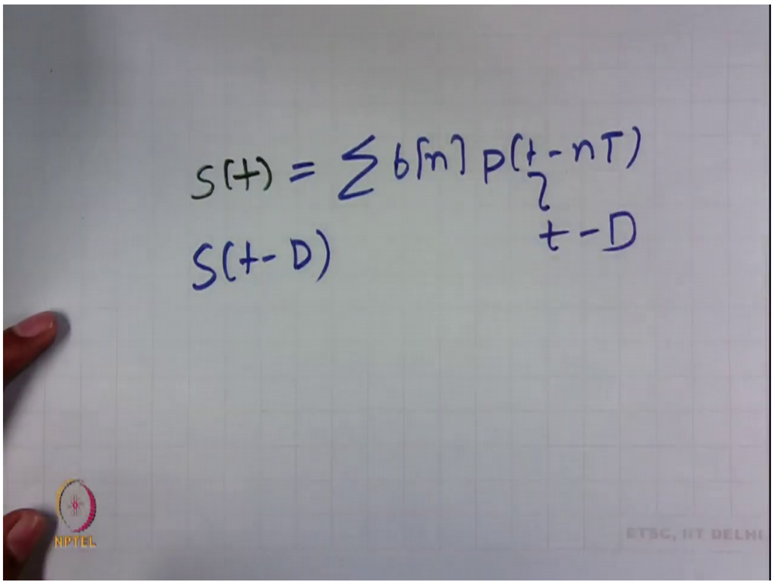

And we know what is this S of t minus D is simply, let me write what is this S of t S of t as I have said is nothing, but b n p t minus n T. So, if we want to find S of t minus D what do I need to do is, I simply have to change t to t minus D as a random variable. So, replacing S of t minus D by this replacing S of conjugate t minus tau minus D by this. What is the difference? Instead of t now, I have t minus tau, instead of n I have m and this I have said several times when you are multiplying two summations it is a good idea to have different running variables. So, that you can have the cross terms.

And this b n here comes out with a conjugate because, we have a conjugate here and the pulse shapes I am assuming to be real so, there are no conjugates in here. Furthermore what you can see is I can pull this to summation together. So, I have a double sum together and what more I can assume that this b n and b m are statistically independent of the pulse shapes, that is quite obvious. What numbers you would choose to map your binary sequence will be completely independent of the pulse shapes that you choose. So, these are statistically independent and thus I can use that expected value of XY is nothing, but expected value of X into expected value of Y.

So, I have expected value of these numbers I have taking the expected value of these pulse shapes. Furthermore, for simplification what I am assuming is that b n and b m are statistically independent, they are statistically independent as I have such several times and moreover I am assuming that they are with zero-mean.

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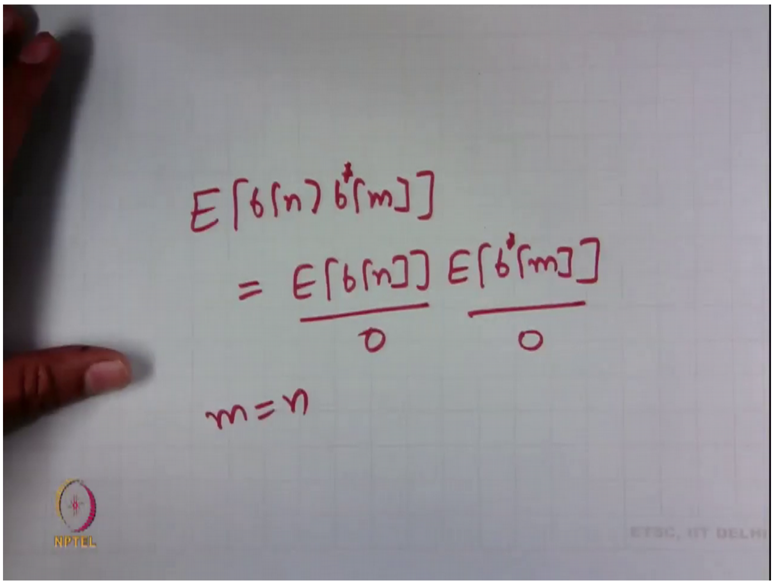

So, let me work this out. So, if I have expected value of b n times b m if there statistically independent I can write this as this and if they are with zero-mean this will be 0, this will also be 0 right. So, m is different from n the expected value of b n times b m will be 0 and that is the only case in which expected value of this product will be non-zero is when n is same as m ok. So, m has to be same as n otherwise expected value b n times b m conjugate would be 0.

So, thus this will be non-zero only when n is same as m, if they are statistically independent with zero-mean and thus I can replace these two summations with a single summation. And, I can assume that m is same as n otherwise this is going to be 0 there will be no contribution of those terms. So, I can write expected value of mod of b n square so, if b m is same as n. So, we will have bn times b n conjugate b n times b n conjugate is nothing, but mod of b n square. So, from here we can easily obtain this.

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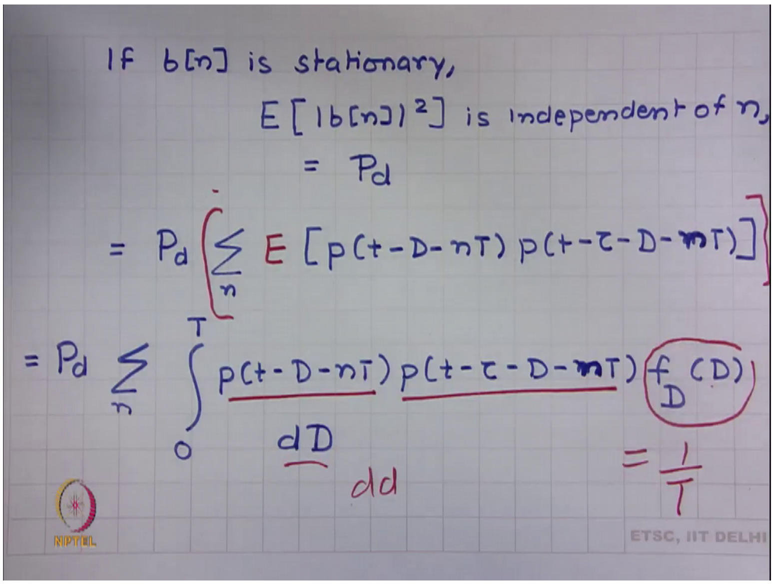

Now, further because b n is a stationary expected value of this quantity is independent of n and is some constant let us call this is P d; we have already used this notation before. So, I can pull this again out of this summation, I have P d times this. So, I am left with finding the expected value of this quantity. Now, if you look at the expected value of this quantity by definition what is this. So, you have to take this a function, you have to take another function the random variable here is D. So, I have to find the pdf of D taking a numerical value D. So, here I am taking the numerical value as also capital D because, otherwise I would end up with a very inconvenient dd which I wanted to avoid.

So, I have assumed the numerical value also to be capital D ok, this is just notation and then I have to integrate it for all values of D. A value of D can lie between 0 to T, remember D was a uniformly distributed random variable which took a value between 0 and T. If this random variable D was not added here what would have happened, this was a then a deterministic function right.

And, you would not have expected value then and this would then be a function of time and that is why we said that, if you do not have this then the autocorrelation function that you would get would be a function of the absolute values of time and this would then be inconvenient ok. And, that is why we need to introduce this D; let us see to solve this, solving this just involves some change of variables.

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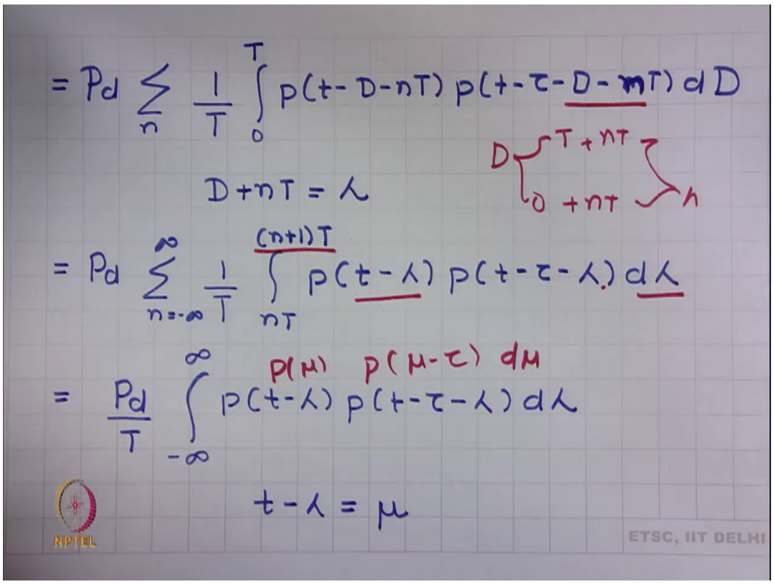

So, first thing that I did is I have replaced f D by D; if it is a uniformly distributed random variable between 0 to T this f D D would just be 1 by T ok. So, putting this value there what we get is 1 by T, the same thing as before now changing D plus n T to lambda. So, first let us see where the limit would go. So, D goes from 0 to T, D goes from 0 to T. So, lambda would go from lambda would go from n T to n plus 1 T then D plus n T is lambda. So, this is t minus lambda, this is again t minus lambda so, this is lambda. So, this is t minus tau minus lambda and I need to integrated for d lambda ok. Now, something important that this is an integration so, let me explain this.

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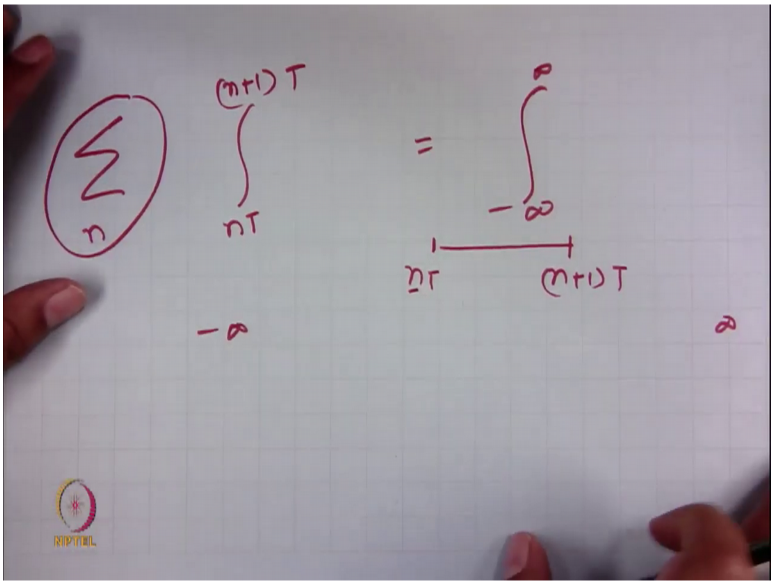

So, I have an integration from n T to n plus 1 T and the summation over all n. So, what this integration is you need to integrate this function from n T to n plus 1 T. And, if you vary this n from minus infinity to plus infinity in this summation actually what you can do is, you can replace these two things by just one integration which goes from minus infinity to plus infinity and this is a trick that we employ. So, this summation and this integration has an effect that the resultant integration goes from minus infinity to plus infinity, rest everything is same.

Then again doing another change in variable I am substituting now, t minus lambda as mu and when I do this; what I can find is this becomes p of mu, this becomes p of mu minus tau, this d lambda becomes d mu minus sign but, the limits will also get flipped and in end net result in would be same.

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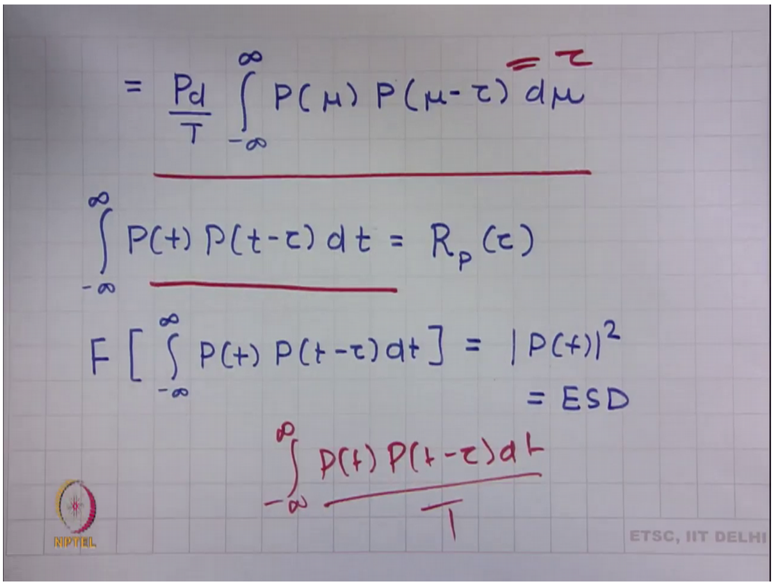

So, I have this equation and now you see what has happened, now this is only a function of tau. This is not a function of mu because, mu is a running variable it goes from minus infinity to plus infinity, is a definite integral. And, hence this is not a function of mu, this is simply a function of tau. And what is this quantity? This quantity P t into P t minus tau dt is the autocorrelation function of pulse P t. And, if I take the Fourier transform of this, the Fourier transform of this is energy spectral density of the pulse.

So, it might be confusing that why we have defined the autocorrelation function like this because, sometimes I define autocorrelation function sometimes I define it like this. And, now I am not having any t and the Fourier transform of autocorrelation functions sometimes I said it has power spectral density and sometimes I said is as energy in spectral density. So, what is going on here right? So, let us look at it so, there is no confusion.

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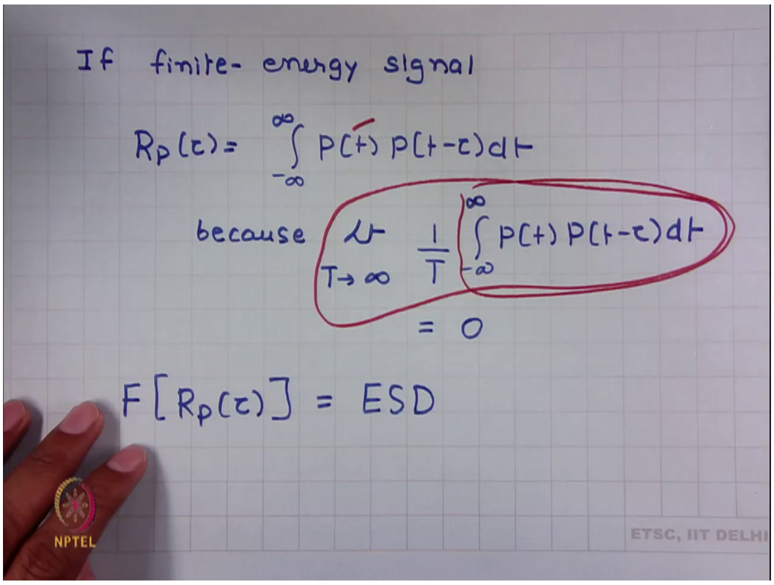

So, if I have finite energy signal p t is a finite energy signal, it is a pulse and we said that is a pulse with a finite energy right. You do not want to use a pulse shape where this infinite energy because, then you would be transmitting infinite energy right. So, pulse p t is of finite energy and when I have a finite energy signal, the autocorrelation function is defined using this expression. Because, if I would have used this expression; let us say if I use this what would happen because, this quantities is kind of finite and T is infinite this quantity would go to 0

So, there would be no meaning of that autocorrelation function. So, once I have finite energy signal, I define autocorrelation function just by using this integration and not dividing it by T, because that would make it 0. And, if I take then the Fourier transform of this autocorrelation function which has this definition then the Fourier transform of this is energy spectral density and not power spectral density ok; let us see why is that let us try to see it quickly.

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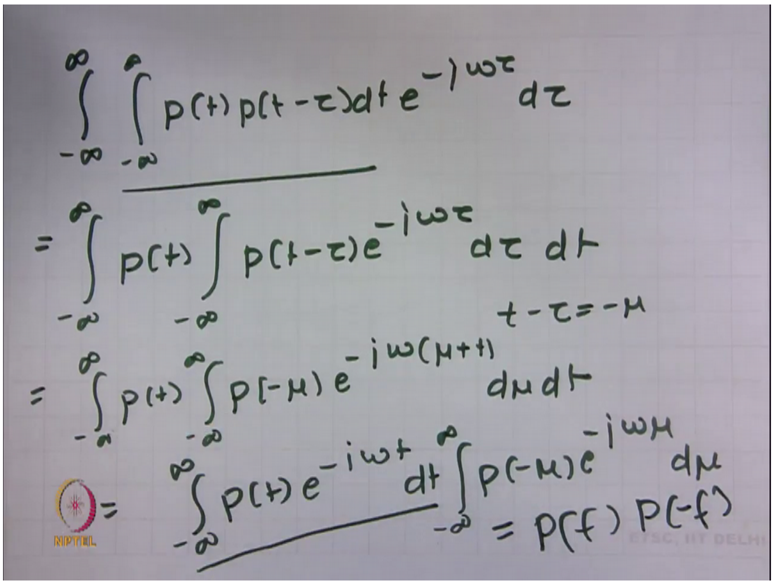

So, I am taking the Fourier transform of I am taking the Fourier transform of this. So, I am taking the Fourier transform of this, what I can do is I can pull this p t here; rest I can collect just rearranging terms nothing else and then what I do let me put t minus tau as minus mu. So, that I get p t this becomes p of minus mu this becomes e to the power minus j omega mu plus t d mu dt. And, this becomes then p t I pull this here e to the power minus j omega t d,t this p minus mu and this you know what is this. So, this is let me write here. So, this quantity is nothing, but p f this is a Fourier transform of p t and this is p of minus f right.

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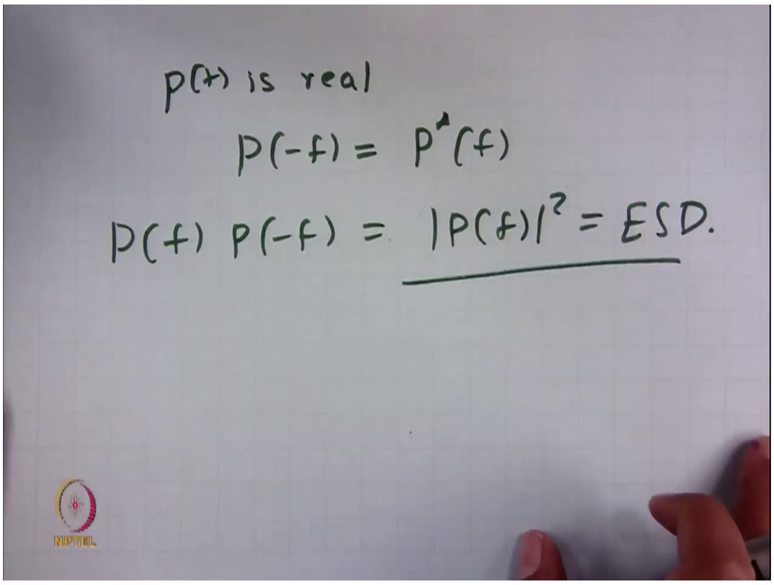

And you know that if p t is real then p of minus f is nothing, but p conjugate of f. So, p f into p of minus f would become mod p f square which is the energy spectral density ok. Hence, is a simple proof that if I take the Fourier transform of this autocorrelation function that what we end up with is energy spectral density.

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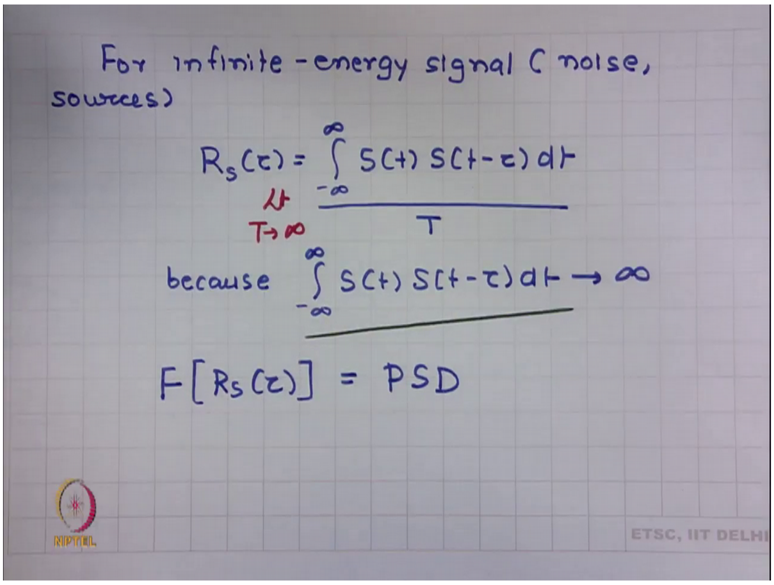

For infinite energy signal like noise and sources or stationary sources the best definition to choose autocorrelation function is this definition because, this quantity is tending to infinity because, it is of infinite energy. So, when you are talking about infinite energy signals the good definition is this one and when you take the Fourier transform of autocorrelation function which is defined using this relationship, then we get power spectral density ok. Having understood this let us come back to the result that we were trying to derive.

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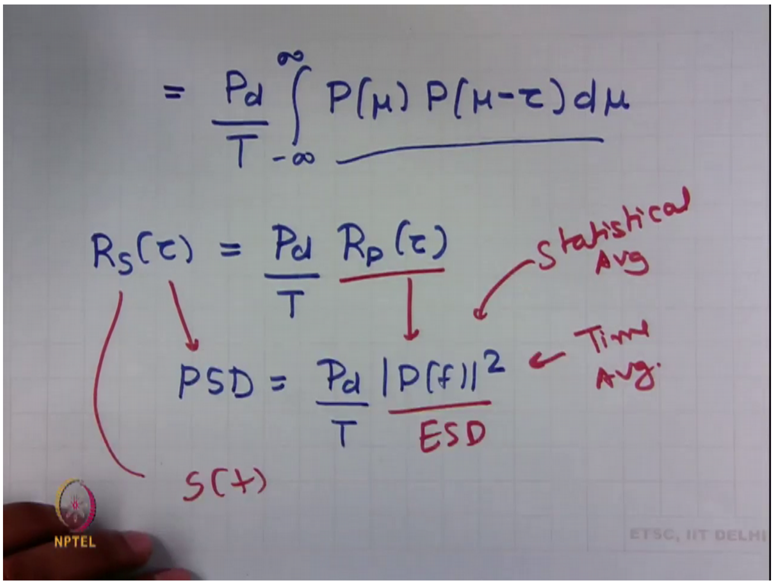

So, what we have caught now is this expression and what is this is nothing, but it is the autocorrelation function of the pulse. So finally, what we have got is autocorrelation function of S t where, S t is a cyclostationary random process given by P d by T times autocorrelation function of the pulse train. Now, because this S t has infinite energy the Fourier transform of this autocorrelation function will give us power spectral density whereas, Fourier transform of this autocorrelation function will give the energy spectral density.

So, remember this is the energy spectral density of the pulse p t and surprisingly this is the same answer as we have got in the case of polar signaling mechanism. Remember the polar ins polar signaling mechanism also this b n and b m are statistically independent and they are with 0 mean right. So, we also derived this using the same assumption. So, if I have b n and b m is statistically independent with zero-mean this is the answer of power spectral density. So, the interesting thing is here we have obtained this power spectral density using statistical averages. And, before we obtain this power spectral density using time averages and the answer converges right, we got the same answer when we use this time averages approach.

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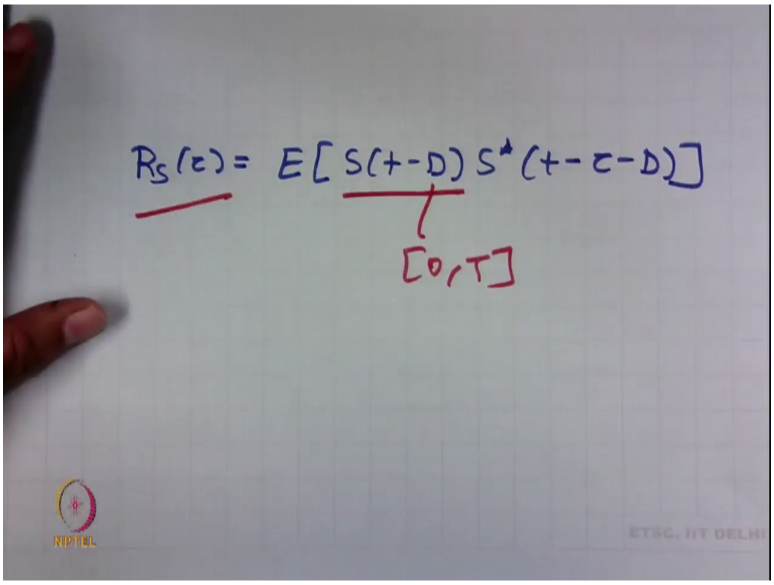

And this answer converges because we have chosen this definition of a stationarized autocorrelation function. So, we have stationarized a cyclostationary random process by adding this random variable D where, D is a uniformly distributed random variable and it takes a value between 0 in T.

So, I have a stationarized this random process and I define this stationarized autocorrelation function and if I choose this definition of the stationarized autocorrelation function; the answer of power spectral density that you would get from time averages would be the same thing as the answer that you would get by following statistical average approach. And, that is why this explains the motivation why this definition of a stationarized autocorrelation function is good for us.

So, we have finished with these power spectral density and we have looked into the power spectral density of basically three line coding mechanisms: unipolar, polar and alternate mark inversion scheme. And, we have seen that this is a very strong function of how you choose to map your binary sequence and what is the pulse shape that you choose. In next lectures what we will see is what are the other criteria for choosing pulse shapes and we will also see some other examples of modulation scheme.

Thank you.