Principles of Digital Communication
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Lecture – 20
Modulation Complex Baseband Representation of Passband Signals (Part-2)

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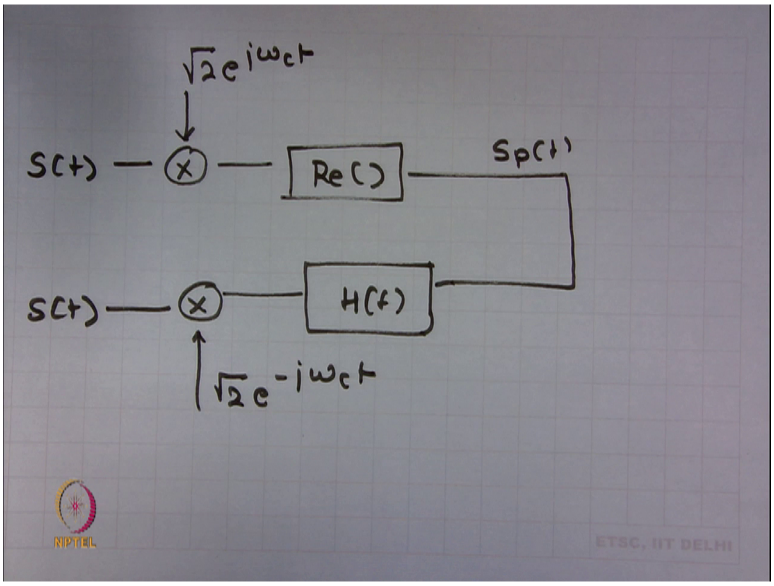

So, good morning, welcome to the next lecture on Modulation. In last lecture, we started with complex baseband to passband representation. And we developed an important picture of that relationship which is here So, what we have said is we have a complex baseband signal which is S t you multiply this with rotating complex exponential take the real part, and you get a passband signal a real passband signal.

And you can go from passband signal to the complex baseband signal by passing this pass band signal through a Hilbert filter and then again multiplying with a suitable rotating complex exponential. And you get back complex baseband signal. So, this is to help us understand analytically what goes on and in terms of reality these things are implemented something like this.

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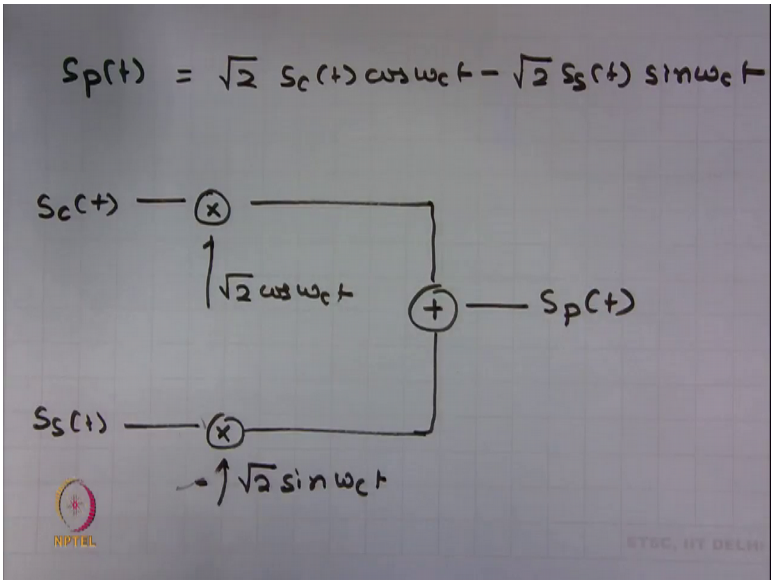

So, you have S c t which is the real part of this complex baseband signal multiply this with cos omega c t multiply S s t with sin omega ct add these two things to get S p t.

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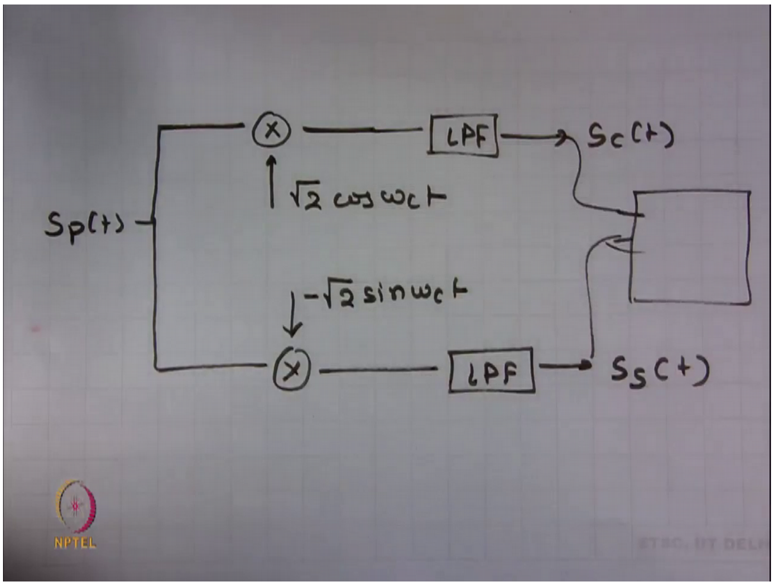

And then we can also go back from S p t to S c t and S s t by having again the multiplication with cosines and passing them through low pass filter. And then you can accept or take this as c t and S s t and a baseband processor can process these signals. Today, we will like to carry forward this idea of establishing equivalence between complex baseband and passband signal.

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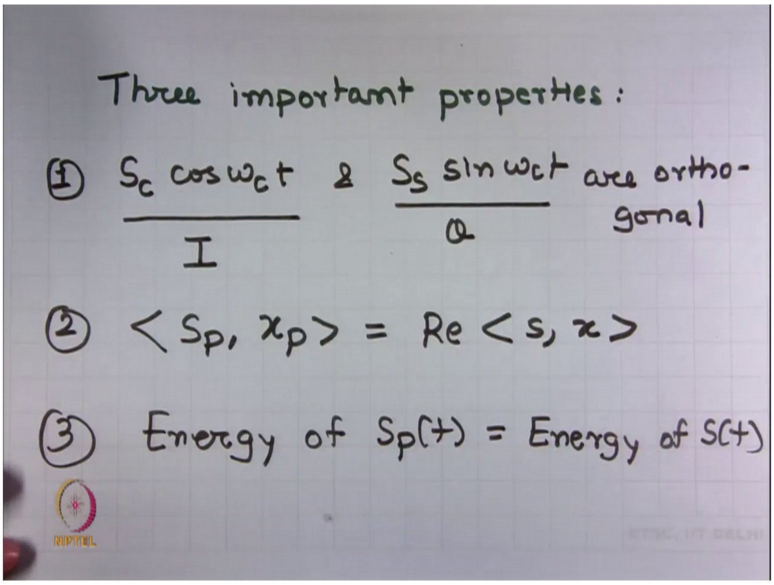

And we like to first study three important properties. So, three important properties that we will study and prove in today's lecture this. So, first property that we will prove is S c cos omega c t which is also known as I component and S s sin omega ct which is the Q component are orthogonal to each other.

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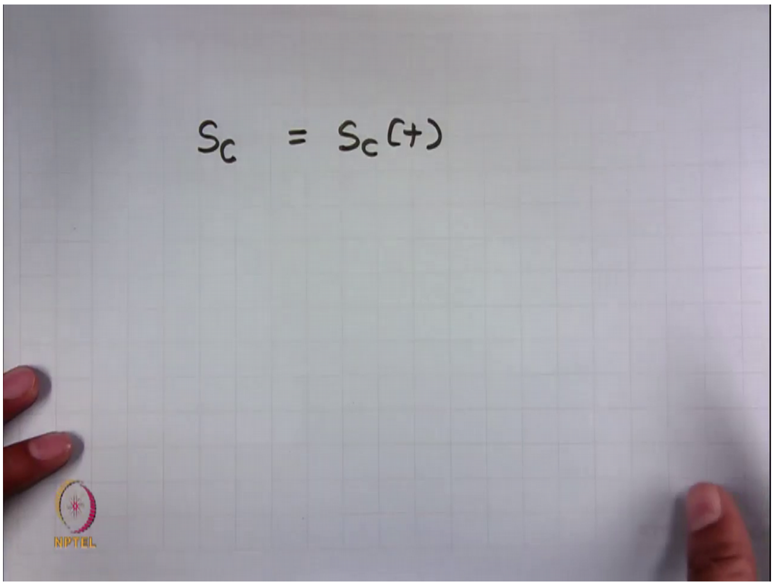

Remember that sometimes I write S c and sometimes I write S c t. So, S c is a function of time right and sometimes for type of graphical reasons I simply write this as S c, it is more convenient right. Your equation does not become cluttered with this t. So, whenever you see S c, you should understand that this is a function of time right, we do not have to have this t at all times. So, S c is nothing but S c t and S s is S s t ok. So, that is the first property that we have to prove that I and Q components are orthogonal to each other.

The second property that we have to prove is inner product of S p with x p, S p and x p are the passband signals, p represents passband signals. And again I do not have t in here is same as the real part of inner product of s and x, s and x the complex baseband equivalent of these passband signals. So, this is the second property that we have to prove that inner product of passband signals is same as the real part of inner product of their equivalent complex baseband signals, s and x are the complex baseband signals of S p and x p respectively.

The third property that we have to prove is energy of S p t passband signal is same as energy of baseband signal ok. And this will follow because of the normalization factor that we used, we use the normalization factor of root 2 and this is to make the energy of the passband signal same as the energy of its equivalent complex baseband signal. And later when we do detection we will see that the performance of the detector depends upon the energy of the signals right.

And if the energy of the passband signal is same as the energy of its complex baseband signal then we do not lose anything right. So, whether you detect S t or S p t it does not lead to any deterioration in the performance. So, these are the three important properties that we will first derive and then we will move to other things.

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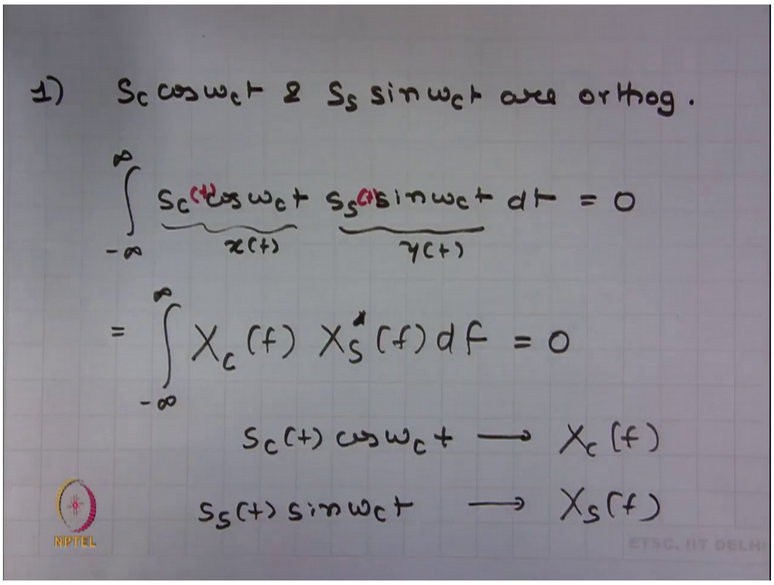

So, let us start with the property number 1. So, we have to prove S c cos omega ct and S s sin omega c t are orthogonal. And to prove this what we need to do is that their inner product is 0 that is what you require to prove the orthogonality between the two signals. So, we take their inner product which is S c cos omega ct S s sin omega c t dt is we want to prove that this is 0.

Well, what you can also do is, so I have omitted t we can put it t here. So, we have to prove that this inner product is 0 ok. Now, now S c t cos omega c t and S s t sin omega c t everything is real right that is why in this inner product, there is no conjugate or anything everything we have assumed to be real.

Now, if you want to prove this to be 0, a more instructive way to do this is by proving this is 0. What is this X c f? X c f we have defined it to be Fourier transform of S c t cos omega c t, and X s f we have defined to be the Fourier transform of this. So, we have considered two signals. So, let us say that this let us treat it as x t and y t. So, what we are saying is x t has a Fourier transform X c f y t has a Fourier transform X s f. And so from Parseval’s theorem we can write this.

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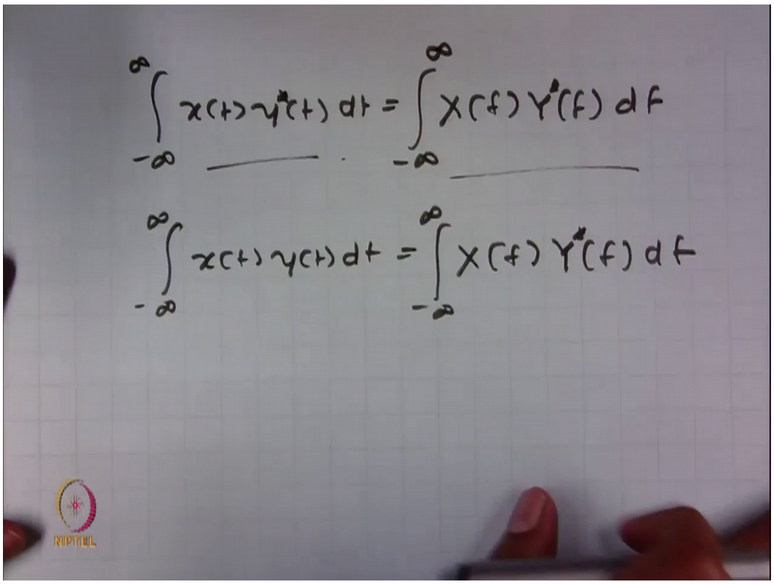

Remember Parseval's theorem, what is Parseval's theorem let me rewrite Parseval's theorem for you. Parseval's theorem says that x t y t conjugate dt is X f Y f conjugate df, when it ran down its integration from minus infinity to plus infinity these two quantities are same. So, this is the inner product of the two signals. And this inner product this is also the inner product in the two signal is a spectrum right. So, whether you take the inner product of the signals or you take the inner product of the spectrum of the signals, inner product is same, inner product is preserve when you move when you go from time domain signal to spectrum.

Now, because my y t is real, so I can say x t y t dt should be same as minus infinity to infinity X f Y f conjugate d f and this is exactly what we have written there. So, this we have considered is to be x t; this we have considered to be y t. We wanted to prove that this is 0. So, we can prove this is 0 by saying that X c f into X s f conjugate df is 0 from Parseval's theorem ok.

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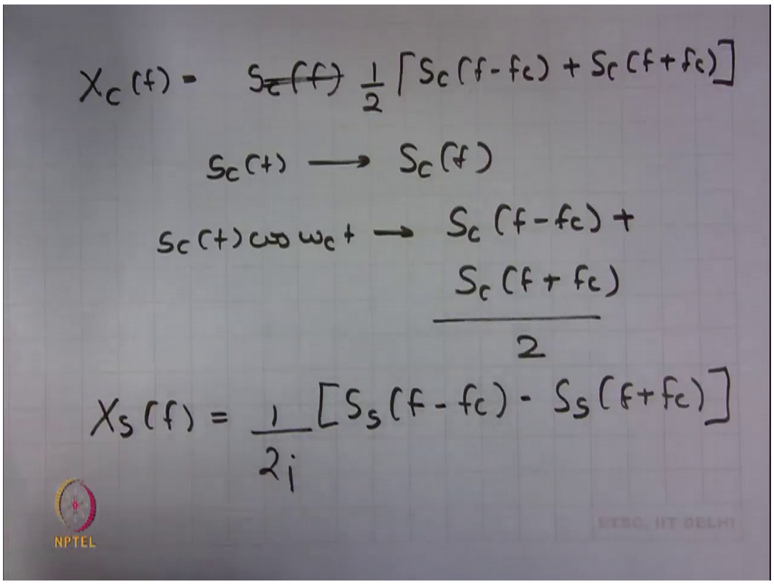

So, now let us see what is this X c f? And there is also a one line proof for this, but why we are doing such a lengthy prove is because just proof is more instructive right. You, you learn more when I carry out this lengthy proof. So, X c f if you look at this, it is nothing but. So, let us let us define S c f let me also define that S c t, let us say that this has a Fourier transform S c f.

So, S c t into cos omega c t would have a Fourier transform S c f minus f c plus S c f plus f c by 2. So, this is from signals and systems goes very basic stuff. And I assume that you know this. So, X c f would then be half S c f minus f c plus S c f plus f c by 2. Similarly, we can say that x S f is 2 j S s f minus f c minus S s f plus f c ok, because instead of cos omega ct it has sin omega c t right. So, based on this we can get xs f as this.

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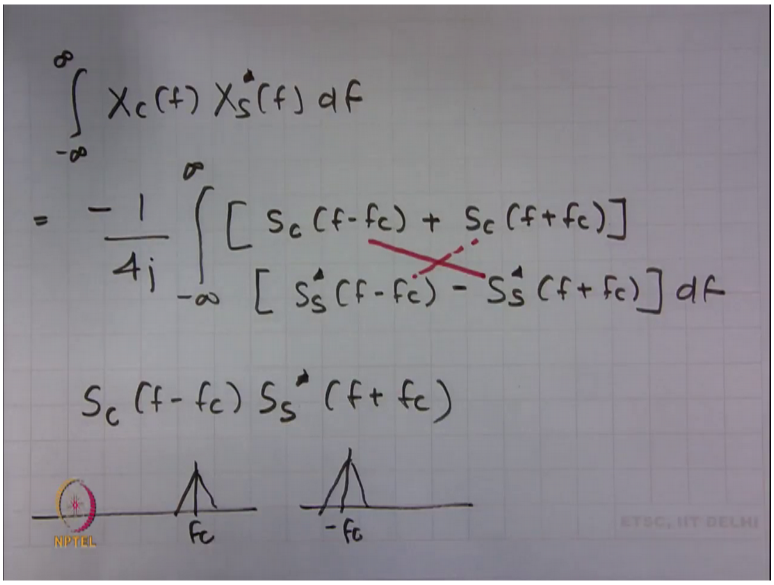

Now, let us try to take this with the conjugation. What you would end up with is X c f into X s f conjugate df would be nothing but it is minus 1 by 4 j; I am not explaining the constants, because they are not so important right, because we have to prove anyway it is 0 right. So, the constants values are not so important as such. So, what you would have is S c f minus f c plus S c f plus f c multiplied by S s f minus f c minus S s f plus f c. And these things would come with conjugation.

We have to see what this integration is. And what I have done is nothing I have just replaced the values of X c f and X s f. If you look at this now, the big idea comes that if you look at this thing, let us worry about what is the multiplication of this with this. So, let us see what is S c f minus f c into S s conjugate f plus f c, now if we look at this and this is a big idea because most of the time we would be using this idea is this is the spectrum which is centered around f c. So, if I look at this, this would be some spectrum which is centered around f c. And this is a some spectrum which is centered around minus f c right.

So, one is this part of the spectrum is centered around minus f c, this part of the spectrum is centered around f c. When I multiply these two things there is no overlap and hence this multiplication gives us 0 ok. So, this times this would give me 0. Similarly, if I multiply this with this, it also will give me 0.

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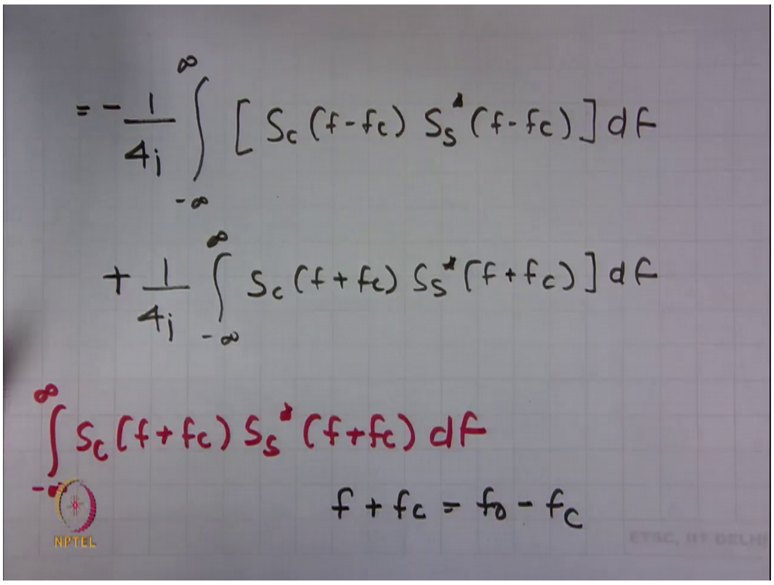

So, in the end, I can reduce this integration by having minus 1 by 4 j what it would end up with is let us try to see you yeah. So, first part you would have S c f minus f c times S s f minus f c conjugate. Let me write it as two integration minus 1 by 4 j minus infinity to infinity let us see what we have is you would have S c f plus f c into S s conjugate f plus f c d f. Let me check the signs. So, this would come with the positive sign. So, this is what we would have finally.

Now, let us look at this expression let me use different pen and let us see what S c f plus f c into S s conjugate f plus f c d f S. So, to understand this let us change the integration variable. So, let me define f plus f c is some f naught minus f c ok, the change in variables.

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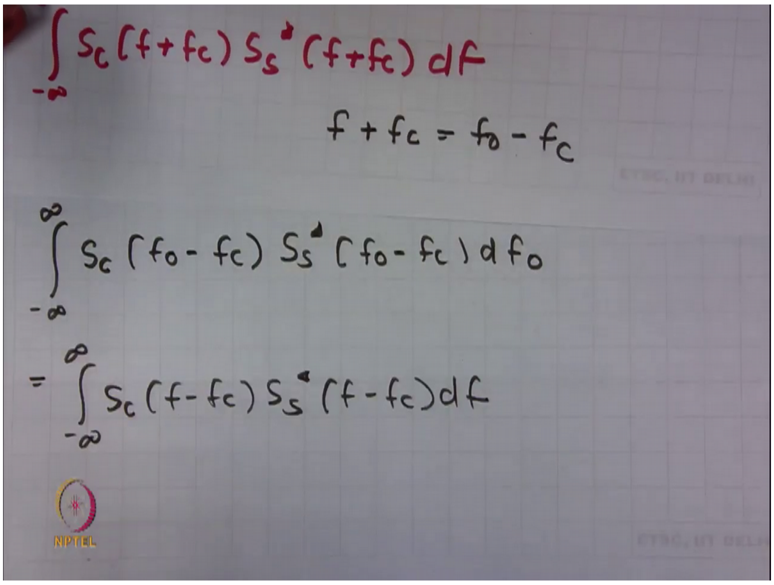

So, I can write this integration. Let me write this what we would have is S c so f plus f c can be replaced by f naught minus f c. So, we would have S s conjugate f naught minus f c d f is nothing but d f naught. And because f was varying from minus infinity to plus infinity, f naught will also vary from minus infinity to plus infinity. So, we have this is same as this. And this in fact is nothing but it is same as just replacing f naught by f.

So, this quantity what we had is same as this quantity ok and they comes with the opposite sign. So, this integration poised down to 0. So, hence we have proved that the inner product of I and Q components is 0. And this is an interesting fact. Let us try to prove the second point or let us try to see with the second point the inner product of the passband signals and the inner product of the baseband signals.

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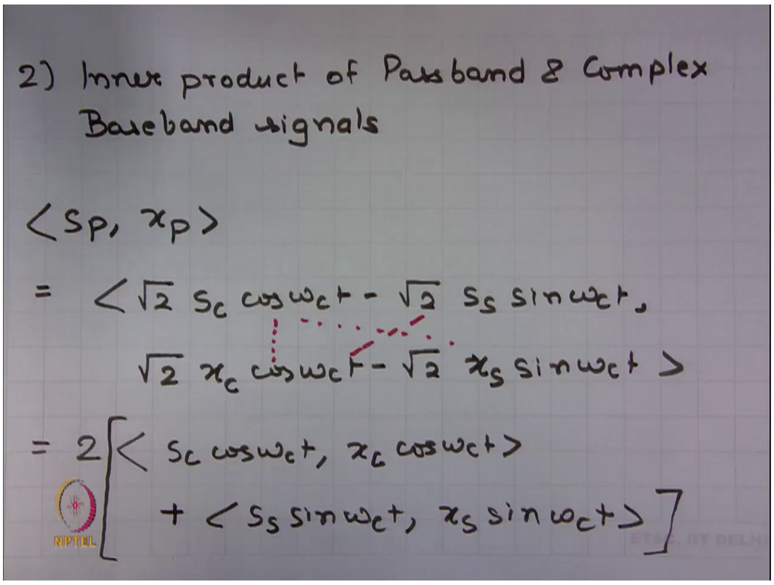

So, let us try to look at the second point. We talk about the inner product of passband and complex baseband signals. So, let us see what it is. So, let us try to think about taking a inner product of S p and x p. And S p and x p are the passband signals I am not writing the t here because I know that you can assume that there is a t, and it is always inconvenient to write every time this t. So, for that simple reason for this typographical reason only I have omitted t there is no technical or deep reason why I have omitted t just for writing convenience.

So, this inner product if you see is then I write it in full that means, I want to take the inner product of this quantity with the passband x p. So, I have to take the inner product of this quantity within with this quantity. So, x c and x s as you would guess these are the cosine and sin parts of the passband and signal x p ok. Now, because inner product is a bilinear operation, I can take inner product of this term with this term and this term with this term and so on and so forth right.

So, let us start. So, first thing that we can do for convenience is to pull out these root 2s and make it 2. So, I have 2 then let us see. So, first I take the inner product of this with this. So, I have S c cos omega c t inner product with x c cos omega c t. Then I need to take the inner product of this with this. And this is 0 because what I am doing is I am taking the inner product of I with the Q component. And we have already derived these I and Q components are orthogonal and hence the inner product is 0 right. So, this gives us 0 and so I do not do it.

Similarly, the inner product of sin omega c t with cos omega c t is also 0. So, what you end up with is simply you need to just multiply this term with this term and this term with this term. And 2 is a already been pulled out. So, I can simply write that this is S s sin omega c ts inner product with x s sin omega c t all right.

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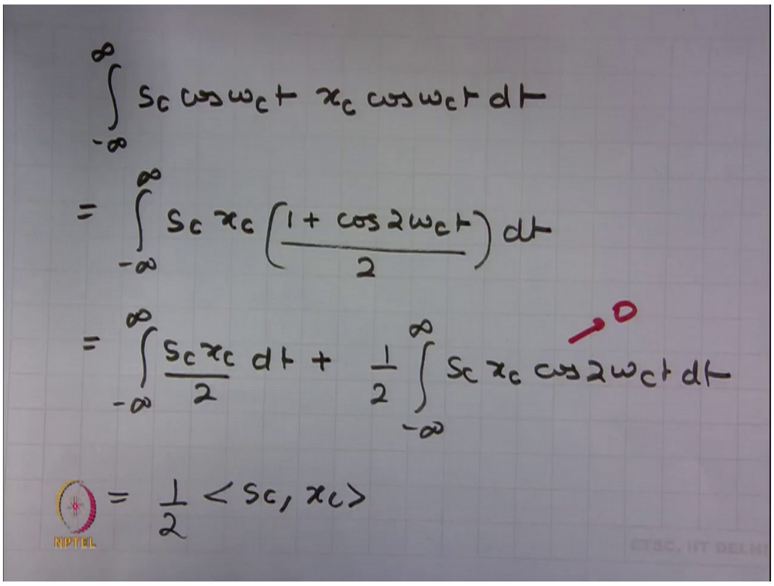

Now, let us see what is the inner product of S c cos omega ct with x c cos omega c t. And as everything is real, I do not unnecessarily put a conjugate. Now, this is very simply S c xc 1 plus cos 2 omega ct by 2 dt and this is minus infinity to infinity S c x c by 2 dt plus half minus infinity to infinity S c x c cos 2 omega c t dt.

Now, you have to very quickly judge that this part should be 0. Why is this 0, because S c and x c are the baseband components, baseband components means their frequency spectrum is centered around dc. This is a component whose frequency spectrum lies at around 2 omega c right. So, there is no-overlap in the spectrum of these two signals. And if you invoke Parseval's theorem and if you have done the proof of the property one, it is should be very clear to you that this integration should go to 0. If it is not clear, I would urge you to look at the proof of the property one ok. The idea is very simple baseband signals this is a passband signal at 2 omega c, no-overlap in a spectrum and hence this integration would turn out to be 0.

What you then end up with is this quantity. And what is this, this quantity is nothing but it is half times inner product of S c with x c. So, what I can write is using this idea that inner product of S c cos omega ct and x c cos omega c t is nothing but it is it is half times inner product of x S c with x c and there is already 2 here, so you can multiply this 2 with half.

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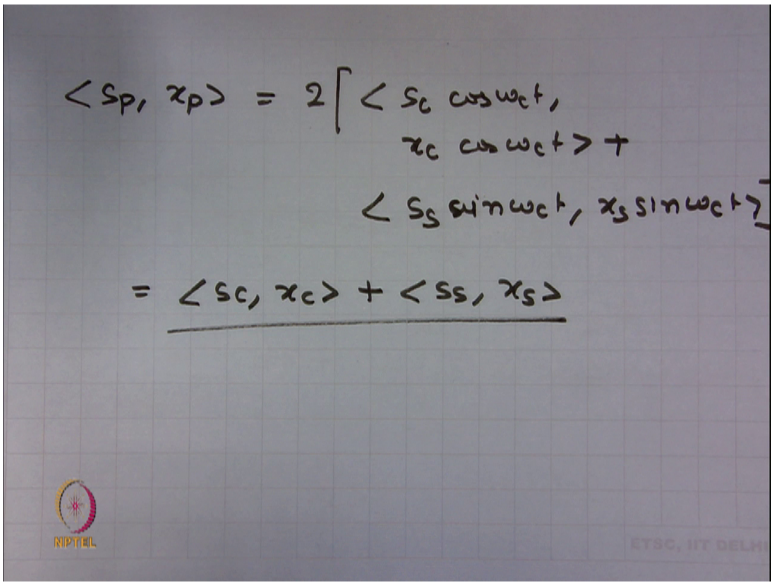

And what you can simplify is what you can simplify it 2 is so inner product of S p x p I said it is 2 times the inner product of S c cos omega c t with x c cos omega c t plus inner product of S s sin omega c t and x s sin omega c t. And this is S c x c inner product and S s x s inner product.

So, this relationship establishes the link between how can we think about the inner product of two passband signals by just carrying out this inner product operation in baseband domain, because all processing happens in the baseband domain. So, if you want to take the inner product of the two passband signals just do it and baseband right likes take the sin and cosine part of S p in the way we have done before. And extract the sin and cosine part of x p the way we have done before and then take the inner product on these parts in this way ok.

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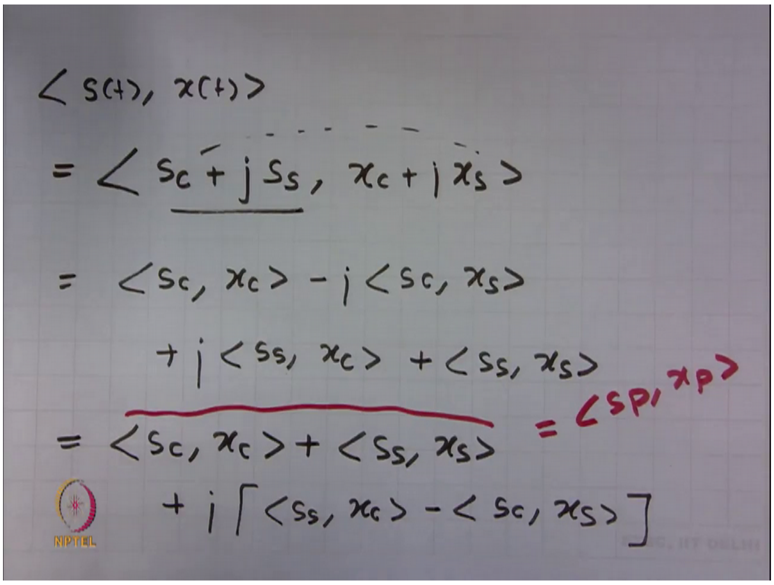

Now, let us see what is the inner product of S t with x t? So, what is S t and x t, they are not passband signals. But they are the complex baseband representation of the pass band signals. Remember this is the notation that we are following. S p t is a passband signal s t as a complex baseband representation of the passband signal. So, now, we are interested in taking the inner product of s t with x t. And let us see what it is. So, again this can be written as S c plus j times S s. So, again I have omitted the t for typographical reasons only, and x t is x c plus j x s is the complex signal.

Now, taking the inner product again term by term we can take. So, this would give us S c with x c inner product. This then we have to multiply let us say with this the j with come with a minus sign S c x s to multiply this with this j will come with positive, remember the bilinear properties of inner product if you have forgot forgotten amount them just revise that is simple. Then I have to multiply this term with this term I get minus and this j comes with a conjugate. So, there would be a plus S s in x s.

So, I can write this as S c x s plus S s x s plus j times S s x c minus S c x s ok. The most interesting thing to see that if I take the real part of this quantity, what I end up with is this. And this quantity as I have seen is nothing but it is the inner product of S p with x p ok and hence now I have the relationship between the inner product of passband signals and its complex baseband representation.

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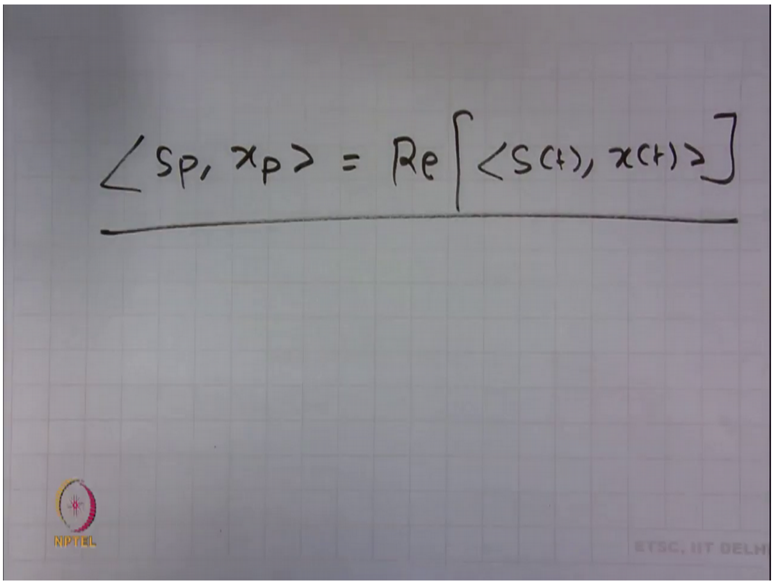

So, there is a big idea that inner product of S p with x p is nothing but it is the real part of the inner product of S t and x t ok. This is the relationship between the inner product of passband and inner product of baseband signals.

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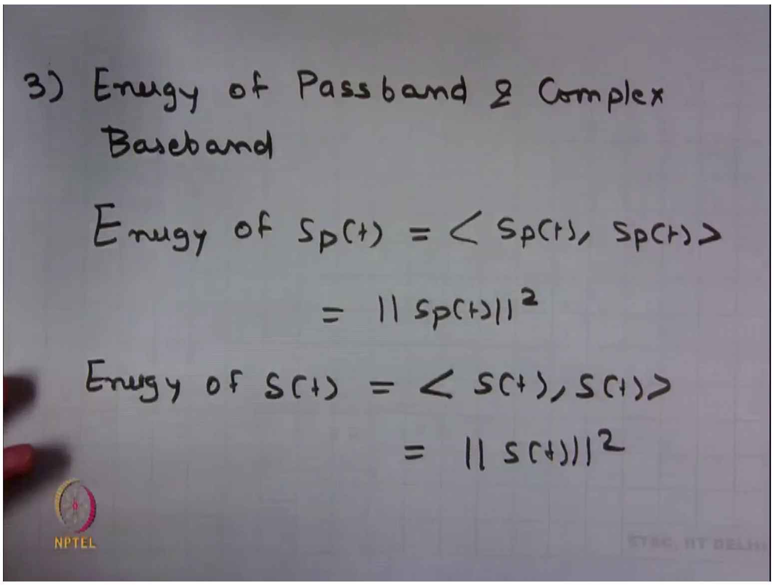

Let us prove property number third. And this property is about energy of passband and baseband signals. Baseband I mean complex baseband representation of the corresponding passband signal. So, complex baseband; now, what is the energy let us say of S p t of the passband signal energy is nothing but you need to take the inner product of the signal with itself right.

So, this is S p t with S p t you take the inner product of the signal with itself you get the energy of the signal. And this is nothing but it is norm of S p t square right. This is typically the representation this we have already seen that when you take the inner product of the signal with itself what you get is the norm square of the signal right.

What is the energy of S t, it is the again the inner product of S t with S t and this is nothing but again the norm of S t square, very simple a straightforward. Now, the question is what is the relationship between this quantity and this quantity? Can we relate it? Yes, we can relate because we have already seen that.

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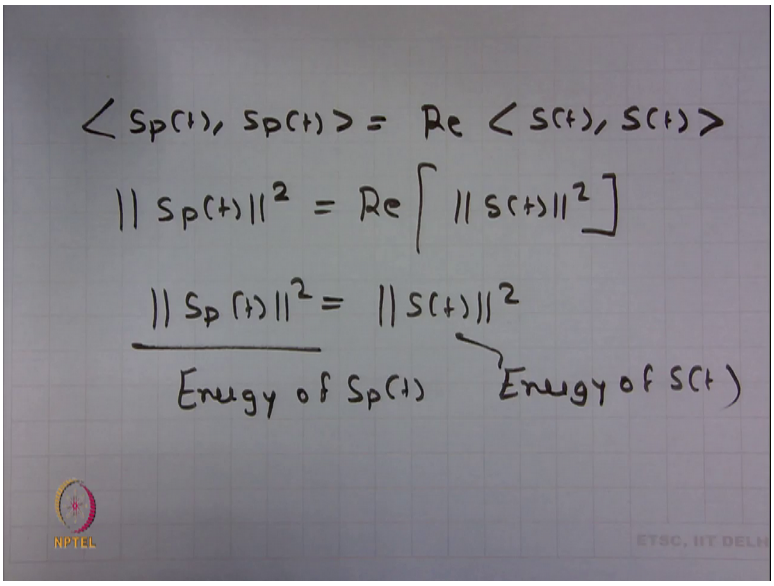

Inner product of S p t with S p t is nothing but it is a real part of inner product of S t with S t is not it from the previous relationship. And this quantity is nothing but S p t square which is the real part of norm of S t whole square. And amazingly norm is a real quantity right, because norm denotes the length of a signal and length is a real quantity. So, this is a real quantity anyway. So, taking the real part does not mean anything. So, I can simply write this as norm of S t square.

So, this is the energy of S p t and this is the energy of S t. How did we get the energy of a passband signal same as the energy of the baseband signal, this is because of the normalization factor that we have used in our expression. So, we have used a normalization factor of root 2. Now, the reason why we wanted to do that is because we want to get the equivalence between the energy of a passband signal and the baseband signal and that is why we have introduced that complexity, certain books which do not use this normalization factor of root 2 energy of passband signal will not be same as the energy of the baseband signal ok. In our case, this has turn out to be true because of that normalization factor ok.

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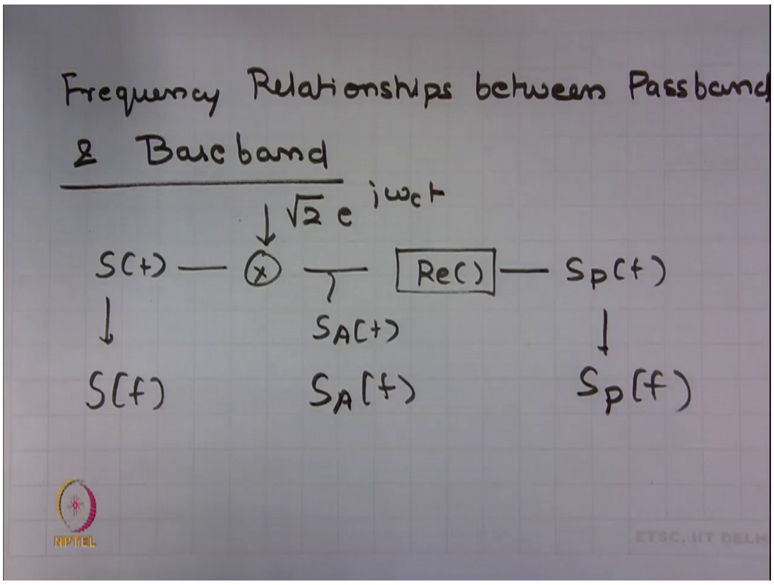

Let us now go and try to establish frequency relationships between passband and baseband signals. Let us try to see that what it is. And before we see this let me ask you to have another look at the picture that we developed for relationship between S t and S p t as I have said before this picture is really useful picture. Let us look at this picture again ok. Using this picture we can easily see what is the relationship between the spectrum of x t and S p, S p t?

So, let us do it. So, let me draw one by one again these blocks. So, let us start with S t multiply it with root 2 e to the power j omega c t take the real part and what we get is S p t this we know right. So, let us see now what is the relationship between the spectrum of S p and spectrum of S t? So, let us assume that the spectrum of S t is S f. Let me call this quantity as S A t. What is the spectrum of S A t? Let me call this as S p f. And let me call this spectrum as S p f.

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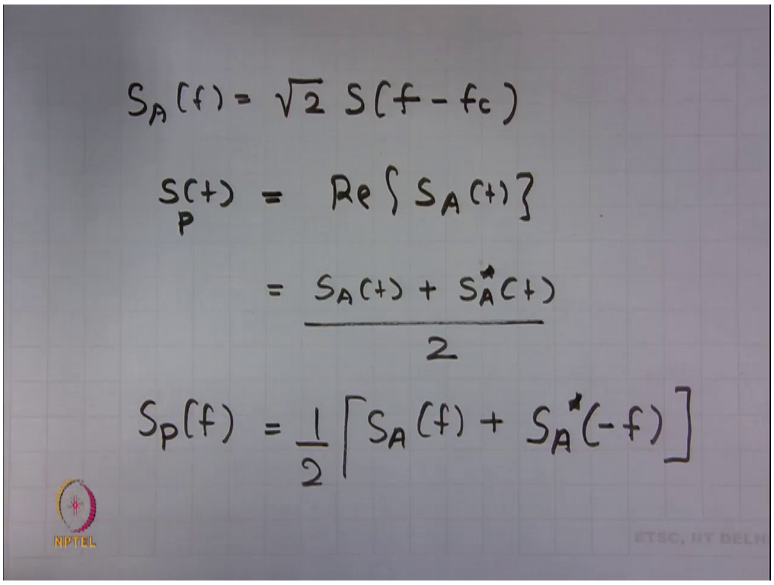

So, what is the spectrum S A f? S A f would be nothing but because S A t is S t multiplied by this quantity. So, spectrum of S A f would be simply root 2 times S f and because S t is multiplied by rotating complex exponential is a spectrum would just shift right and it would get positioned at around f c from baseband it would go to passband. So, this is a spectrum S A f.

Now, what is the relationship between S t and S A t, S A t, S t is real part of S a t sorry S p t the pass band. So, this passband signal is a real part of S A t ok. So, if I want to take a real part of a quantity what should I do, the real part of a quantity is just the quantity and you add to this quantity its conjugate divided by 2 ok. When you add the conjugate the imaginary part cancels out and what you end up with is 2 times the real part and that is why you have this addition divided by factor 2, very straightforward kind of things that should be and better for you once you have finished the course on signals and systems.

Now, if I know this the spectrum of S p t which is S p f should be half the spectrum of sat we have called this as S A f a spectrum of S A t conjugate now this is again trying to use the property from signals and systems this spectrum of this signal put p S A minus f conjugate. So, you must have learned about conjugate symmetry and things like that and this is the invoking and using the same idea. So, if you want to take the Fourier transform or the spectrum of S A t conjugate, this would be S A minus f conjugate. So, I will not prove this, I would simply use this.

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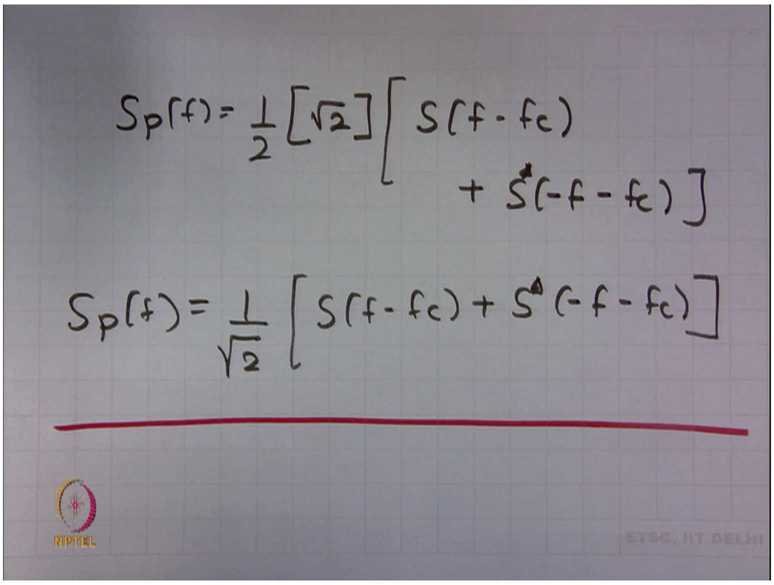

And from thus we can say S p f would be half S A f, what is S A f, we have already derived. So, there is a root 2 factor which I would take together. So, this would be S f minus f c. And then S A minus f conjugate. So, this would be S of minus f. So, f becomes minus f minus f c. And this quantity comes with a conjugation. So, what we have done is 1 by root 2 times S f minus f c plus S conjugate minus f minus f c. So, this relationship establishes the link between the spectrum of the passband signal and the spectrum of the complex baseband signal ok.

Let us now try to do the things in the other way. So, let us try to see what is how can we obtain the spectrum of complex baseband signal from the spectrum of the passband signal.

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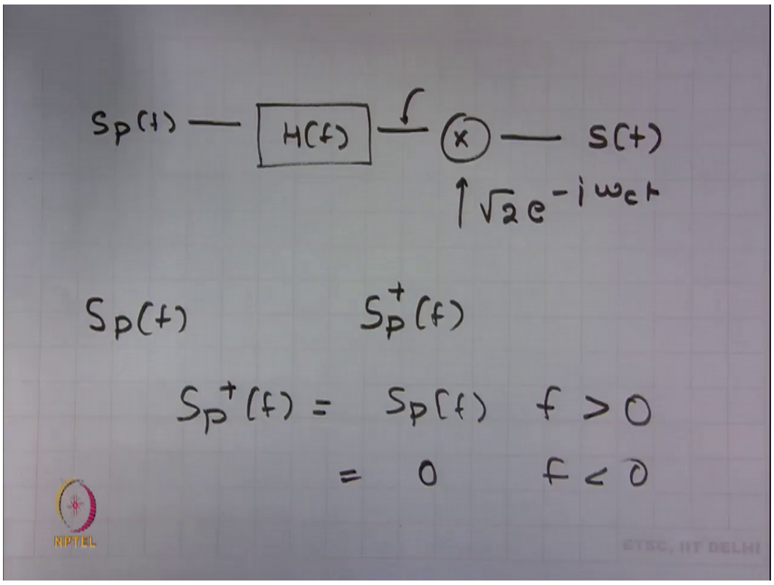

So, I would use this picture that we have S p t we pass it through Hilbert filter and then I multiply this thing with root 2 e to the power minus j omega c t and I get S t. Now, is the things are easy. So, this spectrum here, so if my spectrum here is S p f the spectrum here is the positive part of this. So, I denote this with S p f plus S p f plus means simply that this is S p f for f greater than 0, and this is 0 or f less than 0 ok. So, it is just taking the positive part of the a spectrum of S p f and that is what the Hilbert filter does, it just kills the negative spectrum.

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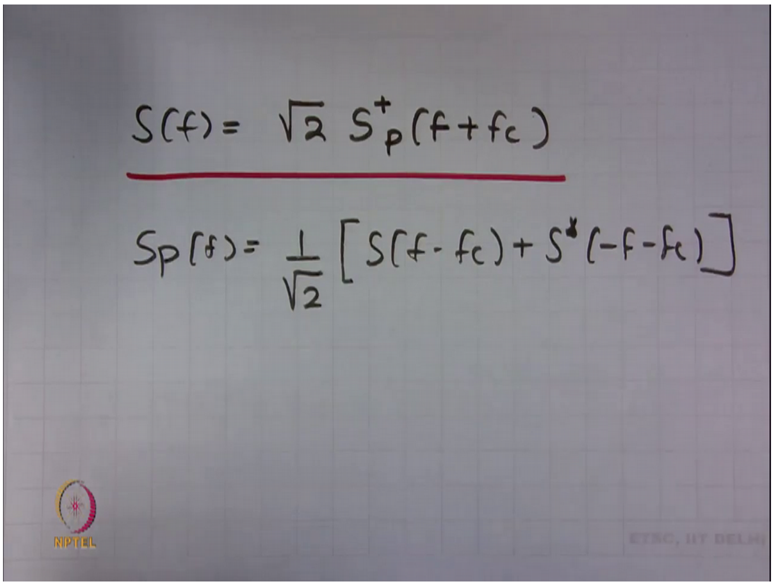

So, the spectrum of S t which I call S f should be root 2 times S plus f, now because of this complex exponential we have the spectrum of S p f shifted to minus f c. So, this relationship links the spectrum of the passband signal and a baseband signal. So, let me write down both the equations together. So, the two equations together are these. So, this links the spectrum of the passband and baseband signals. Let us do one example about this a simple example.

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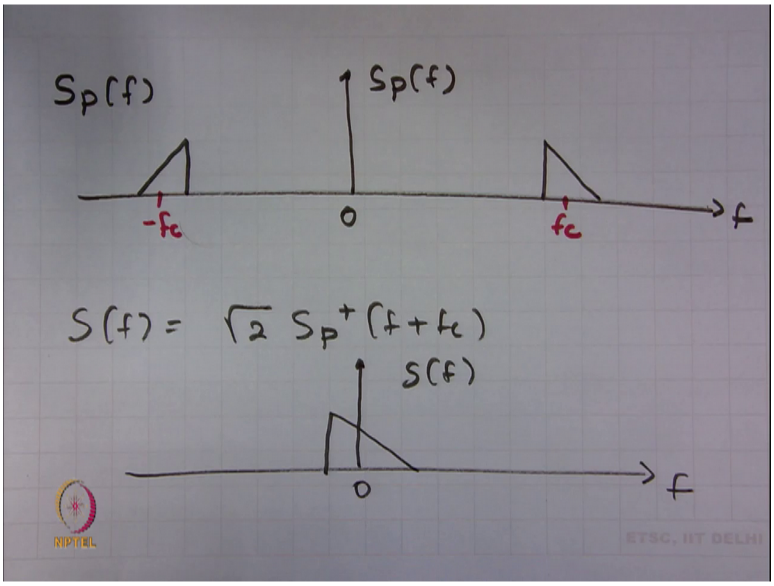

So, let us say that I have passband signal whose spectrum is this. This is f equals to 0 I have such a spectrum. And let us say that my minus f c is here and plus f c is here. Now, first of all looking at this spectrum, we know that S p t must be our real signal. So, this is S p f. Why, because we have seen that if my S p t is a real signal then the magnitude of the spectrum should be symmetric, it should be even and that is the case here the magnitude of the spectrum is symmetric. So, this underlying signal should be a real signal or it might be a real signal at least right so that we get that clue from this picture.

Now, to get the spectrum of S f we can use the relationship. So, the one thing that we have to do is first we have to kill the negative part, and then we can translate the positive part towards negative direction because this plus f c. So, this should be shifted by minus f c towards left. So, resultant spectrum would be something like this. So, this is the spectrum of S f.

Now, if you look at this spectrum, you can notice two important things. Now, the magnitude is spectrum is not even, it is not symmetric and if the magnitude the spectrum is not symmetric we know that the underlying signal must be a complex signal. And that is the case S t in general is a complex baseband signal and as a spectrum would not be symmetric around 0. And now we know that this is for analytical convenience.

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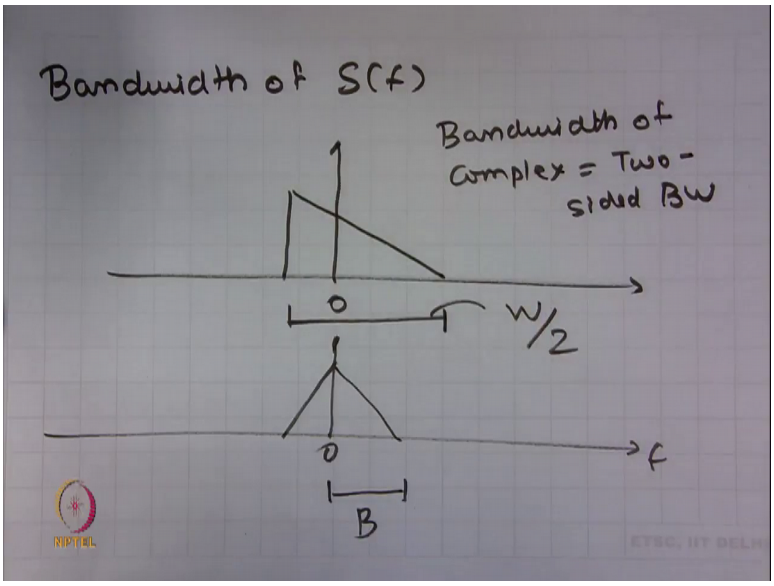

One thing that I would like to point out at this stage is if suppose we want to define the bandwidth of S f. And suppose S f is a given like this right. So, I have some signal and whose spectrum is given like this how do we define the bandwidth of such a signal. In case of the signals which are real for example there is spectrum would be symmetric around dc. And we say that the bandwidth definition that we normally choose is the one-sided bandwidth. So, meaning that if you want to talk about the bandwidth of this signal this bandwidth would be one-sided bandwidth means the range of positive frequencies.

And the idea that we have why we use this definition there is any way this is just the duplication of this part its symmetric. So, there is no extra information available in this component. And hence we take the bandwidth just to be the bandwidth of the positive part, in this case what should be the bandwidth. Now, there are many definitions that exist in literature which define the bandwidth of such a signal, one idea is that you take the two-sided bandwidth, let us say the two-sided bandwidth is W.

And then you divide this two-sided bandwidth by 2 and you treat this as the bandwidth of this complex signal. There is one definition that exists in literature. However, according to me the better definition of bandwidth for such signals is the complete two-sided bandwidth. So, the definition that we like to use is the bandwidth of a complex signal is two-sided bandwidth. And the idea is very simple. The idea is that this complex signal does not exist in reality; this is just an analytical signal. And this signal corresponds to a real passband signal is not it.

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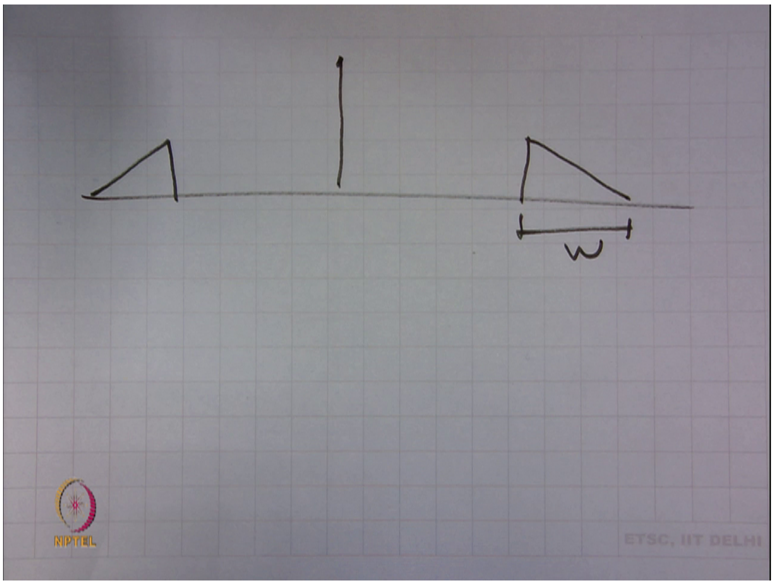

So, this signal actually corresponds to a passband signal which has the spectrum like this. And the bandwidth of this passband signal is the complete range of this triangle. And hence if you want to think about what is the bandwidth of this signal? Analytically the more proper definition of the bandwidth would be two-sided bandwidth, so that because this two-sided bandwidth corresponds to the bandwidth of the physical passband signal to which this complex baseband signal corresponds to. And hence we would use this as a definition of bandwidth of a complex baseband signal.

So, we have developed some key ideas in this lecture. We have understood how to define the bandwidth of a complex baseband signal, we have understood how to take the inner product between passband and a baseband signal. We have looked at the energy equivalence between passband and baseband signal and we have also established the orthogonality between I and Q components. And we have established the relationships between the spectrums of passband and baseband signals.

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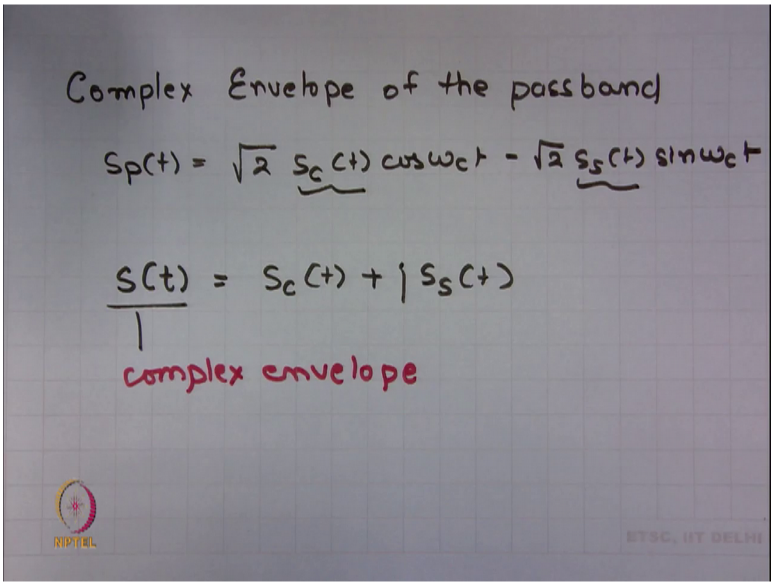

Now, what we would do is we would try to learn how to carry out certain pass band operations in baseband. How to do this, we would learn about these things using some examples so that whatever we have learnt can be made use of. So, things that we would do now is we would try to see basically three important things.

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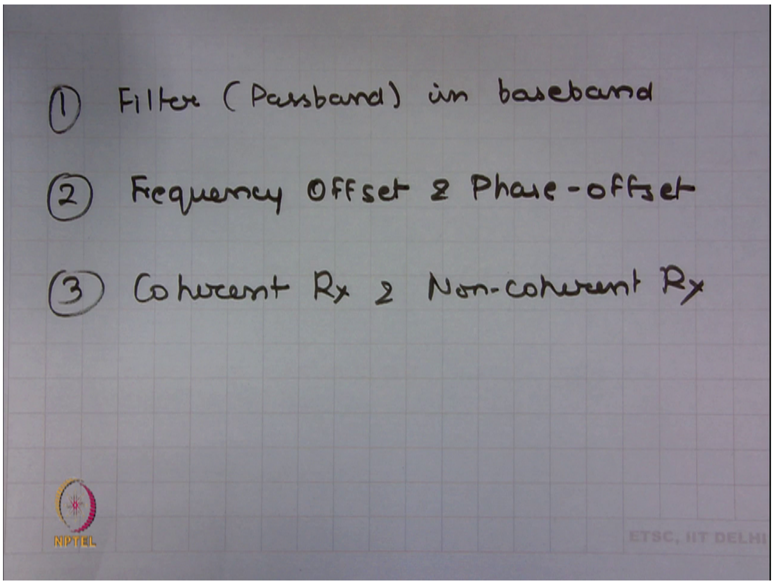

Suppose, I want to implement a filter or let us say passband filter in baseband, how can we do that using the ideas that we have established. So, this is the first thing that we will learn as an example or trying to use things that we have learnt. The second thing that we would see is so there are certain effects which we will talk about frequency offsets of phase offsets which reduces the performance of our digital communication system receiver or a system in general. So, we would try to see can we get rid of these effects by doing some processing in baseband and we would see that.

Third idea that we would like to develop is how can we think about this coherent receivers and non-coherent receivers in terms of the inner products and things like that that we have discussed and how can we think about these receivers in baseband. So, with these three examples, we would complete this discussion on this complex baseband and passband equivalence. And then onwards we would start with this spectral description of sources. So, these examples we will discuss in the next lecture.

Thank you.