Electrical Equipment and Machines: Finite Element Analysis
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Lecture 04
Revisiting EM Concepts: Vector Calculus and Electrostatics

So, now, we would go to the fourth lecture of this course, we will talk more about vector calculus in this lecture.

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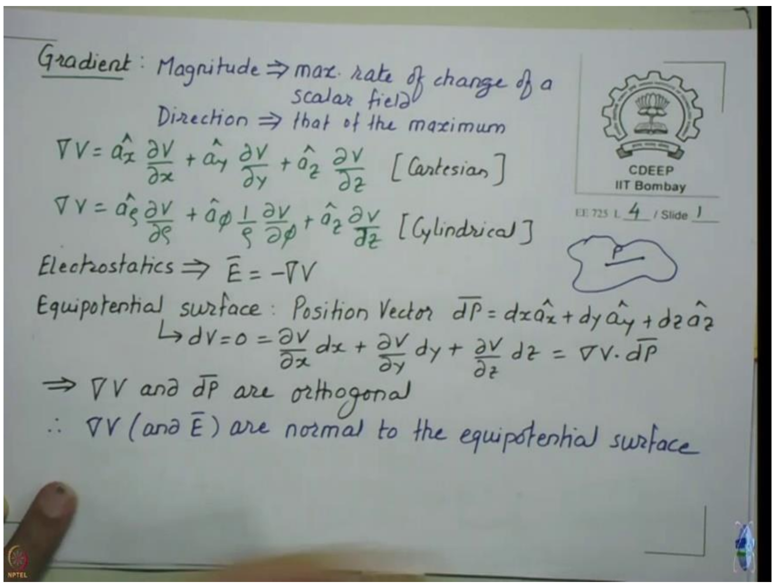

So, first is gradient concept. The gradient of a scalar has got both magnitude and direction. The magnitude describes the maximum rate of change of a scalar field and direction is the direction of the maximum change. So, we will see an example and more explanation of this in next slide.

So, the gradient of V in Cartesian and cylindrical coordinates are given in the following expressions

$$\nabla V = \hat{a}_x \frac{\partial V}{\partial x} + \hat{a}_y \frac{\partial V}{\partial y} + \hat{a}_z \frac{\partial V}{\partial z} \quad [\text{Cartesian}]$$

$$\nabla V = \hat{a}_\rho \frac{\partial V}{\partial \rho} + \hat{a}_\phi \frac{1}{\rho}\frac{\partial V}{\partial \phi} + \hat{a}_z \frac{\partial V}{\partial z} \quad [\text{Cylindrical}]$$

In electrostatics $\mathbf{E} = -\nabla V$. Now, here of course, units match because the unit of $-\nabla V$ is V/m because the unit of ∇ is 1/m and unit of E is V/m. Now, another thing to note here is the minus sign in the expression of E.

Now, let us understand intuitively why there should be a minus sign. By definition, E is always from a higher potential to a lower potential or say from a positive charge to a negative charge. By convention we take positive charges at a higher potential, and negative charges at a lower potential. So, E is from high to low. When we say gradient, it is always from low to high. So, that is why E and ∇V are separated by a minus sign.

So, if you take an equipotential surface, and suppose there are 2 points as $(x_1,\ y_1)$ and $(x_2,\ y_2)$, then you can write the distance vector P joining these 2 points as $dx\hat{\mathbf{a}}_x + dy\hat{\mathbf{a}}_y + dz\hat{\mathbf{a}}_z$ where $dx = x_2 - x_1$ and similarly other components dy and dz can be calculated. Since it is an equipotential surface, the potential difference between any two points on the surface is 0.

So, that means $dV = 0$ between any two points on this surface. Now dV can be written from total differential formula as the following expression.

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \nabla V \cdot d\bar{P}$$

Now, this expression can be written as $\nabla V \cdot dP$ because you have both the expressions ∇V and dP, as they are vector expressions, if you take the dot product of the two vectors ∇V and dP, you will get the potential difference (dP is a distance vector).

So, that means you have $dV = 0$ and $\nabla V \cdot dP = 0$ because it is an equipotential surface. That means ∇V and dP are orthogonal, because in the previous lecture we saw dot product of two vectors is 0 means the cos of angle is 0 so that means the angle has to be 90 degrees.

So, that means these ∇V and dP vectors are orthogonal and E and ∇V are oriented in opposite directions. It means ∇V as well as E are normal to the equipotential surface.

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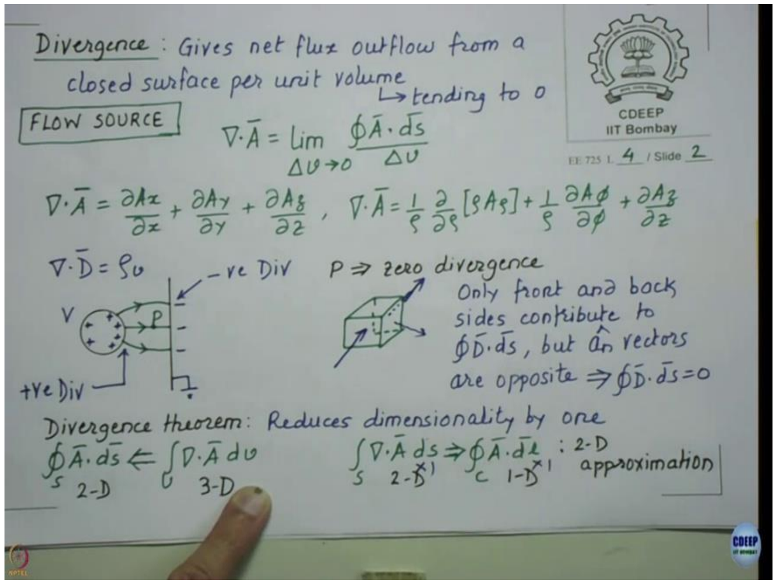


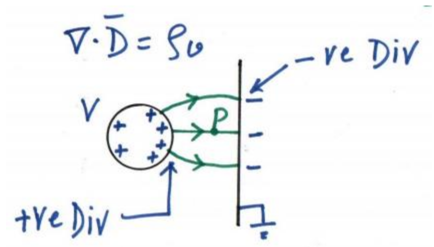
Next comes the divergence. Divergence gives a net flux outflow from a closed surface per unit volume with volume tending to 0. By definition $\nabla \cdot \mathbf{A}$ is given by the limit given below, with the small unit volume tending to 0.

$$\nabla \cdot \bar{A} = \lim_{\Delta v \to 0} \frac{\oint \bar{A} \cdot \bar{ds}}{\Delta v}$$

$\nabla \cdot \mathbf{A}$ in Cartesian coordinates is this and in cylindrical coordinates is given by these following expressions.

$$\nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \quad \nabla \cdot \bar{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}[\rho A_\rho] + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Now, remember divergence of a vector always represents a flow source. If divergence is non-zero that means, there is some source which is responsible for flow of fields. Now, let us again take the example of the lead to ground structure (given below) that we have been seeing quite often since beginning of this course.

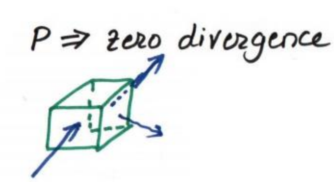


First you notice in the figure that I have purposely drawn the positive charges on the lead little bit closer to each other on the side that faces the ground plane and they are little bit sparsely spaced on the other side of the lead, why that happens? Because of the presence of this other electrode which is at ground potential, negative charges on that electrode will attract positive charges on the lead electrode. If it was an isolated cylindrical conductor, then the charges would be uniformly distributed.

So, whenever you see or draw anything you should consider the corresponding background physics. Similarly, on this ground plane, these charges will be more closely spaced as compared to the charges at the far ends.

Now it is easy to understand that divergence will be positive on the lead, because field vector is originating. Nothing is coming in and something is going out, and therefore net divergence is positive, whereas, on these negative charges, something is coming in, but nothing is going out from those points. So, the divergence is negative. Whereas, if you take some point P in the space between the lead and the ground plane, the divergence is 0.

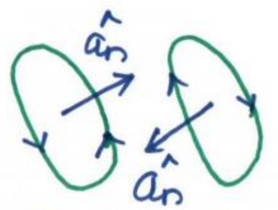
Now, how do we understand that? Now, assume that this middle field line is impinging on some kind of a cube (shown below) and at the center of the cube you have point P, and consider that you are calculating $\oiint A \cdot dS$. Now there are six faces for this cube.



The two side faces and the top and bottom faces will not contribute to the dot product, because this field is perpendicular to the area vectors of these faces. So, the only two faces that will contribute to the dot product are the front and back sides.

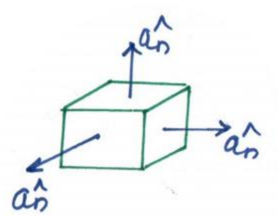
Because in that case, you have field as well as normal in the same direction, but there is a difference: the unit normal on the front phase is going to be outward. We need to understand the concept of the unit normal vector with reference to open surfaces and closed surfaces. Suppose you take this open surface, you have two choices for unit normal, either it could be like this or it could be going in the opposite direction.

But this choice does not remain with you, if you are considering the corresponding contour integral. The possible directions of the normal and the corresponding contour integrals are shown in the following figure.



Open surface

But if you take closed surface something like the one shown in the following figure.



Closed surface

Now for all faces of this closed surface the unit normal is always in the outward direction. So, for closed surfaces it is always an outward normal, for open surface, if you are considering the corresponding contour integral, then the direction of the unit normal gets fixed. If contour integral is not being considered, then you can take the unit normal in either of the two directions.

So, the unit normal in the case of the small cube is opposite to the direction of the field for this front face. Whereas for the back face the unit normal is along the direction of the field and that is why those dot products cancel each other and the overall dot product for the entire closed surface is 0 and that proves that since this is 0, divergence is 0.

Now, the divergence theorem is $\nabla \cdot \mathbf{A}\, dV$ evaluated over volume is equal to $\oiint A \cdot dS$. Now, effectively this theorem is useful to convert 3D (volume) into 2D (surface) integral. This is very useful particularly in integral equation numerical techniques, where the dimension of integral is reduced by 1. Similarly, suppose if you are doing a 2D analysis, mostly in this course also we are doing 2-dimensional analysis. So, when we are doing 2-dimensional analysis. we take the cross section which is perpendicular to the direction of current. Always, whenever you take a cross section you will find that we show the current direction using either dot or cross, that means you are taking always cross section which is perpendicular to the direction of current and we approximate that the 2D condition you see on the paper continues infinitely in Z direction. So, that is the approximation we do. For that approximation also if you apply this divergence theorem, effectively what you are doing is dv is effectively $dS \times 1$ which means you are taking per unit depth in the Z direction.

Although Z is infinite, we do calculation per unit depth. So, effectively dv is $dS \times 1$ and dS is $dl \times 1$. So, and dS in 3D case is same as $dl \times 1$ in 2D as given in the following equations. So, both the following equations are same but since we are actually doing a 2D approximation already, we have reduced the dimension by 1, and by further invoking divergence theorem, we have further reduced the dimension of the integral by 1, again this is very useful when you do 2-dimensional analysis.

$$\underbrace{\oint_S \bar{A}\cdot d\bar{s}}_{\text{2-D}} \Leftarrow \underbrace{\int_v \nabla\cdot\bar{A}\, dv}_{\text{3-D}} \qquad \underbrace{\int_S \nabla\cdot\bar{A}\, d\bar{s}}_{\text{2-D} \times 1} \Rightarrow \underbrace{\oint_C \bar{A}\cdot d\bar{l}}_{\text{1-D} \times 1} \;:\; \text{2-D approximation}$$

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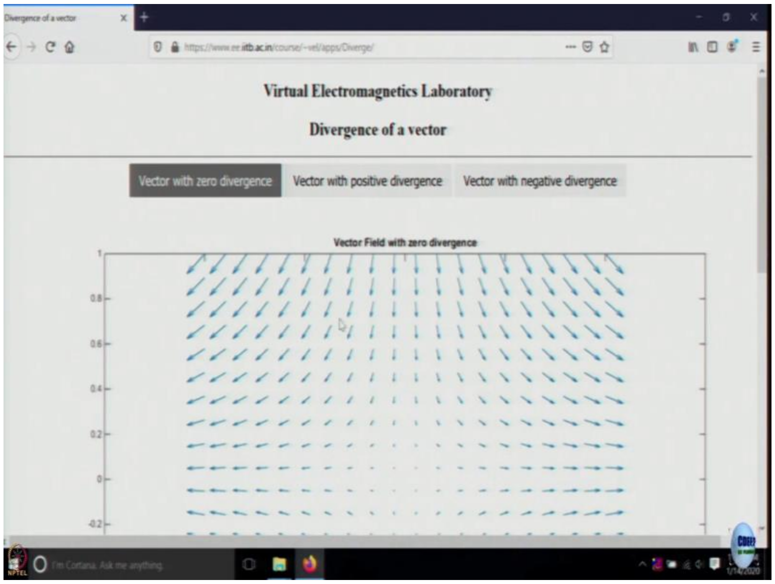

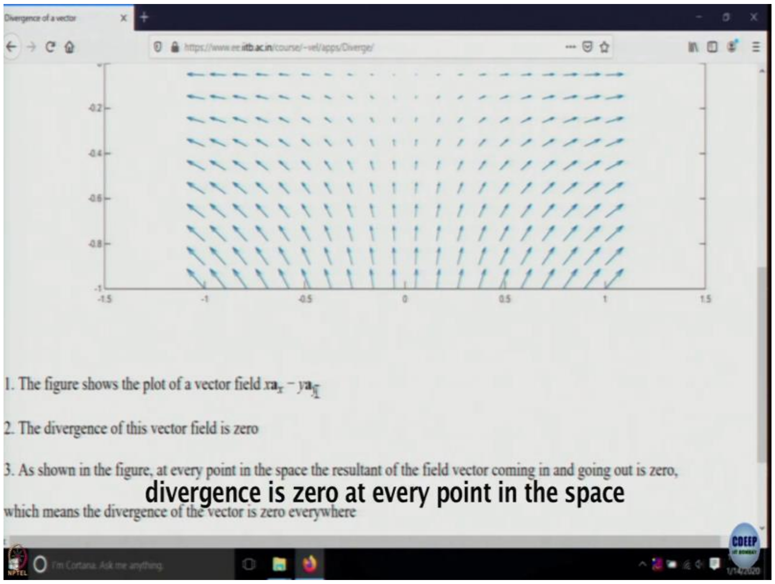

We again go to the virtual electromagnetic laboratory where divergence is explained through some field plots. Now, the field plots in the above slides are for divergence of the equation $x\hat{\mathbf{a}}_x - y\hat{\mathbf{a}}_y$. So, by definition (formula) divergence is 0. But then how do we understand this?

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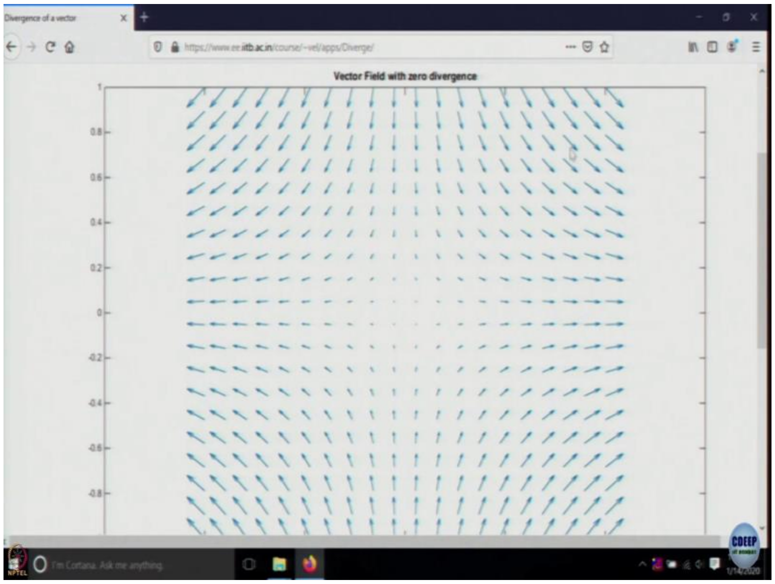

So, for example, you can take any point in the above field plot and you will understand that the rate of change of field with x and rate of change of field with y exactly cancel out. Also, you can observe form the above vector plot, if the field slope in x direction is increasing, the field slope with y direction is reducing by the same amount. So, that makes the divergence equal to 0.

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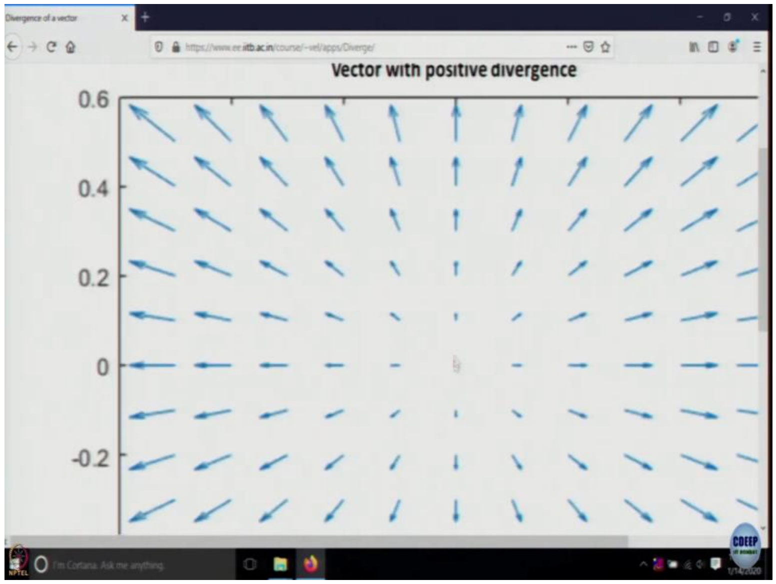

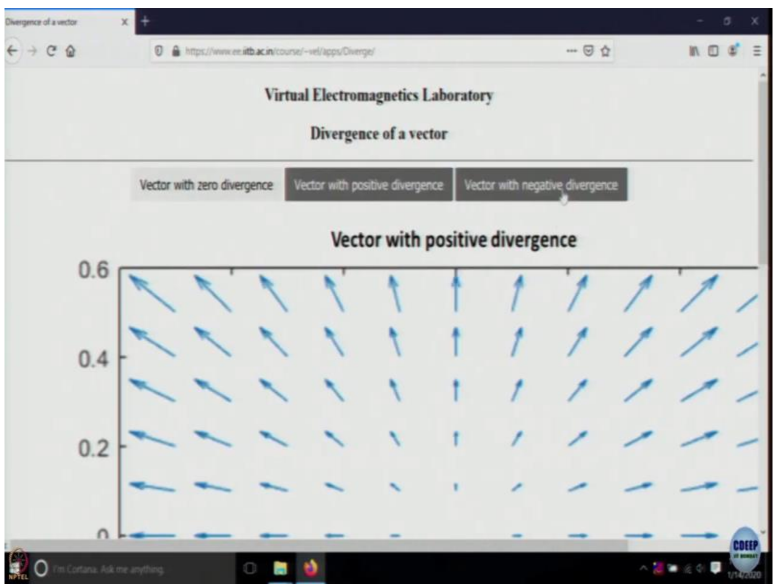

Now, vectors with positive divergence, for example, consider that the vectors shown in the above figures correspond to electrostatic field produced by positive volume charge density then there is a positive flow source. Similarly, for negative divergence, at every point there will be negative volume charge density that is negative flow source. Now we will go to the next topic which is curl.

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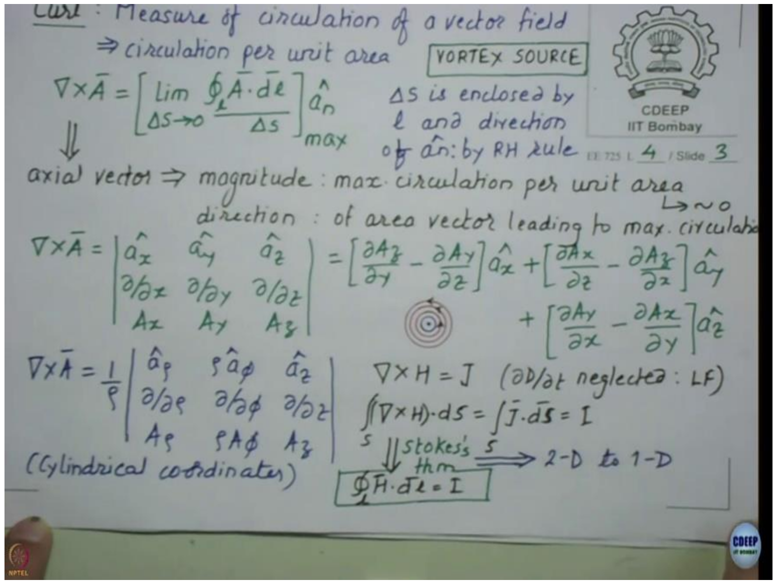
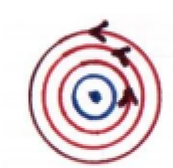


Curl is a measure of circulation of a vector field and it is defined as circulation per unit area and it represents a vortex source and by definition $\nabla \times \mathbf{A}$ is given by the formula given below

$$\nabla \times \bar{A} = \left[\lim_{\Delta s \to 0} \frac{\oint_l \bar{A} \cdot \bar{dl}}{\Delta s} \right]_{max} \hat{a}_n$$

Similar to the case of divergence, where volume was tending to 0, here the corresponding surface is tending to 0. So, here a surface which is bounded by a closed contour.

And the most common example of a vortex source that we can think of is a current source. For the conductor shown in the following figure, current is coming out of this paper and the direction of the corresponding magnetic field is given by the right hand rule.



So, the field direction is in anti-clockwise way. The loop area is normal to the vortex source (current) direction and that will give you the maximum circulation and that is the meaning of the max in the definition of the curl. In this case, area vector of the loop is along the same direction of the vortex source and that will give the maximum circulation. For example, if the area vector of the loop is not in the direction of the vortex source, but it is in some arbitrary direction, then you will not get the maximum circulation which is calculated by using the formula of curl.

Then the expression for curl in Cartesian and cylindrical coordinates are as given below.

In Cartesian system:

$$\nabla \times \bar{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\hat{a}_z$$

$$\nabla \times \bar{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

(Cylindrical coordinates)

So, just to quickly understand the expression of curl in Cartesian system, see the sequence yz, zx, xy. It is better to write in this form rather than putting a minus sign and just interchanging these two terms, because this makes it simple. And then finally, you have the Ampere's law in point form as $\nabla \times \mathbf{H} = \mathbf{J}$.

Remember, probably I did not mention when I was talking about Maxwell's equations. Maxwell's equations that we have seen in the previous lectures are in point form or differential form, because they are involving curl, divergence operations and therefore, they involve partial derivatives. That means, Maxwell's equations that we have seen till now are basically point form of equations, they are telling you what is happening at a point because derivatives change from point to point.

$\nabla \times \mathbf{H} = \mathbf{J}$ is point form of Ampere's law and when you actually take the surface integral on both sides of this equation you obtain the following equation

$$\int_S (\nabla \times \bar{H}) \cdot d\bar{s} = \int_S \bar{J} \cdot d\bar{s} = I$$

$\Downarrow$ Stokes's thm $\Longrightarrow$ 2-D to 1-D

$$\boxed{\oint_l \bar{H} \cdot d\bar{l} = I}$$

Then you have $\iint \mathbf{J} \cdot d\mathbf{S} = I$ on the right hand side and apply Stokes' theorem which converts the open surface integral on the left hand side into the closed line integral. So, you will get $\oint_l \mathbf{H} \cdot d\mathbf{l} = I$. Again, here Stokes' theorem is useful in converting the 2D integral into the 1D integral.

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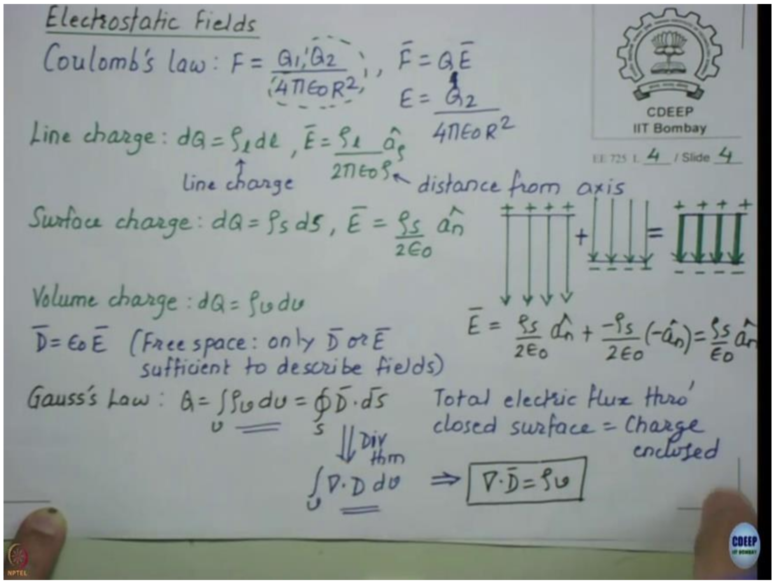


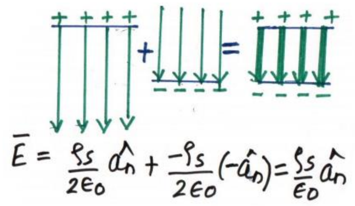
Similarly, you can actually see there is some experiment in the virtual lab on curl of a vector field also, you can see those experiments in your leisure time. If you go to electrostatic fields, we start with Coulomb's law, which is

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}$$

wherein this $\frac{Q_2}{4\pi\epsilon_0 R^2}$ is the electric field intensity due to charge Q_2 and if that electric field intensity is acting on Q_1, then you will get force on the charge Q_1 as $Q_1 E$.

Now, in case of electric fields you can either have line charges, surface charges or volume charges. So, line charge is expressed as $\rho_l dl$ and the corresponding electric field intensity is $E = \frac{\rho_l}{2\pi\epsilon_0 \rho}\mathbf{a}_\rho$ and again remember ρ_l unit is C/m which is line charge per unit length, whereas, the ρ in the denominator of the E expression is the distance from the axis in cylindrical coordinates.

When it comes to surface charge, dQ is represented as $\rho_s dS$ and the corresponding electric field intensity is given by $\frac{\rho_S}{2\epsilon_0}\hat{\mathbf{a}}_n$. Now, here how do we understand this? You have the capacitor problem shown in the following figure, we know the field distribution is like the way shown in the right most part of the figure.



Now, understand the field distribution of the capacitor using the formula $E = \frac{\rho_S}{2\epsilon_0}\hat{\mathbf{a}}_n$. If you take only the positive plate of the capacitor (left most part of the figure), then you have the field lines as well as going in upward and downward directions from the plate, remember because the charges on the plate are positive, the field will emanate from top and bottom surfaces of the plate, but I have shown only the field lines from the bottom surface.

Similarly, for the negative plate you have the field lines terminating on that. So now, if you superpose these two fields, and there are no nonlinearities in this problem, you will get the resultant field distribution as given in the right most part of the figure. So, only in the region between the two plates, you will get addition because both the fields are in the same direction, whereas above and below, you will have cancellation of fields because they are in opposite directions and the corresponding formula of resultant E is $\frac{\rho_S}{\epsilon_0}\hat{\mathbf{a}}_n$.

And remember the magnitude of the D vector is nothing but ρ_S (in this case) because the unit of ρ_S is C/m^2 and unit of D is also C/m^2, D can be related to surface charge density on the conductor surface or to volume charge density in non-conductors through the equation $\nabla \cdot D = \rho_v$.

As mentioned here in free space, $D = \epsilon_0 E$ and only one of the two vectors **D** and **E** is sufficient to represent electrostatic fields. Only when you have a material piece and corresponding polarization represented by P vector which we will see later, you need both vectors to differentiate between **D** and **E** and you have to have a separate treatment given to both vectors.

The Gauss law is defined as total electric flux through closed surface is equal to charge enclosed within it. That can be mathematically expressed as $Q = \oiint_S \mathbf{D} \cdot d\mathbf{S}$ and $Q = \iiint \rho_v dv$, which is nothing, but volume charge density integrated over the volume.

Again, if you apply the divergence theorem, you will get $\nabla \cdot \mathbf{D} dv$ and that is why now, if you correlate these two expressions, you will get $\nabla \cdot \mathbf{D} = \rho_v$. So, in a way we have derived the first Maxwell's equation from the Gauss law.

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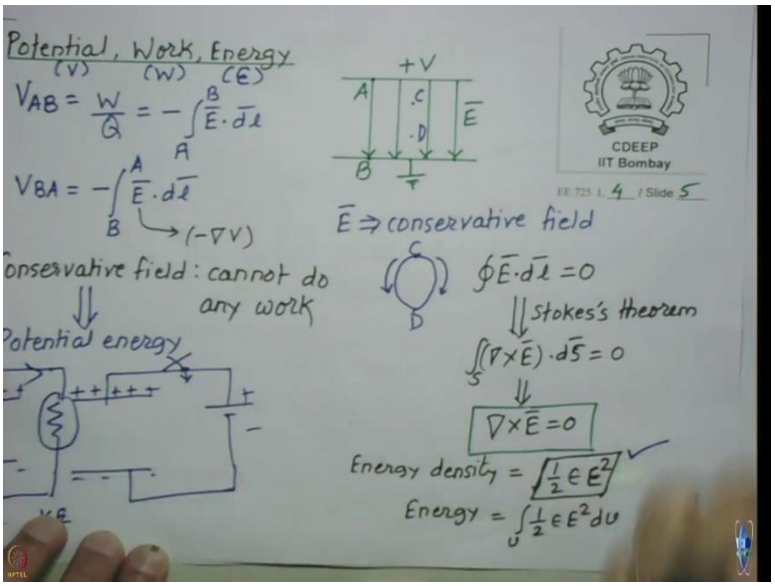
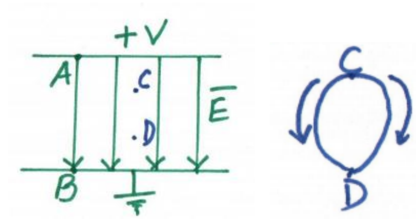


Now, next discussion is on potential, work, and energy. The definition of potential difference V_{AB} is $-\int_A^B \mathbf{E} \cdot d\mathbf{l}$ and potential is also defined as work done per unit charge $\left(\frac{W}{Q}\right)$. So, if you see here, if V_{AB} is this, then V_{BA} is obtained by changing the limits of the integral as given below.

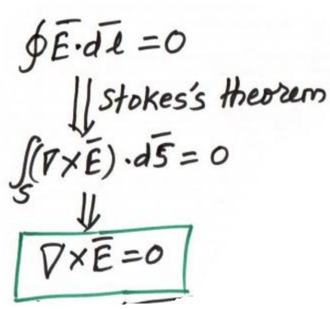
$$V_{BA} = -\int_B^A \mathbf{E} \cdot d\mathbf{l}$$

Now in this equation, the integral is from B to A and E is $-\nabla V$. So, the two minus signs result into a positive value. So, when you go from B to A, you will get positive voltage and that is what it should be because there is a gradient.

That way we have just verified that the sign convention is correct. The next important thing is conservative nature of electrostatic field. That means if you take 2 points C and D as shown in the following figure



The $\int \mathbf{E} \cdot d\mathbf{l}$ along these two paths shown in the above figure is same, because the potential difference is same. And now since the value of the integral along both these paths are same, the value of the integral in this closed path will be 0. And now you apply Stokes' theorem as explained in the following equations you will finally get $\nabla \times \mathbf{E} = 0$.



Now, in case of time varying fields it will become, $\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$, which we will see later, but for electrostatics this $\frac{\partial B}{\partial t} = 0$. Now, what is this conservative field? Conservative fields cannot do any work. So, what is the meaning of this statement? Suppose, you have a capacitor which is connected to a battery to charge it by closing the switch. So, in certain time constant capacitor will get charged and the corresponding electrostatic energy will get stored, we call this as potential energy, but this potential energy got stored only when some charges move.

So, always remember that the work will be done only when charges move, the static charges will not do any work. Similarly, now, this charged capacitor with this potential energy is not going to do any useful work unless you make those charges move. For example, now this capacitor is connected to a resistor, this capacitor will get discharged in no time and then the potential energy will produce heat in this resistor.

Now, that heat of course is a work, it may be useful or non-useful, but it is a work. It is a useful work for heating applications. So, the point to be noted here is, when the capacitor was charged, the charges were static, and it was only potential energy that is not capable of doing any useful work and it is a conservative field. Charges were allowed to move by shorting the capacitor.

When the charges are moving and the corresponding kinetic energy results into a useful work. So, you can always associate kinetic energy with work. That is why in electromagnetics, moving charges effectively do some work, this is a very important concept to be understood.

Now, the next point here is the energy density. That is energy per unit volume (J/m^3) is $\frac{1}{2}\epsilon E^2$, we will use this expression quite often in the FEM theory when we talk about energy minimization, and total energy is the volume integral of this.

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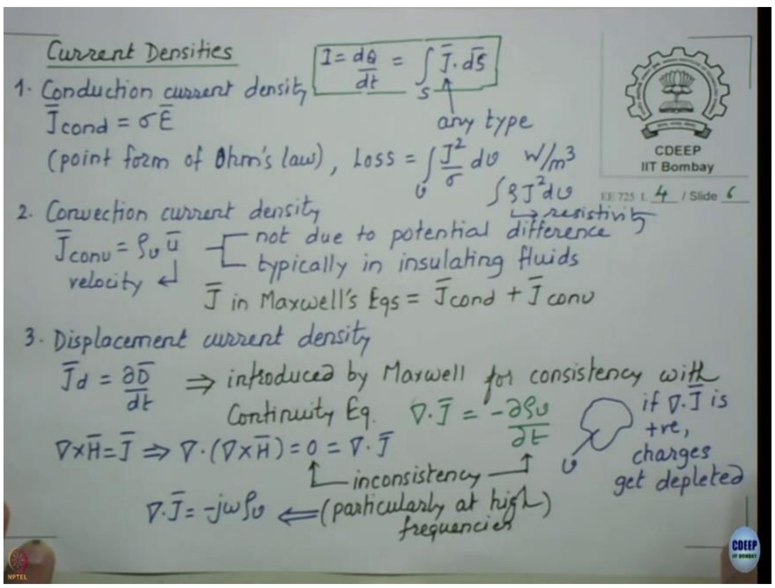

The last topic of this lecture is current densities, three types of current densities are discussed here, conduction current density, convection current density and displacement current density. For all these current densities, the expression of I is same. $I = \frac{dQ}{dt}$ and $I = \int_S \mathbf{J} \cdot d\mathbf{S}$, but $J_{cond} = \sigma E$ is governed by Ohm's law and this $J_{cond} = \sigma E$ is called as point form of Ohm's law which is $V = IR$.

So, $J_{cond} = \sigma E$ is more universally applicable as compared to $V = IR$, because for application of $V = IR$ you need to have the conductor of certain uniform dimensions so that you can calculate resistance as $\frac{\rho L}{A}$, then you can easily apply Ohm's law but $J_{cond} = \sigma E$ can be applied at every point of an arbitrary shaped conductor.

Whereas, $V = IR$ is like an integral form because you are actually finding the potential difference across the ends of a conductor. So, in finite element method, we will use this to calculate eddy current losses or losses in any conducting part, the loss is given by $\int_v \frac{J^2}{\sigma} dv$, this also can be represented as $\int_v \rho J^2 \, dv$ because $\frac{1}{\sigma}$ is the resistivity (ρ).

Note W/m^3 is unit of $\frac{J^2}{\sigma}$ or ρJ^2. So, now again I think, ρ in today's lecture we have used third time for a different quantity, so now this ρ is resistivity. ρ was earlier used for volume charge density, then for cylindrical axis distance. So, please bear in mind the differences. Convection current density is given by this $\rho_v \boldsymbol{u}$, where ρ_v is volume charge density which is multiplied by velocity $\boldsymbol{u}$.

This is not governed by Ohm's law which means potential difference is not forcing this convection current to flow. It is like you can imagine that there is some liquid insulation and by some means charges are generated in the insulation and you make that liquid to flow by some mechanical means like a pump. So, now those charges which you have introduced will flow and constitute current and that will be called convection current.

So, needless to say in this course on machines and equipment, we will not have this convection current density except if you are doing some advanced analysis like for example, in case of transformers wherein you are trying to understand static electrification phenomena wherein static charges get developed due to friction between oil and solid insulation and because of pump action those charges move, and there this convection current would be applicable but otherwise in normal circumstances, this conduction current density would be applicable.

In Maxwell's equation, J is basically combination of both conduction and convection if both are applicable. Remember all Maxwell’s equations are in general form, but then depending upon the problem solved and, depending upon the variables applicable, some variables will get eliminated in the formulation.

Then finally, the displacement current density which is not very easy to understand. So, this was introduced by Maxwell to bring consistency with the continuity equation. Continuity equation is $\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$. First, let us understand the continuity equation. Suppose you have some volume and if you evaluate $\nabla \cdot \mathbf{J}$ for that volume and if it is positive that means some charges would get depleted from that volume. Some charges are getting depleted means $\frac{\partial \rho_v}{\partial t}$ is becoming negative and this negative of negative in the right hand side of the continuity equation is positive and that is why divergence J is positive. So, this is the first explanation. In the starting of next lecture, I will explain the other viewpoint with respect to circuits, but what is the inconsistency here?

If we consider the original Ampere's law in point form ($\nabla \times \mathbf{H} = \mathbf{J}$) and if we take the divergence on both sides of this equation, we know divergence of curl is identically 0. And that means divergence J will be equal to 0 because we are taking divergence of both sides, but divergence J here is given by this continuity equation, that means we are forcing $\frac{\partial \rho_v}{\partial t} = 0$, which may not be the case particularly at high frequencies, because if you convert this into time harmonic form, then it will become $\nabla \cdot \mathbf{J} = -j\omega\rho_v$ and particularly at very high frequencies this is not going to be insignificant. So, this is the inconsistency. For example, whenever you have stray capacitance currents becoming significant at high frequencies, you cannot actually have $\nabla \cdot \mathbf{J} = \mathbf{0}$ because then we are not allowing this term $\left(\frac{\partial \rho_v}{\partial t}\right)$ to be finite. More about this point in the next lecture. Thank you.

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