

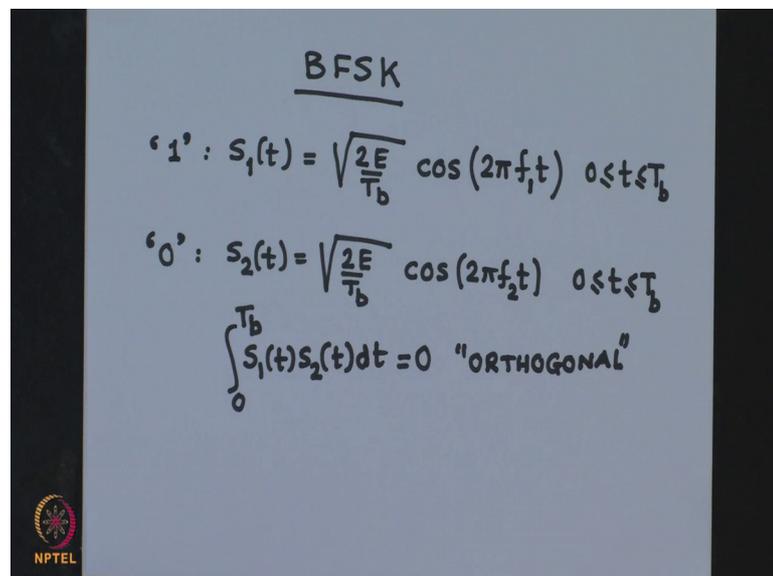
Principles of Digital Communications
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Lecture - 46
Binary Frequency Shift Keying - II

In binary frequency shift keying for the transmission of two symbols 1 and 0; we use two different carriers with two different frequencies f_1 and f_2 . We have studied that when these carriers have different phases, then the difference between these 2 carrier frequencies should be integer multiplication of $1/T_b$; and when the phases for these 2 carriers are the same, then the difference between the 2 carrier frequencies can be chosen to be integer multiple of $1/2T_b$, where T_b is the signaling interval for the symbol 1 or symbol 0. We have studied how to generate binary FSK signals and also studied the optimum receiver for the same.

Now, we will calculate the probability of error for this optimum receiver and also evaluate the power spectral density of the transmitted BFSK signal.

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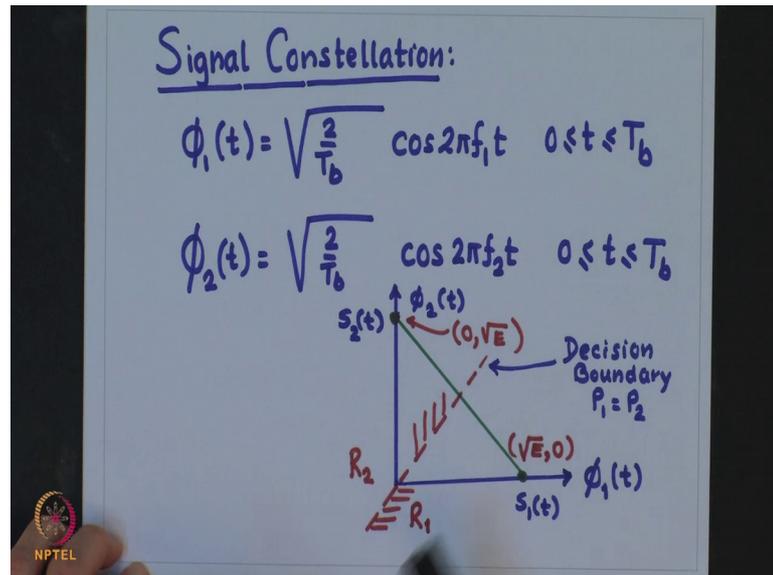
The image shows a slide with handwritten mathematical equations for Binary Frequency Shift Keying (BFSK). The title is "BFSK". The equations are:

$$\text{'1'}: s_1(t) = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_1 t) \quad 0 \leq t \leq T_b$$
$$\text{'0'}: s_2(t) = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_2 t) \quad 0 \leq t \leq T_b$$
$$\int_0^{T_b} s_1(t) s_2(t) dt = 0 \quad \text{"ORTHOGONAL"}$$

There is a small NPTEL logo in the bottom left corner of the slide.

So, our binary FSK signals are s_1 and s_2 and they are orthogonal, then we choose our 2 basis signals as $\phi_1(t) = \sqrt{2/T_b} \cos(2\pi f_1 t)$, and $\phi_2(t)$ as given here.

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Given this basis signal for the signal set $S_1(t)$ and $S_2(t)$ we can get the signal constellation as shown in the figure here.

So, the decision boundary presuming that, both the symbols are equiprobable would be the perpendicular bisector of the line joining the 2 signal vectors or signal points. So, this side it would be R_1 and this side it would be R_2 . Given this we can easily calculate the probability of error as follows.

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Probability of error (BFSK)

$$P_e = Q\left(\frac{d_{12}/2}{\sqrt{N/2}}\right)$$
$$= Q\left(\frac{\sqrt{2E}/2}{\sqrt{N/2}}\right)$$
$$= Q\left(\sqrt{\frac{E}{N}}\right) \quad \epsilon_b = E$$
$$= Q\left(\sqrt{\frac{E_b}{N}}\right)$$

Probability of error is a Q function of the distance between the 2 signals divided by 2, and the standard deviation of the noise. So, distance here is equal to this distance remember this is root E, root E. So, this would be equal to root 2 E divided by 2, italic N by 2 is the power spectral density of the white noise. So, this is equal to.

Now, in this case for the equiprobable symbols, the bit energy is the same as the energy in the signal. So, we can write the probability of error in terms of the bit energy. Now this expression is the same as we got for the binary amplitude shift keying.

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PSD of BFSK

Let $m_1(t)$ denote random binary wave that modulates the carrier freq. f_1

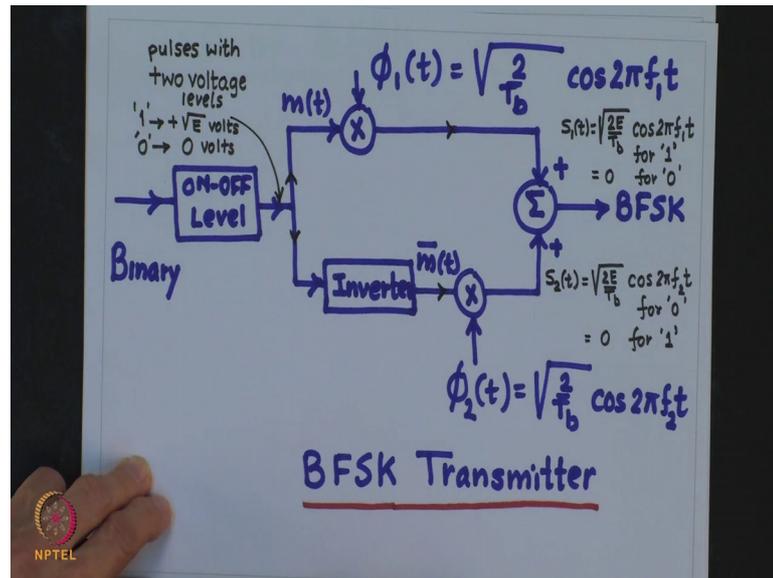
'1': $A p(t)$ where $p(t) = \pi\left(\frac{t}{T_b}\right)$
 $A = \sqrt{\frac{2E}{T_b}}$

'0': $0 \cdot p(t)$

$m_1(t) = \sum_{k=-\infty}^{\infty} \alpha_k p(t - kT_b - \beta)$
 $\alpha_k = 0, A$

Now, let us evaluate the power spectral density for the binary FSK.

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We had studied the binary FSK transmitter, which is as shown in the figure out here. If you have a binary data sequence, it passes through on off level. The output here are pulses with 2 voltage levels, for symbol 1 you get root E volts and for symbol 0 you get zero volts. So, that is indicated as $m(t)$; out here it passes through a inverter, the output here is $\bar{m}(t)$. Here for symbol 0, you will have root E and for symbol 1 you will have 0. So, again pulses with 2 voltage levels, they get to the product modulators with $\phi_1(t)$ and $\phi_2(t)$ and you get the outputs here. So, this is $S_1(t)$. So, here for 1 you will be transmitting this thing signal waveform, for 0 we do not transmit anything and in this branch bottom branch this is $S_2(t)$. So, for symbol 0, we transmit this waveform and for symbol 1 we do not transmit any waveform. So, the combination of this basically generates the binary FSK wave.

Now, since they are 2 oscillators, which are used to represents symbol 1 and 0 are independent, we may view the resulting binary FSK wave has the sum of 2 binary ASK signals or 2 on off keying signals. One bask signal operates with the oscillator of frequency f_1 and the second bsk signal operates with the oscillator of frequency f_2 and both are independent.

So, let us try to calculate the power spectral density look at the top branch, in the top branch consider this signal out here. This signal $S_1(t)$ could be considered as a signal which has been modulated by a random binary wave, which takes value of root 2 E by T

b for symbol 1 and takes value 0 for symbol 0. Whereas, if you look in the bottom branch you have the signal $S_2(t)$, you could consider that this signal out here $\cos 2\pi f_c t$ is being modulated by a random binary wave $m_2(t)$ we could call it, which is root $2E_b$ by T_b for symbol 0 and it is 0 for symbol 1.

So, let us try to analyze the power spectral density for the top branch and then we can extend that result to the bottom branch. So, in the top branch let $m_1(t)$ denote random binary wave that modulates the carrier frequency f_c . So, for symbol 1 we have the output as A times $p(t)$ where $p(t)$ is a rectangular pulse, which we can denote as shown here. And the value of A is equal to root of $2E_b$ by T_b and for symbol 0 we have 0 multiplied by $p(t)$. So, your $m_1(t)$ is of the form $\alpha_k p(t - kT_b)$ minus some random variable β_k , this is a pulse train for $m_1(t)$ and α_k is a random variable, which can take the value equal to 0 or A , A has been given as root of $2E_b$ by T_b .

Now, we know how to calculate the power spectral density for this type of a random wave.

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$$\therefore S_{m_1}(f) = \frac{|P(f)|^2}{4T_b} \left[1 + \frac{1}{T_b} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_b}\right) \right]$$

$$p(t) = \Pi\left(\frac{t}{T_b}\right)$$

$$\updownarrow$$

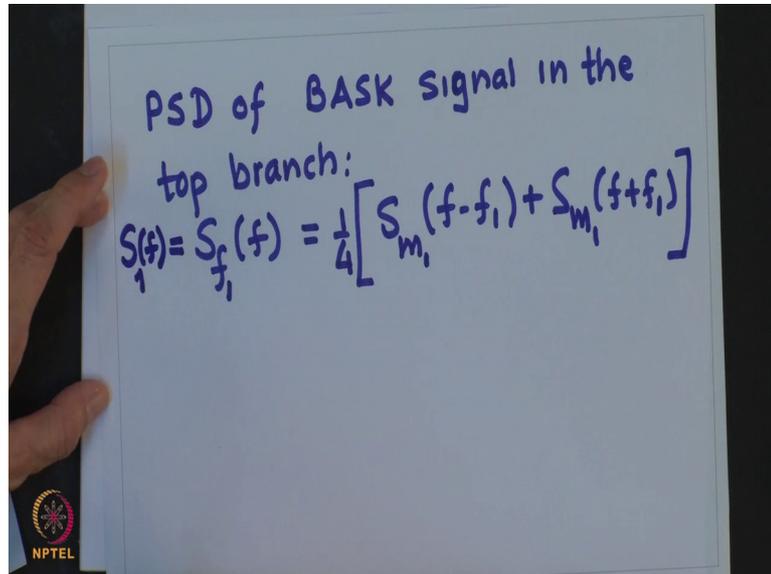
$$P(f) = T \text{sinc}(T_b f)$$

$$S_{m_1}(f) = \frac{A^2}{4} \left[\delta(f) + T_b \text{sinc}^2(T_b f) \right]$$

This is nothing but we had studied this form for unipolar line code. So, immediately we can write the power spectral density for this random wave $m_1(t)$ as follows. Now if we choose our $p(t)$ to be rectangle pulse of duration T_b , then for this we know that Fourier transform is $T \text{sinc} T_b f$, and considering this we can write the power spectral density for

the result as the following expression. Because of the property of the sinc function, which is 0 at the multiples of $1/T_b$, all other terms in this summation series will vanish.

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PSD of BASK signal in the top branch:

$$S_{f_1}(f) = S_{f_1}(f) = \frac{1}{4} [S_{m_1}(f-f_1) + S_{m_1}(f+f_1)]$$

The image shows a hand holding a whiteboard with the above text and equation written in blue ink. An NPTEL logo is visible in the bottom left corner of the whiteboard.

Given this now, we can easily calculate the power spectral density of the binary ASK signal in the top branch as follows, will indicate that as $f = f_1$ is coming from the top branch related to the frequency f_1 and we know that this is equal to one fourth S_m .

Now, given this, we can calculate the power spectral density on the top branch. So, that would be by plugging in this expression into this. So, let us call this as $S_{f_1}(f)$ to simplify our notation correct. So, we can immediately get that and that would be as shown here correct.

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$$S_{m_i}(f) = \frac{A^2}{4} \left[\delta(f) + T_b \operatorname{sinc}^2(fT_b) \right]$$

$$S_1(f) = \frac{1}{4} \left[S_{m_i}(f-f_1) + S_{m_i}(f+f_1) \right]$$

$$= \frac{A^2}{16} \left[\delta(f-f_1) + \delta(f+f_1) + \frac{\sin^2[\pi T_b(f-f_1)]}{\pi^2(f-f_1)^2 T_b} + \frac{\sin^2[\pi T_b(f+f_1)]}{\pi^2(f+f_1)^2 T_b} \right]$$

So, this expression we just plugging into this and we get this; so having got the power spectral density for the top branch, similarly we can get the power spectral density for the bottom branch and that we indicate as $S_2(f)$ similar as what we got for the $S_1(f)$, except that f_1 has been replaced by f_2 .

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$$S_2(f) = \frac{A^2}{16} \left[\delta(f-f_2) + \delta(f+f_2) + \frac{\sin^2[\pi T_b(f-f_2)]}{\pi^2(f-f_2)^2 T_b} + \frac{\sin^2[\pi T_b(f+f_2)]}{\pi^2(f+f_2)^2 T_b} \right]$$

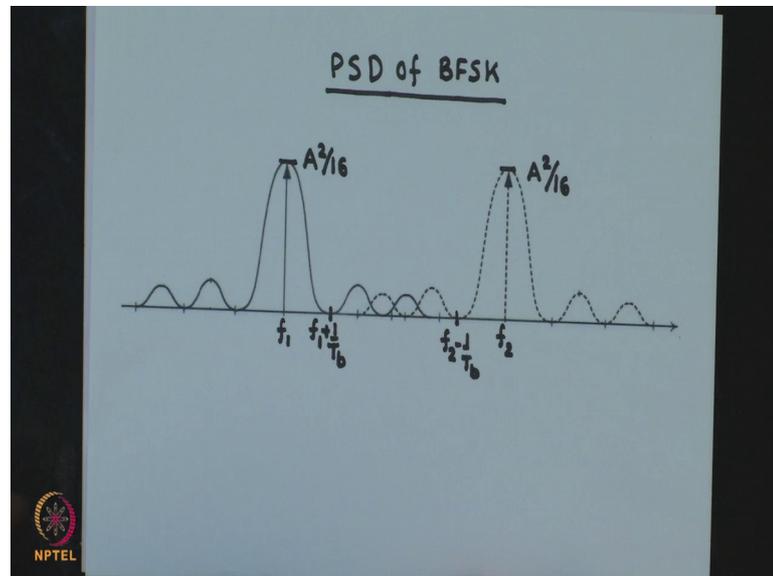
$$S_s(f) = S_1(f) + S_2(f)$$

$$= \frac{A^2}{16} \left[\delta(f-f_1) + \delta(f+f_1) + \delta(f-f_2) + \delta(f+f_2) + \frac{\sin^2[\pi T_b(f-f_1)]}{\pi^2(f-f_1)^2 T_b} + \frac{\sin^2[\pi T_b(f+f_1)]}{\pi^2(f+f_1)^2 T_b} + \frac{\sin^2[\pi T_b(f-f_2)]}{\pi^2(f-f_2)^2 T_b} + \frac{\sin^2[\pi T_b(f+f_2)]}{\pi^2(f+f_2)^2 T_b} \right]$$

And now the total power spectral density of the output binary FSK wave would be the sum of this 2 power spectral density, because remember that 2 branches are independent

and if they are independent then power spectral density this sum up. And if you look at the plot for this power spectral density is this is what we get correct.

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Now, remember that sinc x falls off as $1/x$ for high value of frequency. So, what this means that, this power spectral density will fall off as $1/f^2$ for frequencies far off from f_1 or f_2 . Now there is a form of binary FSK, which is of interest in practical application and that is special form of BFSK is known as Sunde's binary FSK.

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The figure shows a hand-drawn slide titled "Sunde's BFSK". The equations are:

$$f_1 \text{ and } f_2$$
$$(f_2 - f_1) = \frac{1}{T_b}$$
$$\frac{f_2 + f_1}{2} = f_c$$

The NPTEL logo is visible in the bottom left corner.

What is the property of this? It is as follows, we have to transmitted frequency f_1 and f_2 and they differ by an amount equal to the bit rate and the arithmetic mean equals the nominal carrier frequency f_c . For transmission it is assumed that phase continuity is always maintained I repeat please, for transmission it is assumed that phase continuity is always maintained including interpret switching times and this is the definition for Sunde's BFSK. And we will try to evaluate the power spectral density for this special case of BFSK that is Sunde's BFSK.

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The image shows a handwritten derivation on a whiteboard. The title is "PSD of Sunde's BFSK". The derivation starts with the signal equation $s(t) = \sqrt{\frac{2E}{T_b}} \cos\left(2\pi f_c t \pm \frac{\pi t}{T_b}\right)$ for $0 \leq t \leq T_b$. This is then expanded using the product-to-sum trigonometric identity: $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$. The final expression is $s(t) = \sqrt{\frac{2E}{T_b}} \cos\left(\frac{\pi t}{T_b}\right) \cos 2\pi f_c t \pm \sqrt{\frac{2E}{T_b}} \sin\left(\frac{\pi t}{T_b}\right) \sin 2\pi f_c t$. An NPTEL logo is visible in the bottom left corner of the whiteboard.

So, let us evaluate that. Given this information for the choice of f_1 and f_2 and the average of f_1 plus f_2 is equal to f_c , we can express this special binary FSK signal as follows. Where for example, minus sign could be used for the transmission of symbol 1 and plus sign could be used for transmission of symbol 0 and using a well known trigonometric identity, we can simplify this to the following expression \sin of $2\pi f_c t$ and this can be rewritten as \cos is an even function. We assume now that symbols 1 and 0 are equiprobable and the symbols in the adjacent time slots are statistically independent.

So, based on this we can make the following observations pertaining to the in phase component and the quadrature components, of the binary FSK signal correct. So, remember this binary FSK signal has continuous phase.

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Observations:

(i) In-phase component

- completely independent of the input binary wave
- for all time it is $\sqrt{\frac{2E}{T_b}} \cos\left(\frac{\pi t}{T_b}\right)$
- PSD: $\frac{E}{2T_b} \left[\delta\left(f - \frac{1}{2T_b}\right) + \delta\left(f + \frac{1}{2T_b}\right) \right] = S_i(f)$

So, the first observation is pertaining to in phase component T is that, this is completely independent of the input binary wave correct. Irrespective of the binary wave this is the in phase component and so, for all time it is this quantity. So, the power spectral density of this is very easy to calculate. We indicate this power spectral density for the in phase component as $S_i(f)$ and this is the spectrum for the same.

Let us look at the quadrature component; we see that the quadrature component is directly related to the input binary wave.

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(ii) The quadrature component is directly related to the i/p binary wave

- $\rightarrow 0 \leq t \leq T_b : \pm \sqrt{\frac{2E}{T_b}} \sin\left(\frac{\pi t}{T_b}\right)$
- \rightarrow Polar wave:

$$h(t) = \begin{cases} \sqrt{\frac{2E}{T_b}} \sin\left(\frac{\pi t}{T_b}\right) & 0 \leq t \leq T_b \\ 0 & \text{otherwise} \end{cases}$$

\Rightarrow PSD: $S_q(f) = \frac{|H(f)|^2}{T_b}$

During the signaling interval plus minus will depend on which symbol we are transmitting symbol 1 or symbol 0 but important to note that this form is nothing but a polar wave, which can be generated by using the impulses of strength plus minus 1 passing through a filter $h(t)$, whose impulse response is given here correct.

Given this now we can easily calculate the power spectral density for this: this also we have done it earlier. So, the power spectral density for the quadrature component would be related by this expression correct. The straightforward extension of what we have done earlier.

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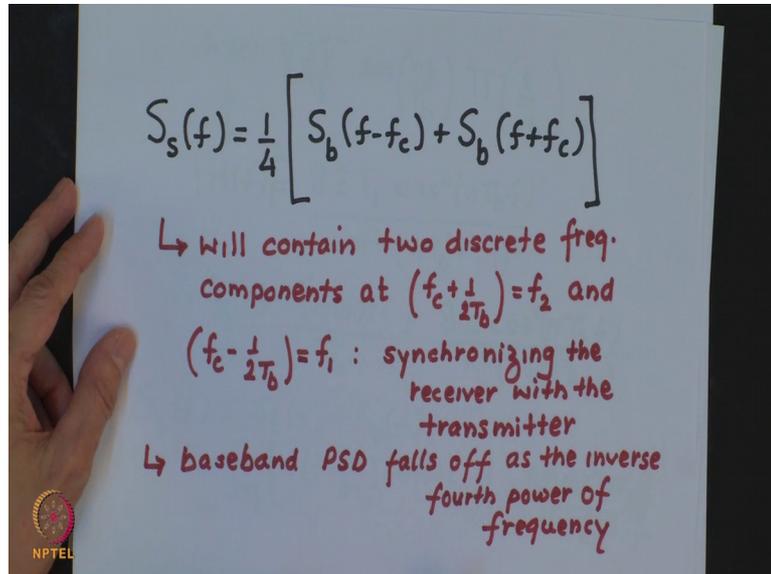
The image shows a whiteboard with handwritten mathematical derivations. At the top, the impulse response $h(t)$ is given as $\sqrt{\frac{2E}{T_b}} \sin\left(\frac{\pi t}{T_b}\right) \Pi\left(\frac{t}{T_b}\right)$. A double-headed arrow indicates the Fourier transform relationship to the magnitude squared of the transfer function, $|H(f)|^2 = \frac{8ET_b \cos^2(\pi T_b f)}{\pi^2 (4T_b^2 f^2 - 1)^2}$. Below this, the quadrature component power spectral density is derived as $S_q(f) = \frac{|H(f)|^2}{T_b} = \frac{8E \cos^2(\pi T_b f)}{\pi^2 (4T_b^2 f^2 - 1)^2}$. The final equation shows the total baseband power spectral density: $S_b(f) = S_i(f) + S_q(f)$, where $S_b(f)$ is labeled as Sunde's BFSK Baseband. The expression for $S_b(f)$ is $\frac{E}{2T_b} \left[\delta\left(f - \frac{1}{2T_b}\right) + \delta\left(f + \frac{1}{2T_b}\right) \right] + \frac{8E \cos^2(\pi T_b f)}{\pi^2 (4T_b^2 f^2 - 1)^2}$. An NPTEL logo is visible in the bottom left corner of the whiteboard image.

So, to get your $H(f)$ again remember your $h(t)$ is equal to this form. So, you integrate this quantity after multiplying by $E e^{-j2\pi f t}$, integrate over the time from 0 to T_b that is how you will calculate a Fourier transform of this and its very simple to show that the Fourier transform turns out to be this. And from there the quadrature components power spectral density is equal to this quantity, which if you plug in this expression for $h(t)$ you get equal to this ok. So, this is equal to square piece, the squared of this is equal to this ok.

Now, so, the base band power spectral density, which consists of the in phase component and the quadrature component can be obtained by addition of the 2, because both are independent. So, if I do that, I get the base band representation for the Sunde's BFSK

case correct. And from here we can get the power spectral density for the modulated waveform that is binary FSK by using this relationship.

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$$S_s(f) = \frac{1}{4} \left[S_b(f-f_c) + S_b(f+f_c) \right]$$

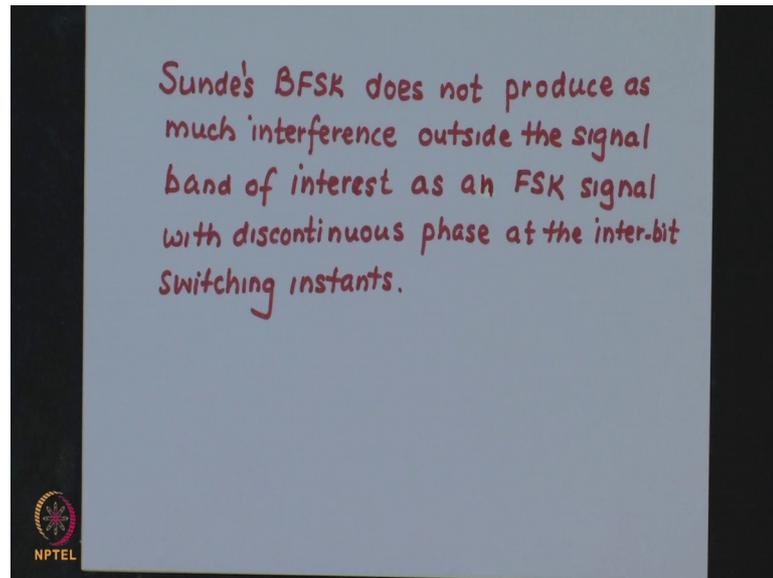
↳ will contain two discrete freq. components at $(f_c + \frac{1}{2T_b}) = f_2$ and $(f_c - \frac{1}{2T_b}) = f_1$: synchronizing the receiver with the transmitter

↳ baseband PSD falls off as the inverse fourth power of frequency

So, this expression out here is plugged into this expression. What is more important to know that, once I do this I take this and put it here this will contain 2 discrete frequency components 1 at f_c plus $\frac{1}{2T_b}$ which I can call it f_2 , and other would be f_c minus $\frac{1}{2T_b}$ which I can call it f_1 .

Now, it is important to know that this 2 discrete components help in synchronizing the receiver with the transmitter. And another important thing to observe is that, if you look at the baseband power spectral density then it falls off as the inverse fourth power of the frequency. So, where as in the earlier case of BFSK though we had only sinc function squared. So, the fall off was at the rate of $\frac{1}{f^2}$.

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So, what this implies that Sunde's BFSK does not produce much interference outside the signal band of interest as an FSK signal, with discontinuous phase at the inter bit switching instance correct.

So, this is an important result and this is because of the type of the shaping pulse which we have used. In ordinary binary FSK the shaping pulse is a rectangular shape whereas, in the Sunde's BFSK case we have sin function as a shaping pulse.

Now, quickly let us summarize the performance of the three modulation scheme which we have studied. So, we have studied binary amplitude shift keying, binary phase shift keying and binary fish frequency shift keying. So, as far as the performance probability of error is concerned, we find that performance of binary ASK and binary FSK they are the same whereas, performance of binary psk is better. In the sense that binary ASK and binary FSK would require three db mope power then for the case of binary psk for the same probability of error. But binary FSK will require much more bandwidth compared to binary ASK and binary psk. Binary ASK and binary psk both require same amount of bandwidth the difference between the 2 spectrum is that there is a discrete component at f_c in the case of binary ASK case.

Now, if you want to transmit the symbols at higher rate, this would necessitate that we reduce the signaling interval. But now in all this study which we have done so far, it is clear that with the decreasing signaling interval that is T_b , the bandwidth requirement

will also increase. So, the question is, is it possible for us to transmit at a higher rate without increasing the bandwidth. There are techniques available, which are known as bandwidth efficient signaling or modulation schemes and we will study this next.

Thank you.