Principles of Digital Communication
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Lecture – 13
Random Variables & Random Processes: Gaussian Random Process (Part-1)

Good morning. Welcome to a new lecture on Random Processes. And in this lecture, we will be discussing a very important concept and that is about Gaussian Processes. So, let us get started.

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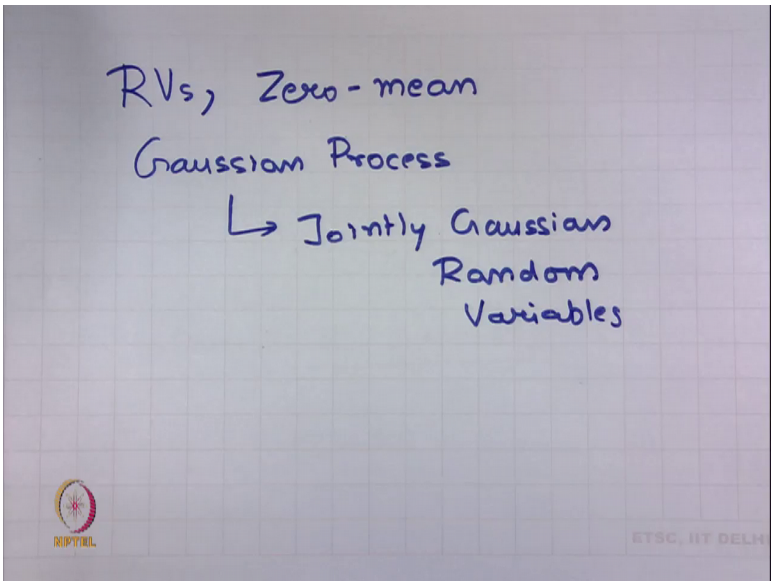

So, at the onset of the lecture I would like to point out that the random variables and the random processes that we will consider will always be with 0 mean ok. So, this is what we will assume. Why do we want to assume this? As I have already stated several times we want to assume random variables with 0 mean, because physical noise processes are with 0 mean and secondly, it makes my equations simplified ok. So, these are the two important reasons why we want to assume that ok.

So, let us get started and let us start talking about Gaussian processes by first discussing what is this jointly Gaussian random variables? So, we have to first understand jointly Gaussian random variables and once we have understood these jointly Gaussian random variables, we will be able to talk about these Gaussian processes. So, this is the outline of this lecture we will talk about jointly Gaussian random variables and then we will start thinking about Gaussian processes ok.

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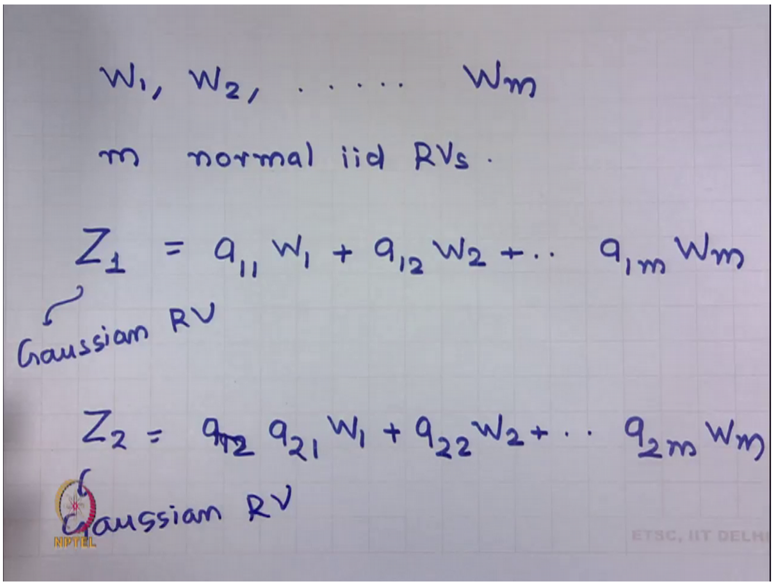

So, what we have is let us assume that I have m normal iid random variables ok. So, W 1, W 2 and w m are m normal iid random variables ok. Then what I do is I make a linear combination of these random variables, once I make a linear combination of these random variables I will end up with another random variable I call this random variable as Z 1 ok.

Now, what is this Z 1? Because Z 1 is a linear combination of m normal iid random variables Z 1 will be a Gaussian random variable. So, we have already seen previously if you have a linear combinations of normal statistically independent random variables what you end up with is a Gaussian random variable, right. So, this we have seen before. So, Z 1 is a Gaussian random variable. What I do is I obtain a second random variable which is obtained again by the linear combination of these m normal iid random variables, but now I have a different linear combination ok. So, let me have ok, so Z 2 is again a Gaussian random variable by the same logic as Z 1 is a Gaussian random variable, so, Z 2 is also a Gaussian random variable.

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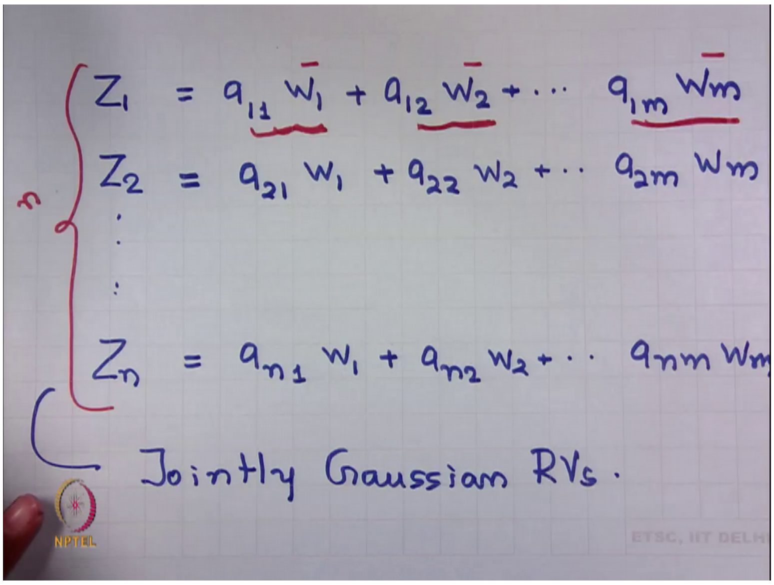

Similarly, I can extend this concept further and I can have let us say n random variables which are obtained; so I have now n random variables and these n random variables are obtained by different linear combinations of common underlying set of normal iid random variables ok. So, they are obtained by different linear combinations of W 1, W 2 and W m ok. These random variables which are obtained by different linear combinations of common underlying set of normal iid random variables are in fact, jointly Gaussian random variables ok.

So, what are jointly Gaussian random variables? Jointly Gaussian random variables are the random variables which are obtained by different linear combinations of common underlying set of normal iid random variables ok. So, this is what is jointly Gaussian random variable ok.

So, why these jointly Gaussian random variables are so useful? Because they depict a physical noise process. So, what is the noise? Noise is nothing but it is a collection of many independent random disturbances and this is exactly what is happening here. So, if you see that these components denote random disturbances, right, these are also independent components.

And we have also seen in the central limit theorem if you add a bunch of many independent random disturbances we get a Gaussian random variable of course, the sponge size has to go up to infinity ok. So, this is exactly what is happening here, we are having various independent components the sum of these independent components is giving us a Gaussian random variable ok.

So, these jointly Gaussian random variables does model physical noise processes and why this W 1, W 2 and W m are assumed to be normal iid, this is just to keep a model simple and tractable ok. And that is why they are of interest. Now, let me write this equation or these set of equations in a simpler way. Let me write it like this ok.

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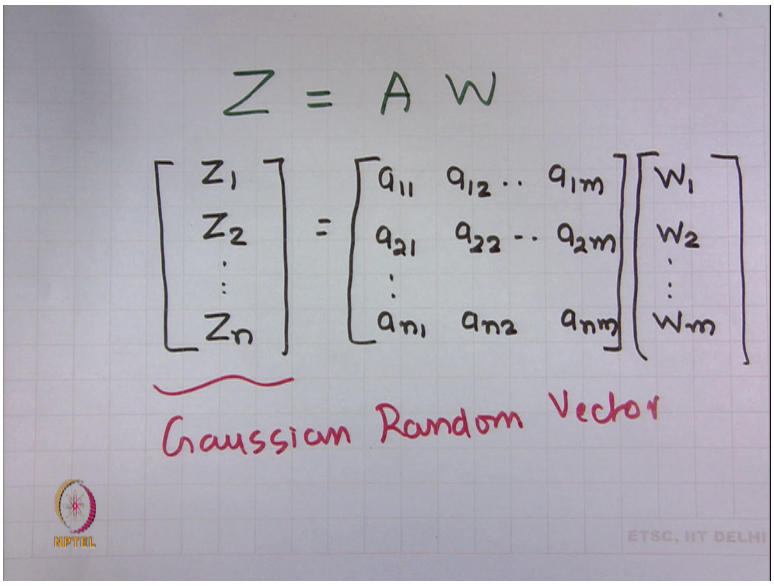

So, what is Z? Z is a vector column vector which consists of random variables Z 1, Z 2 and Z n and as we have seen these Z 1, Z 2 and Z n are jointly Gaussian random variables, and a is a matrix a is this matrix and if I multiply this matrix a with random vector w whose elements are normal iid random variables then I can obtain this random vector Z ok. So, what I am doing is I am just expressing these set of linear equations compactly in this way.

Now, something interesting this random vector Z has its elements which are jointly Gaussian random variables and such a vector is known as Gaussian random vector. What is a Gaussian random vector? A Gaussian random vector is a vector whose elements are jointly Gaussian random variables and because these Z 1, Z 2 and Z n are jointly Gaussian random variables this vector Z is a Gaussian random vector ok. So, what I am saying is if you have a random vector whose elements are normal ideal random variables you multiply this random vector with a suitable matrix then you end up with a Gaussian random vector ok.

So, what is a definition of Gaussian random vector? A Gaussian random vector can be obtained by linear transformation of a random vector whose elements are normal iid random variables ok. So, this is another way in which you can talk about this.

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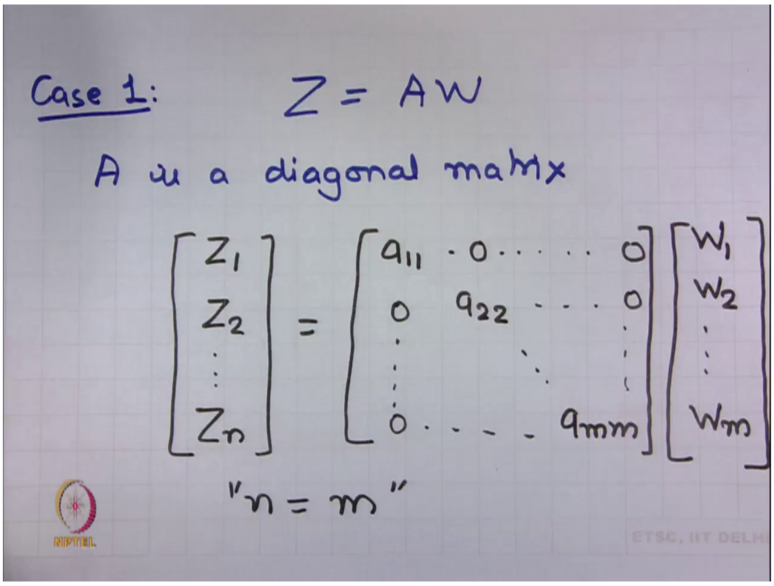

So, let us continue with our model and let us explore some cases in this equation. So, the first case is when A is a diagonal matrix. Let us see what happens. If A is a diagonal matrix I have ok, so, all other elements are 0. So, all other elements except the elements along the diagonal are 0 in a diagonal matrix and also remember the diagonal matrix is a square matrix ok. So, in this case n should be same as m ok.

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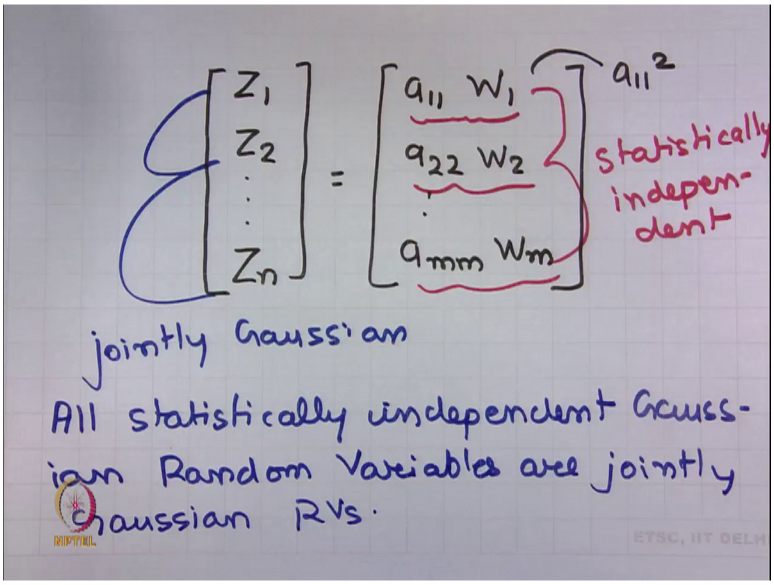

So, now, if A is a diagonal matrix what happens is; remember that n is same as m. So, how have you obtained this is by simply multiplying this matrix with this column matrix, ok.

Now, this W 1 and W 2 and W m are statistically independent, right we have always assumed them to be iid random variables. So, this a 11 w 1, a 22 w 2, a mm W m are also statistically independent. Are they iid? They are not iid because even though they have a mean of 0 the variance has changed. So, what is the variance of this element? So, the variance would be a 11 square ok. So, the variance of these random variables have changed because of this multiplication with these real numbers, but they are independent and also they have a mean of 0.

Now, we have already said that Z 1, Z 2 and Z n will be jointly Gaussian random variables ok. So, this is the definition. If you have a vector w which has its elements as normal iid random variables if you multiply with a suitable matrix such that this matrix multiplication is defined then the random vector that you have is a Gaussian random vector and if it is a Gaussian random vector its elements are jointly Gaussian random variables. So, what we can say is a 11 W 1, a 22 W 2 and a mm W m are also jointly Gaussian random variables and thus all statistically all statistically independent Gaussian random variables are jointly Gaussian random variables ok. So, this is an important idea all is statistically independent Gaussian random variables are jointly Gaussian random variables. Now, let us consider case 2, let us go to case 2.

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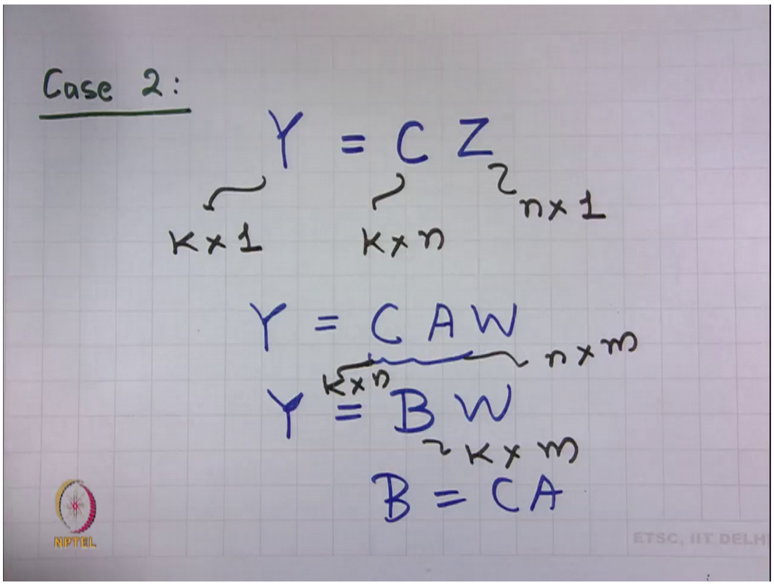

In case 2, let us consider a matrix Y which is obtained by multiplying this random vector Z with a matrix Cok. And of course, we are multiplying it with the matrix C such that this matrix multiplication is defined. As I have said before this Z is offer dimension and by 1 let us assume C to be off dimension k by n and so Y will be of dimension k by 1 ok.

Now, we already know that Z is A times random vector W. So, we have Y equals to C times A times W ok. Now what is the CA? Let us consider that this CA is nothing, but it is some matrix B and so I have Y as B times W, where B I have assumed is C times Aok. So, this A was n times m, C is k times n, so matrix B will be of dimension k times m ok right.

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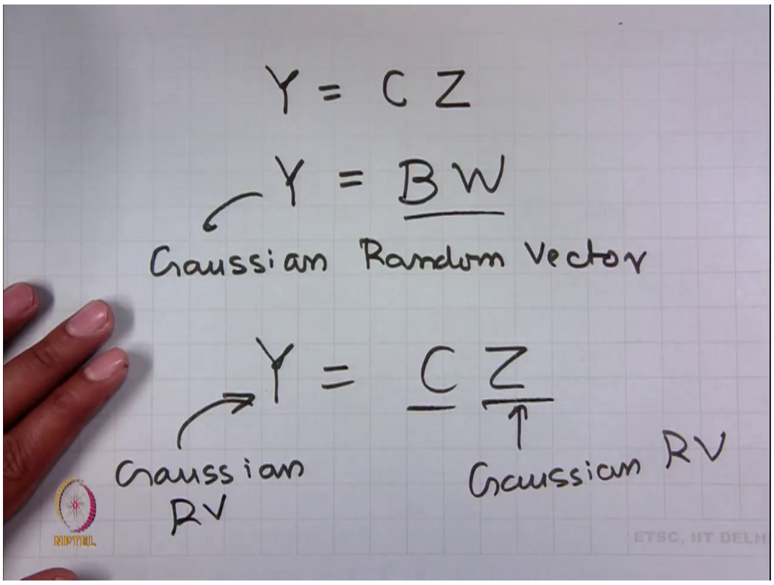

So, what equation we have got is, if Y is C times Z I can also write Y as B times W and this Y is also Gaussian random vector. Why is this a Gaussian random vector? Because it has been obtained by multiplying this random vector W, where W has elements as normal iid random variables with a matrix B ok. And we have said that any such operation should lead to a Gaussian random vector and its elements will be jointly Gaussian random variables ok.

So, in fact, what we are saying is if you take a Gaussian random vector, if you have a Gaussian random vector you multiply a Gaussian random vector with a matrix then what we get is a Gaussian random vector. And this is an important idea. So, linear transformation of a Gaussian random vector gives you another Gaussian random vector, ok.

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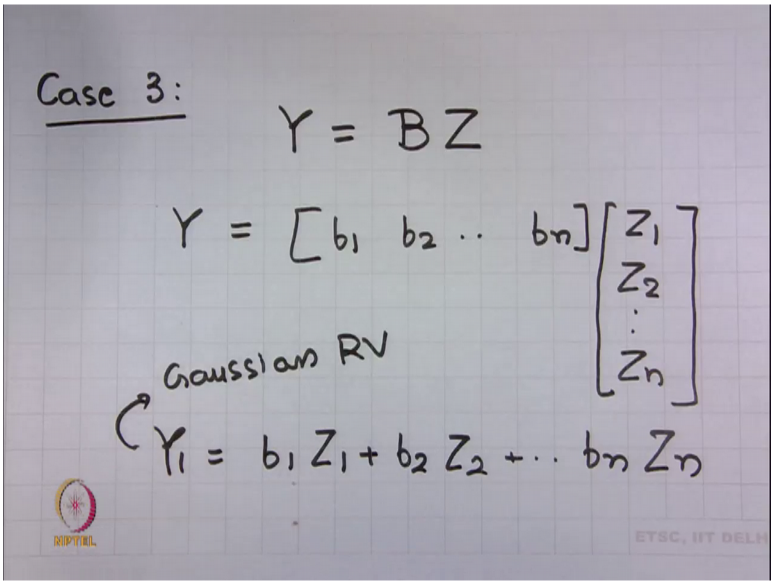

Let us now see case 3 and it is also going to be an important case. So, we have said Y is B times Z, right, where Z is a Gaussian random vector Y is also a question random vector. And let us now assume a special case of a matrix B, where B is a row matrix. So, let me assume b to be a row matrix ok. So, what I get is Y 1 is b 1 times Z 1 plus b 2 times Z 2, b n times Z n ok. And what is Y 1? It is a jointly Gaussian random variable, but since we have just one random variable, we can say that Y 1 is a Gaussian random variable. Jointly Gaussian random variables are also Gaussian random variables, but they are sometimes more than just being Gaussian random variables ok.

So, Y 1 is a Gaussian random variable and what we are saying is a linear combination of jointly Gaussian random variables gives us our Gaussian random variable ok. And this is an important result because sometimes these jointly Gaussian random variables are also defined using this idea.

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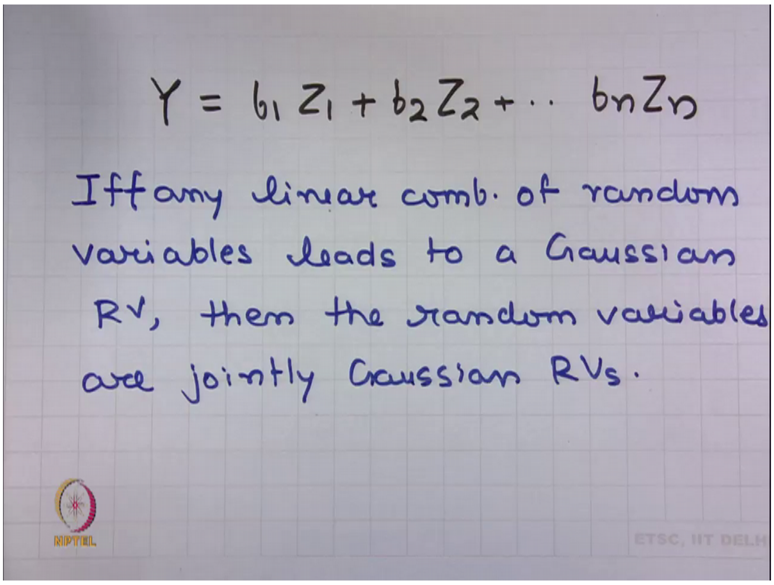

So, this leads to another definition of jointly Gaussian random variables. And what is this? Let me first try this equation again. So, if any linear combination; if any linear combination of random variables leads to a Gaussian random variable then the random variables are jointly Gaussian random variables ok.

So, this is another way in which we can define jointly Gaussian random variables. So, if any linear combination of random variables gives us a Gaussian random variable then the underlying random variables are jointly Gaussian random variables. And remember this is if and only if condition, right. If any linear combination does not lead to a Gaussian random variable then the underlying random variables will not be jointly Gaussian random variables. So, I can add one more if to indicate that this is if and only if condition ok.

So, we have learned the two ways in which we can define jointly Gaussian random variables. The first idea is jointly Gaussian random variables are the random variables which are obtained by different linear combinations of common underlying set of normal iid random variables, and the second interpretation is if any linear combination of random variables is a Gaussian random variable then the underlying random variables are jointly Gaussian random variables ok.

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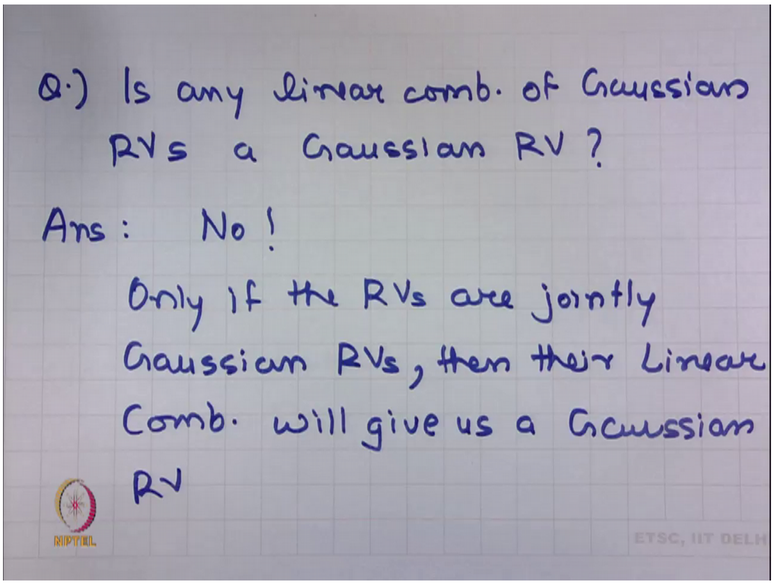

So, let us close this topic by asking another question because sometimes there exists a confusion in the head of students and that question is any linear combination; is any linear combination of Gaussian random variables, a Gaussian random variable ok. So, this is the question that we are asking, right. And what is the answer? Answer is no.

So, not any linear combination of Gaussian random variables is a Gaussian random variable. Only if the random variables are jointly Gaussian random variables then their linear combination; then their linear combination will give us a Gaussian random variable. So, this is important do not get carried away by thinking that any linear combination of Gaussian random variables will be a Gaussian random variable. This is not true the underlying random variables has to be jointly Gaussian random variables ok.

If these random variables are statistically independent random variables, if the linear combination that you are forming you are considering the random variables to be statistically independent Gaussian random variables then of course, their linear combination would be a Gaussian random variable because statistically independent random variables are nothing, but also jointly Gaussian random variables ok. So, the underlying random variables has to be jointly Gaussian random variables.

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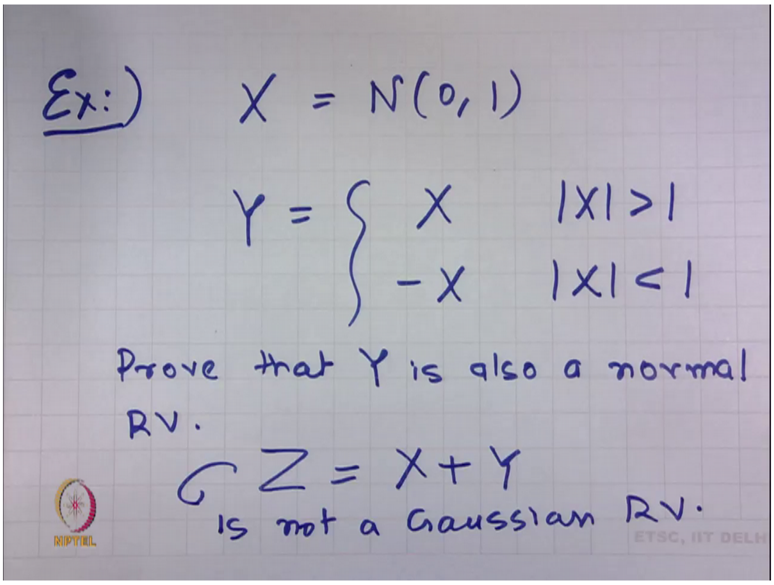

Let us see one example where the linear combination of Gaussian random variables is not a Gaussian random variable. Let us see one example, we will not derive this completely I will leave it to you to think about this.

So, let us consider X as a normal random variable with mean 0 and variance 1 and let us consider Y as a random variable which is defined like this. So, it is X if mod of X is greater than 1 and it is minus X if mod of X is less than 1. First you prove that Y is also a normal random variable ok. That first thing you have to think about and you can prove that Y is also a normal random variable. You can prove it intuitively as well you do not have to recourse to rigorous maths ok. Then let us consider Z, another random variable which is nothing, but sum of X plus Y then you should also prove that the Z is not a Gaussian random variable ok. You will see that that will not be a Gaussian random variable ok.

So, this gives you this example gives you some idea why a linear combination of Gaussian random variables is always not a Gaussian random variable ok. So, work this out and this will help you in clearing up this doubt ok. So, let us now start talking about something else.

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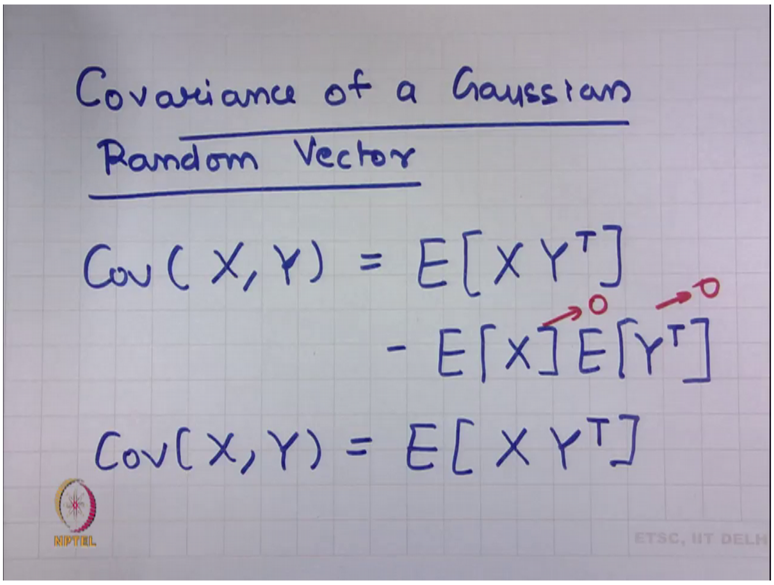

And this is the covariance of a Gaussian random vector. So, we have already defined what is a Gaussian random vector? And now let us talk about what is the covariance of this Gaussian random vector ok. So, let me revise what was covariance anyway. If I have two random vectors X and Y we define their covariance as, so, this we have already seen, right. So, I am not going to repeat this. So, this is how we define the covariance of two random vectors.

And if the random vectors that we are assuming are with 0 mean I have already said that that this is what we will assume throughout this course until and unless stated. So, we can always assume that these random vectors are with 0 mean, so this goes to 0 and the covariance of two random vectors with 0 mean is nothing, but expected value of X into Y transpose ok. So, let us see what happens to this covariance if the involved random vector is a Gaussian random vector.

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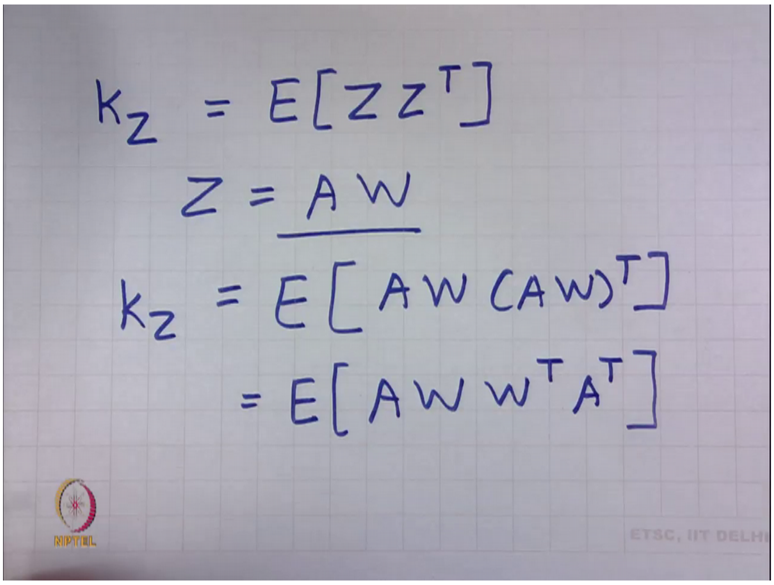

So, we define another notation k of Z, where k of Z is covariance of random vector Z where Z is a Gaussian random vector. So, using the same ideas we can write that this is nothing, but expected value of Z into Z transpose. And we have said that Z is what? It is some matrix times W, where W is a random vector consisting of normal iid random variables. So, substituting this value of Z in here we get k of Z is expected value of A W times A W transpose. And from here we get this is expected value of A W and this will be W transpose into A transpose.

And A and A transpose are deterministic quantities, so expected operator would not influence them, right and we can pull this expected operator inside this product. And what we will end up with is nothing, but A times expected value of W into W transpose times A transpose ok.

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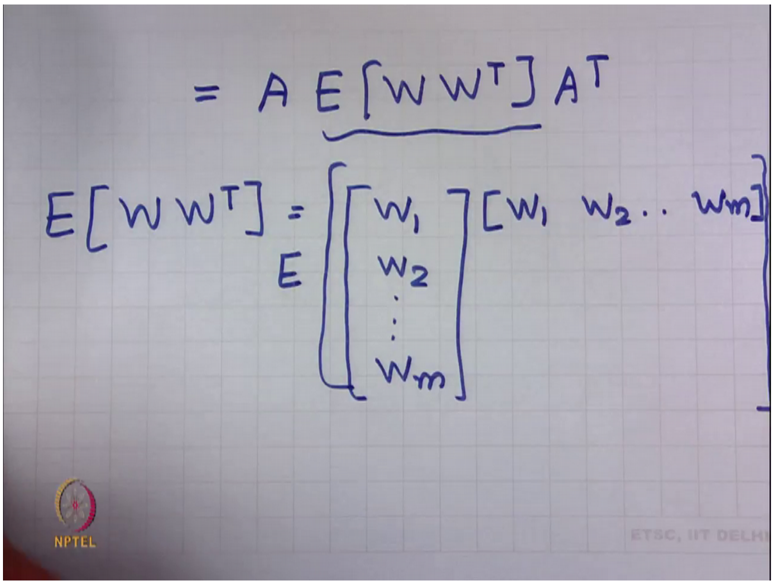

Because A and A transpose are deterministic quantities and we have seen in one of the previous lectures that expected value of a constant times a random variable is nothing, but its constant times expected value of that random variable ok. So, this is what we get.

Let us now see what is this. So, what is expected value of W into W transpose ok? So, as I have already pointed out several times that when we are considering random vectors they are assumed to be column vectors. So, W is a column vector which is like this and W transpose will be this and of course, we have to take the expected value of the product of these two random vectors ok.

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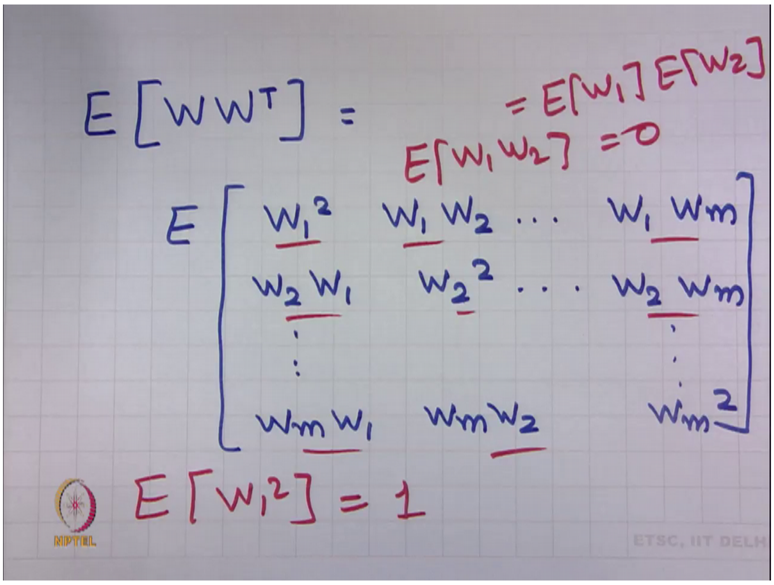

So, multiplying term by term what we get is expected value of W into W transpose is nothing but expected value of W 1 square, W 1 times W 2, W 1 times W m, we get W 2 times W 1 and rest could be filled easily ok.

Now, so when we want to take the expected value of a matrix what we can do is we can take the expected value term wise, right. So, this is also what we have seen. So, let us now see quickly what would be the expected value of such cross terms. What is the expected value of W 1 times W 2? Because these random variables are statistically independent this would be nothing, but expected value of W 1 times expected value of W 2 and because they are with 0 mean this would be 0. So, once you take expected value of these terms you will get a flat 0. And so, what we get is expected value of W into W transpose.

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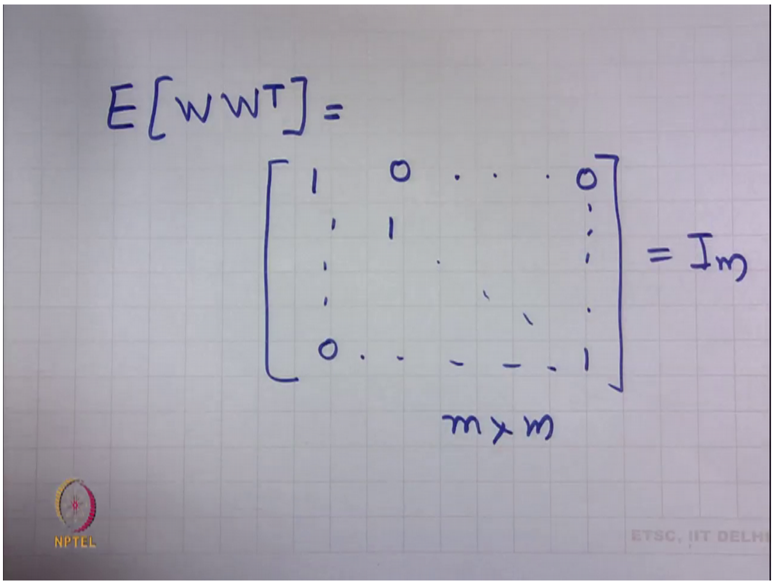

Let me simplify this bit more what is this expected value of these terms squared terms. So, expected value of W 1 square this is nothing but it is the variance of this random variable W 1 because it is with 0 mean and the variance of a normal random variable is 1, right. We have assumed these W 1, W 2 and W m to be normal random variables and so what we get is expected value of W into W transpose is a identity matrix ok. So, this is identity matrix I m where it has a dimension of m by m ok.

So, there is no randomness in this system, right. So, once you are taking expected value of this quantity you get an identity matrix there is no randomness involved here.

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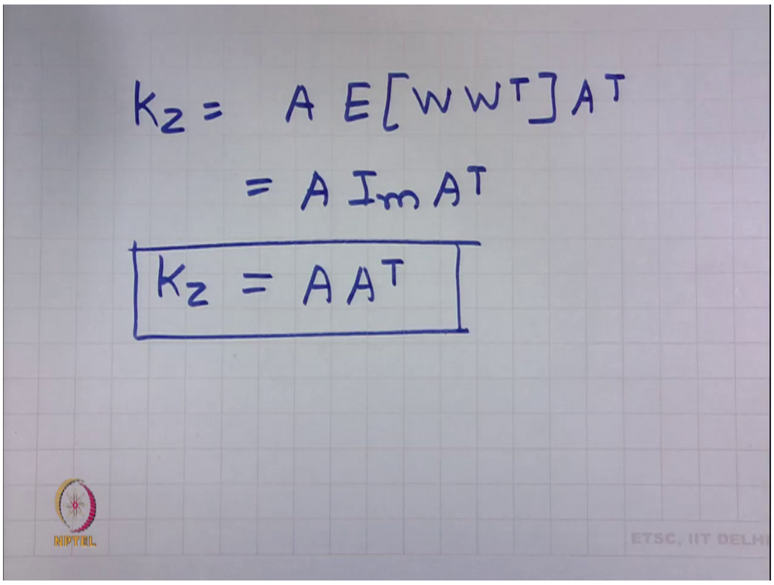

So, finally, what we were trying to find out is the covariance of the Gaussian random vector is said that this would be A times this quantity and this is nothing but identity matrix and A matrix times an identity matrix is nothing but the matrix itself. So, the covariance of a Gaussian random vector is simply A times A transpose, where A is the matrix with which you are multiplying the vector with normal iid random variables to get a Gaussian random vector ok. So, simple the covariance of a Gaussian random vector turns out to be quite simple ok.

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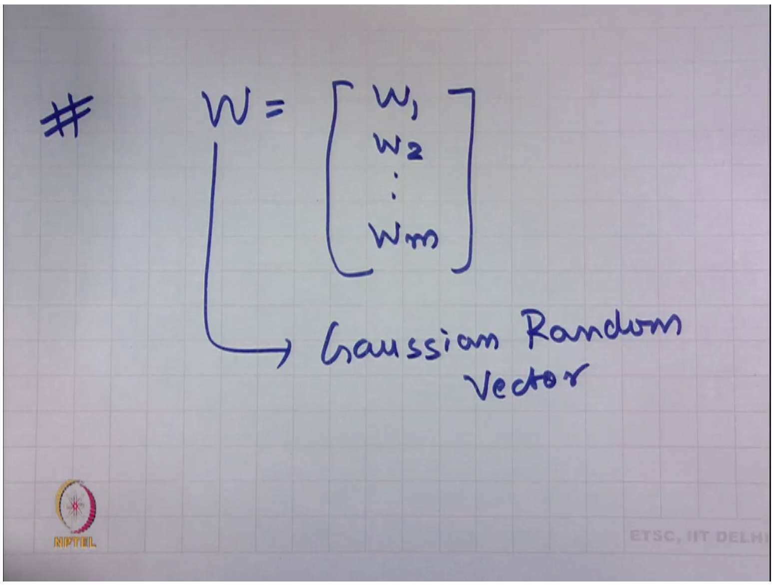

So, just one reminder one point more we are saying W is consisting of m normal iid random variables and because these are normal an iid remember that W is also a Gaussian random vector, is not it. So, it is a special kind of Gaussian random vector where the random variables are independent, right as well ok.

So, we have learned a lot about Gaussian random vectors and what are these jointly Gaussian random variables, we have also seen how to find the covariance of a Gaussian random vector and it is the time now to talk about Gaussian process.

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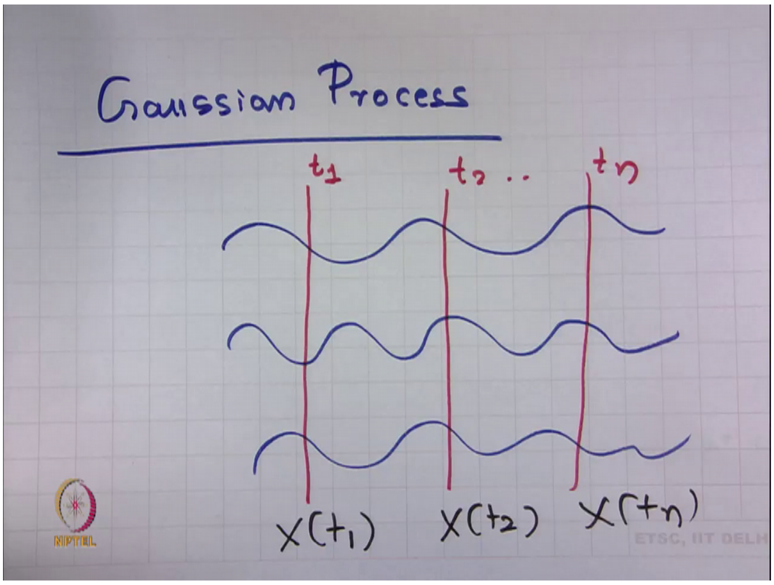

So, let us define the big concept for today and this concept is about Gaussian processes. So, to talk about these Gaussian processes let us first recap what we have learned about random processes. So, we said in the context of random processes that random process is built using several sample functions. And to think about these random processes what we can do is we can take the samples of these random processes.

Let us say I take the sample of this random process at time instant t 1, t 2 and up to t n, and when I take the sample of this random process what I end up with is set of random variables ok. So, we get these set which is consisting of n random variables X t 1, X t 2 and X t n and what is then this Gaussian process? A Gaussian process is a random process whose random variables are jointly Gaussian random variables ok. So, this is a big idea.

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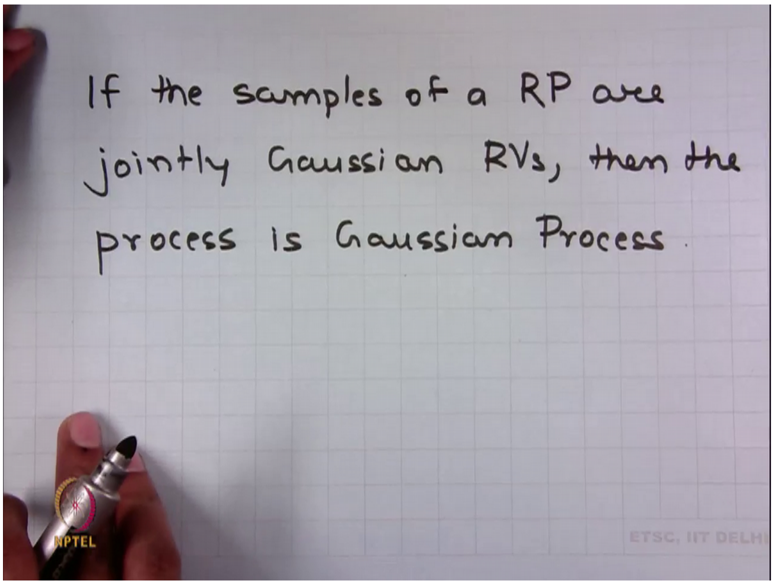

If the samples of a random process are jointly Gaussian random variables then the process is a Gaussian process ok. So, thinking about Gaussian process is easy once we know what are these jointly Gaussian random variables. So, all samples of the random process should give us jointly Gaussian random variables and then the underlying random process is referred to as Gaussian process and this Gaussian process is of lot of interest to us in communication system because noise as we will see is also nothing but a Gaussian process ok.

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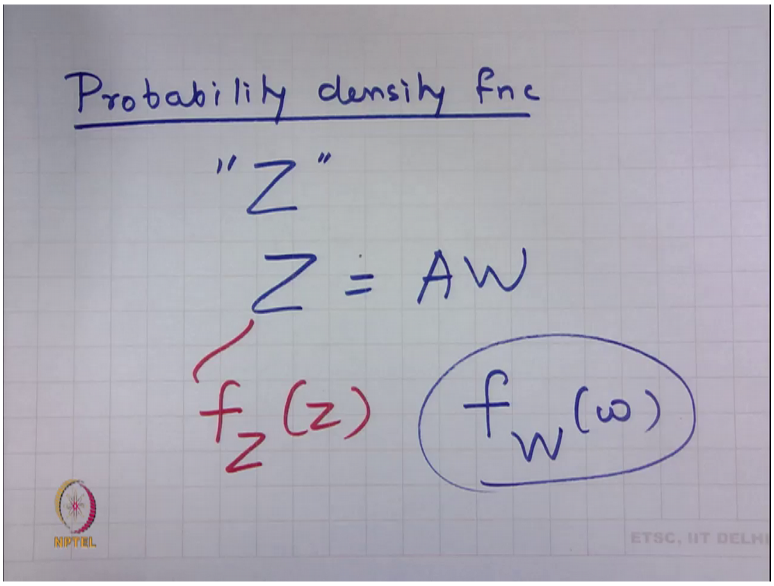

So, what is of interest also to us is to model the probability density function. So, what we are interested in is finding out this probability density function or rather the joint probability density function of this random vector Z, where Z is a Gaussian random vector. Why we want to understand about probability density function of a random vector Z? Because this random process or Gaussian process can be understood by taking the samples, right and the samples will give us a bunch of jointly Gaussian random variables and if I collect this bunch of jointly Gaussian random variables I have a Gaussian random vector ok.

So, instead of talking about a Gaussian process at all time instances, I will like to talk about this random process only at countable time instances and once I want to specify a process in terms of countable time instances I can also talk about this process in terms of a random vector ok. And once I sample this Gaussian process at countable time instances what I end up with is a Gaussian random vector ok. So, that is why we want to think about a Gaussian random vector and that is why now we are also trying to think about what will be the probability density function of this Gaussian random vector Z ok.

So, where to start from? So, we can start from a good old relationship that this Gaussian random vector is nothing, but matrix multiplied with a Gaussian random vector W, where W is special Gaussian random vector where its elements are also independent of each other ok. So, this is the objective trying to find out the joint probability density function of this random vector Z and we already know the way in which we want to talk about this probability density functions ok. So, this denotes the probability density function of random vector Z where random vector takes in numerical value z, ok.

So, to think about this let us first calculate what is the probability density function of this random vector W, ok? To evaluate this is easier; we will start thinking about what is the probability density function of this random vector W ok.

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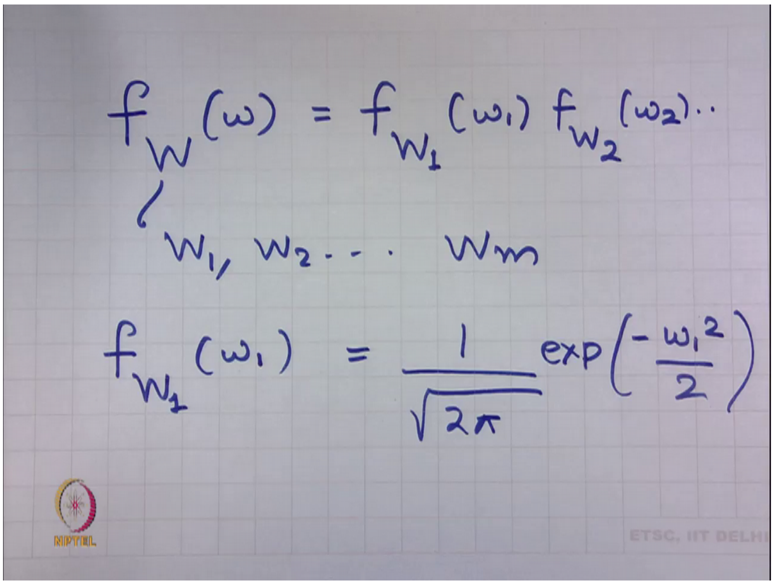

So, let us try to think about this. And what is the special in W? That is elements are independent, if the elements are independent then the joint probability density function is nothing but it is the product of marginal probability density functions. So, let us assume that the elements of W as we have been assuming so far are nothing but W 1, W 2 and W m. So, the joint probability density function for this random vector W is nothing but it is the product of marginal periods, ok up to W m ok.

Now, what is this probability density function of random variable W 1? It is in fact, a normal random variable and if it is a normal random variable we already know its probability density function. What is this? It is nothing but this, right. So, now, I can find the joint probability density function by substituting the marginal probability density functions of W 1, W 2 and so on so forth. So, what I will wind up with is and so and so forth.

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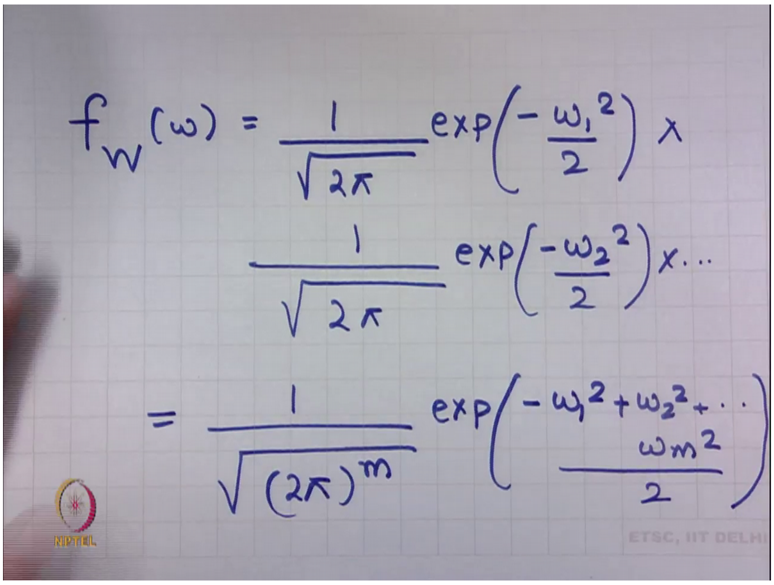

So, I can write this in one go as ok. So, this will be raised to power m because there are m such terms. And I know what is this quantity? So, what is this quantity? This is nothing but norm square of a vector. So, norm of norm square of a vector is nothing but we have already seen this is nothing but it is the sum of square of the components of this vector. So, norm square of a vector is nothing but this quantity where w 1, w 2 and w m are the elements of this random vector w, ok or any vector w.

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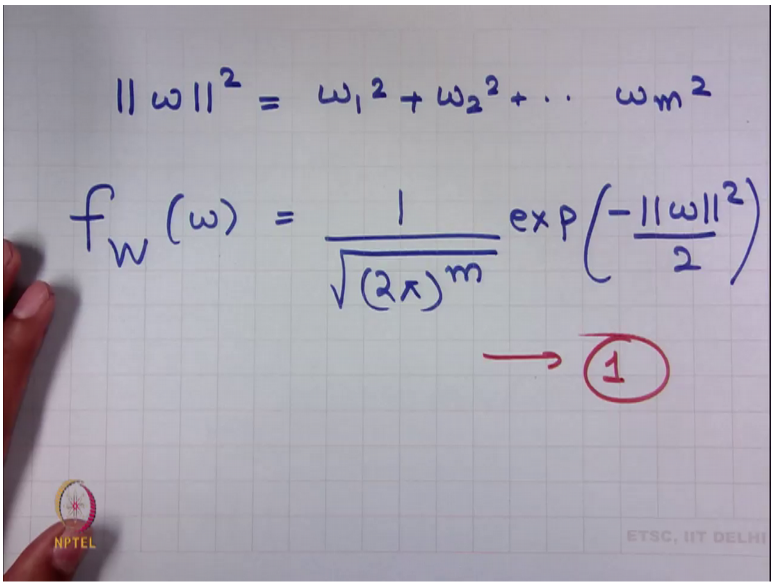

So, what I get is a simple expression like this. So, this is one important expression that we will use let me call this as question number 1 and then I need to have another expression to find out the probability density function of Z in terms of probability density function of W ok. So, our initial objective is to find out this PDF we have already find out the joint PDF for W and now I will use or explore this expression to express the probability density function of Z in terms of royalty density function of W. So, let us see ok.

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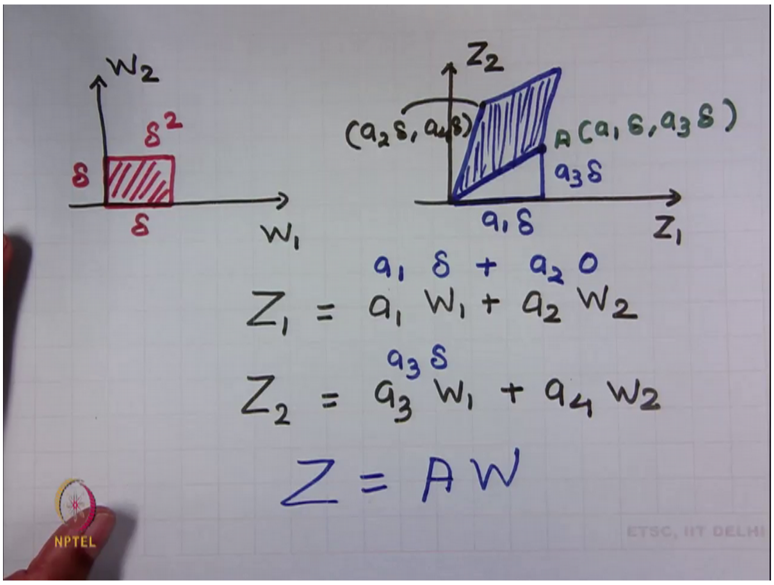

So, to think about this let us invoke geometry and let us assume that I have two random variables W 1 and W 2 and let us also assume that I have another two random variables Z 1 and Z 2. And let us assume that these random variables Z 1 and Z 2 are obtained by linear combinations of these W 1 and W 2.

So, for example, let us assume Z 1 is a 1 times W 1 plus a 2 times W 2 and let us assume that this random variable Z 2 is a 3 times W 1 plus a 4 times W 2. And let us now consider an element of length delta along this random variable W 1. And let us see how this element of length delta is transformed into this Z space. So, if we want to look at this what we are assuming is W 1 as an element of length delta. So, this will be a 1 times delta plus. So, there is no length along W 2, so this would be a 2 times 0. So, that when will thus become a 1 times delta, so, we can draw this here Z 1 would be a 1 times delta. Similarly, Z 2 will become a 3 times delta.

Overall what we will have is that this element of length delta will transform to this element and the coordinates of this point; let us assume that this point is A and the coordinates of this point would be a 1 delta and a 2 delta. Similarly, if I consider an element of length delta along W 2 this element would transform into this space let us assume into this element

And if you look carefully by having the similar logic this point will be a 2 delta and a 4 delta. Now, from these elements let us now look into what happens if I consider area. For example, if I consider a square of area delta square this square will map to a parallelogram of some different area ok. So, this is what happens when you linearly transformed random variables.

So, remember what we are doing here is we are having a random vector Z which is A times random vector W and in this case for example, this random vector W contains two random variables W 1 and W 2. This random vector Z contains two random variables Z 1 and Z 2 and A is a matrix whose elements are a 1, a 2, a 3 and a 4.

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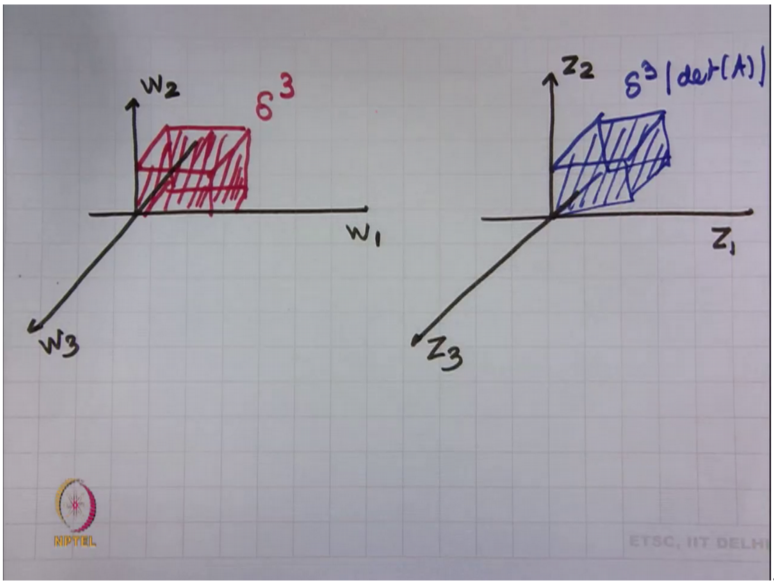

So, I can now extend this concept from two dimension to three dimension and in three dimension; let us assume that I have 3 random variables W 1, W 2 and W 3 and similarly for Z space also I will have 3 random variables Z 1, Z 2 and Z 3. And in two dimension we had a square, in three dimension we will have a cube, let us consider that we have a cube here the volume of this cube will be delta cube and similarly in three dimension this cube will map to a parallel pipe in Z space. So, in that space we will have a parallelepiped, let us consider that we have a parallelepiped here and this parallelepiped will have a volume of delta cube times mod of determinant of A.

And why is this so? This you must have seen in the basics of vectors the most fundamental way in which you can think about determinants is that determinant is nothing but it is the volume scaling factor of the linear transformation described by the matrix A.

If I have such a linear transformation then linear transformation is carried out with this matrix A. If I want to find out how much the volume scales from W space to Z space, the volume in Z space can be simply obtained by multiplying the volume in W space by mod of determinant of A. So, this denotes the volume scaling factor of the linear transformation. So, this is one idea that we will use. The second idea is because each point in this space is getting mapped to a point in this space the probability with which a point lies in this red cube will be same as the probability with which a point will lie in this blue parallelepiped this probability will be same. So, let me write down this.

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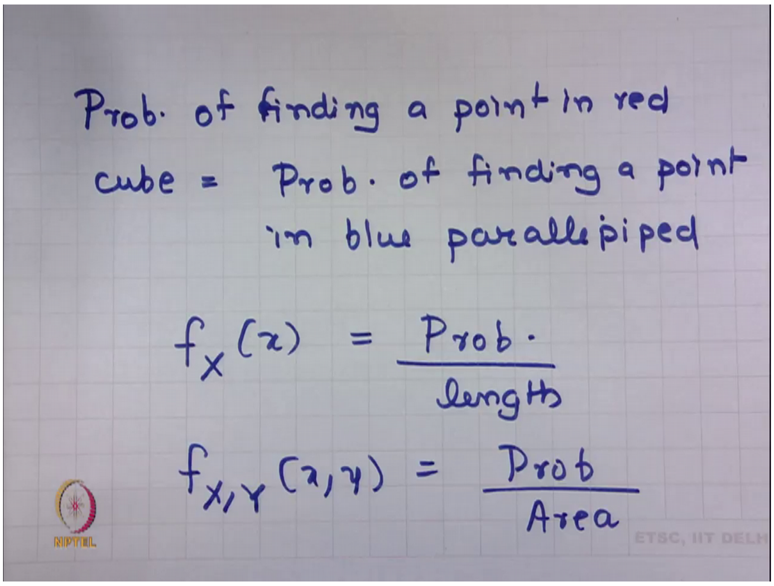

So, probability of finding a point in red cube is same as probability of finding a point in blue parallelepiped ok. Now, what is this probability? We have already seen that this probability can be thought in terms of probability density functions, where if I just consider one random variable then this probability density function of one random variable is nothing but it is the probability per unit length. If I have two random variables involved then the probability density function is nothing but probability per unit area. If I have 3 random variables then the probability density function is nothing but it is the probability per unit volume.

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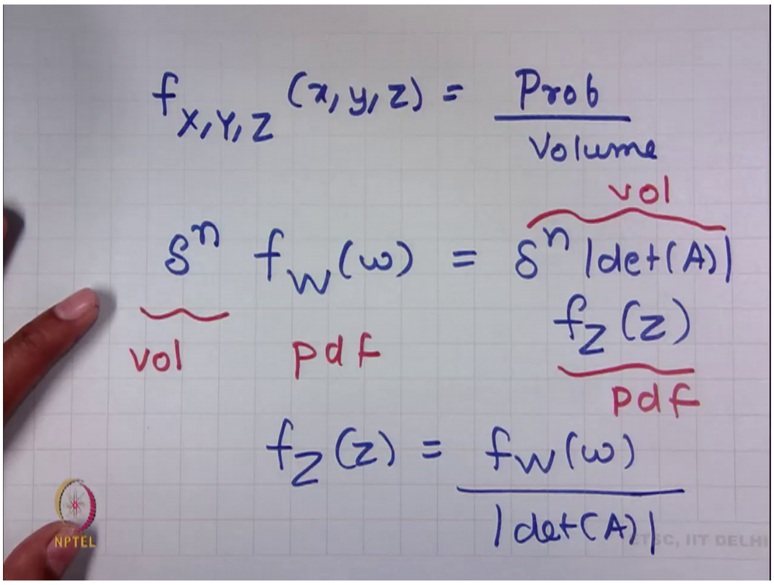

And hence I can think about the probability in three dimension or in n dimension where n is more than 3 by just multiplying the probability density function with the volume. So, probability of finding a point in red cube is nothing but probability density function times volume and let us say the volume is delta to the power n, where I am interested in n dimensional space. So, this is volume and this is probability density function and this will give me probability of finding a point in a red cube. And this thing should be same as the volume of a blue parallelepiped times the probability density function in Z space, ok.

So, again this is volume and this is the probability density function. So, then from this simply what we can obtain is the probability density function in a Z space is nothing but the probability density function in W space divided by mod of determinant of A. So, once I know the probability density function of random vector W, I can easily find the probability density function of a random vector Z by using this expression. So, these are the two important results. What are those?

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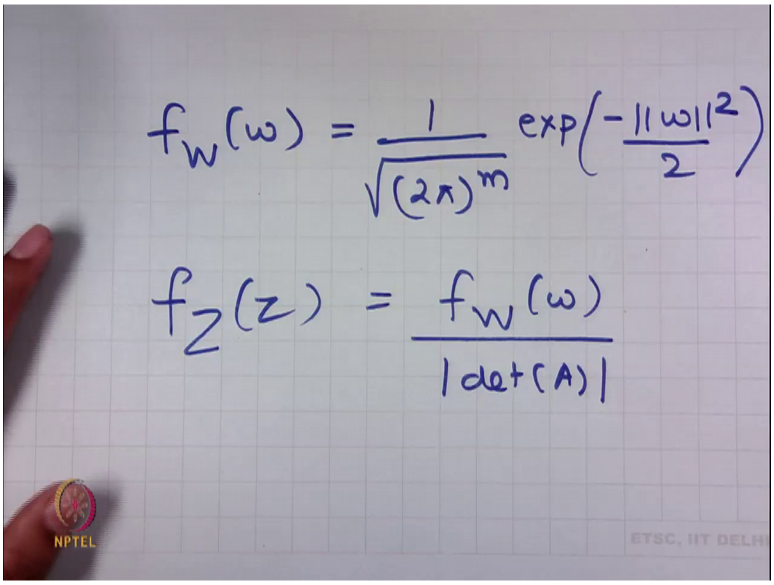

First, we have seen that probability density function of a random vector W is this and then we have seen that probability density function of a random vector Z is nothing but this. And using these two expressions in the next lecture we will derive the probability density function of random vector Z ok.

So, this is what we will carry out in the next lecture. So, we conclude for today in today's lecture we have developed some important ideas and ideas where we have understood what are these jointly Gaussian random variables, we have understood what is this Gaussian random vector, we have understood what is this Gaussian process, we have understood what is the covariance of a Gaussian random vector and we have understood the relationship between the probability density function of random vector Z and random vector W, if Z is A times random vector W.

Thank you.