Principles of Digital Communications
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Lecture – 28
Companded Quantization - I

We have studied two types of Quantizer; one Uniform Quantizer and the other Non-uniform Quantizer. In Uniform Quantizer, decision boundaries and reconstruction levels are equally spaced and implementation of the same is quite easy in practice; whereas, for Non-uniform Quantizer or Lloyd max quantizer, the decision boundaries are closer to those regions that have more probability mass. An implementation of such quantizer is quite difficult in practical applications.

Non-uniform Quantizer or Lloyd max quantizer is optimum quantizer in the sense that it minimizes the mean square quantization noise for the given number of quantization levels and the input pdf. We have also studied that the reconstruction level for the Non-uniform Quantizer turns out to be centroid of the quantization interval or the interval between two decision boundaries and the decision boundaries are the midpoint of the adjacent construction levels.

Now, with this information, it is not very difficult to show that a Uniform Quantizer is an optimum quantizer for a uniform pdf. So, as long as the pdf of the input is uniform, we can use an uniform quantizer. But in a practical scenario as stated earlier also, we would like to implement uniform quantization and unfortunately, in many applications the input pdf is not uniform. Take an example of speech signal or voice signals; it has higher probability in the lower amplitudes and lower probability in the higher amplitude.

So, for such signals, it would be efficient to design a quantizer which has more quantization regions in lower amplitudes and less quantization regions in the higher amplitude correct. So, by this it will overcome the variations in the power level that the quantizer sees at its input. This is the basic principle of Companded Quantization. So, the block diagram of a Companded Quantization is shown here.

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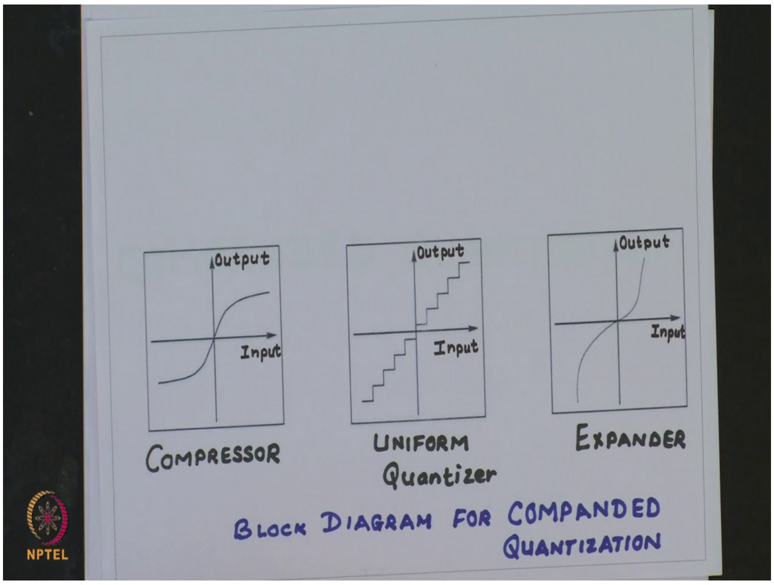

The input passes through the Compressor, the output of which goes to a Uniform Quantizer and then, it follows it is followed by an Expander which has the inverse characteristic to that of the compression.

This Compressor basically stretches the high probability regions which is near the origin and correspondingly compresses the low probability regions away from the origin. So, the regions close to the origin, occupy a greater fraction of the total region covered by the compressor.

So, if the output of the compressor function is quantized using a Uniform Quantizer and the quantized value is transformed via the expander function; then, the overall effect is the same as using a Non-uniform Quantizer. To understand this let us take one simple example.

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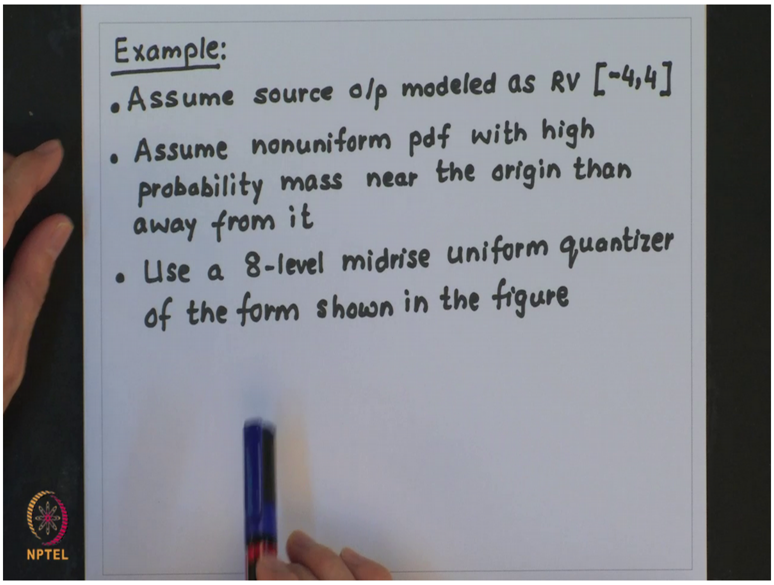

Assume that we have a source output modeled as Random Variable between minus 4 to plus 4, we assume that it has a non uniform pdf with high probability mass near the origin then away from it. And for our discussion we will use a 8 level midrise Uniform Quantizer of the form shown in this figure out here correct.

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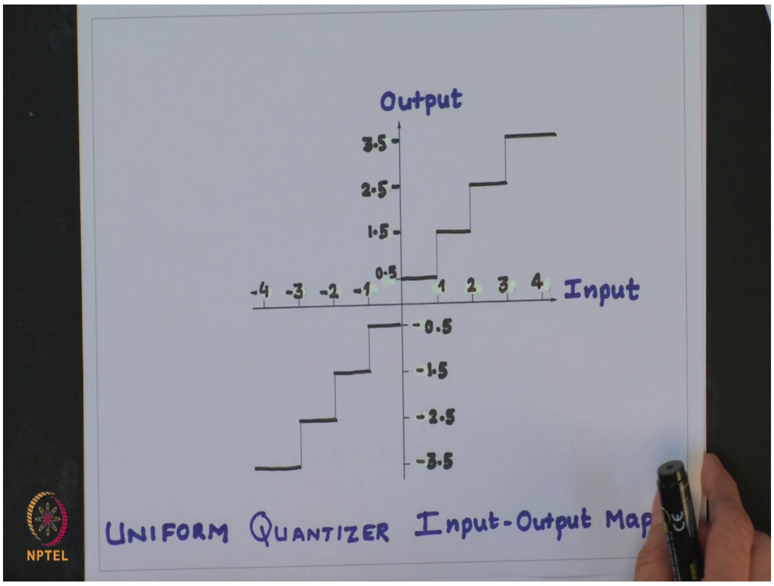

So, this is a Uniform Quantizer and the Input-Output Mapping is given here. So, we are trying to quantize the input values from minus 4 to plus 4. Decision boundaries are all equally spaced and the reconstruction levels are also equally spaced. Now, let us try to flatten out the non-uniform pdf using the Compressor and Expander. Compressor and Expander together is known as COMPANDER.

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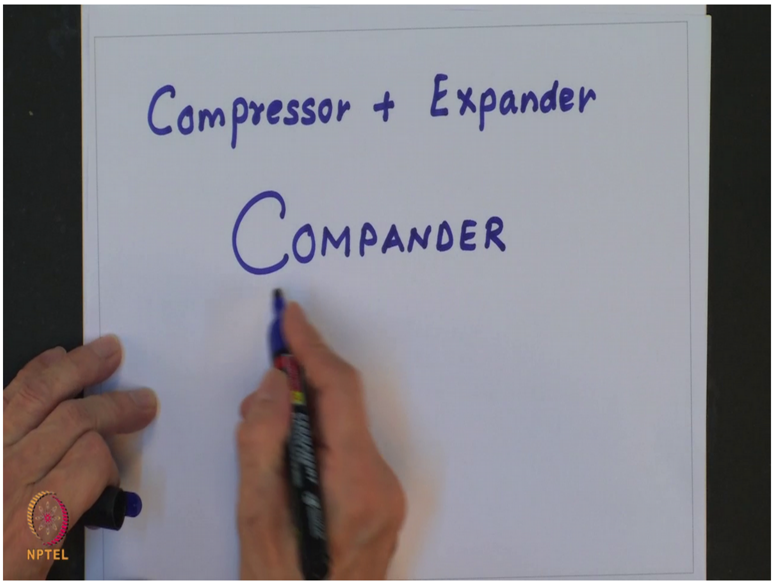

And now, let us use the Compressor mapping as shown in this figure.

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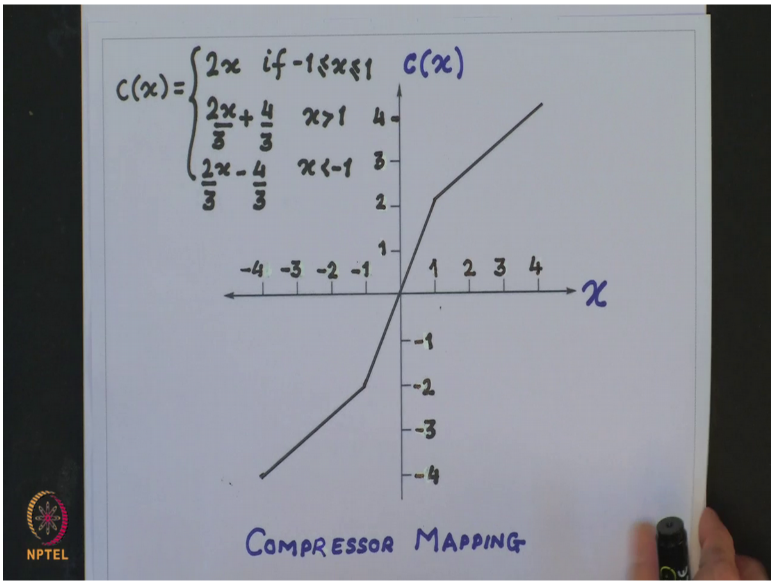

It has a linear region about the origin and there is another linear region beyond 1 to 4 correct and corresponding to this Compressor characteristic, we have the Expander characteristic or expander mapping which is the inverse of compressor mapping as shown here.

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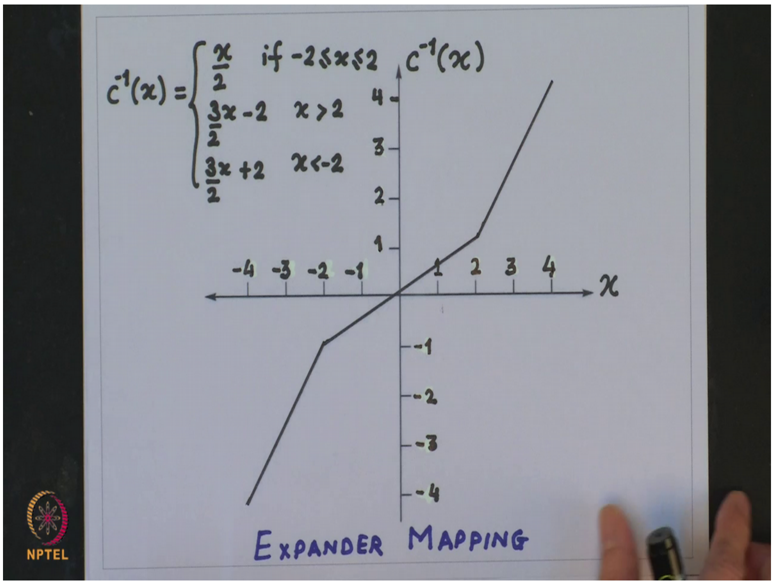

Now, let us see how using this mappings, it affects the quantization error both near and far from the origin.

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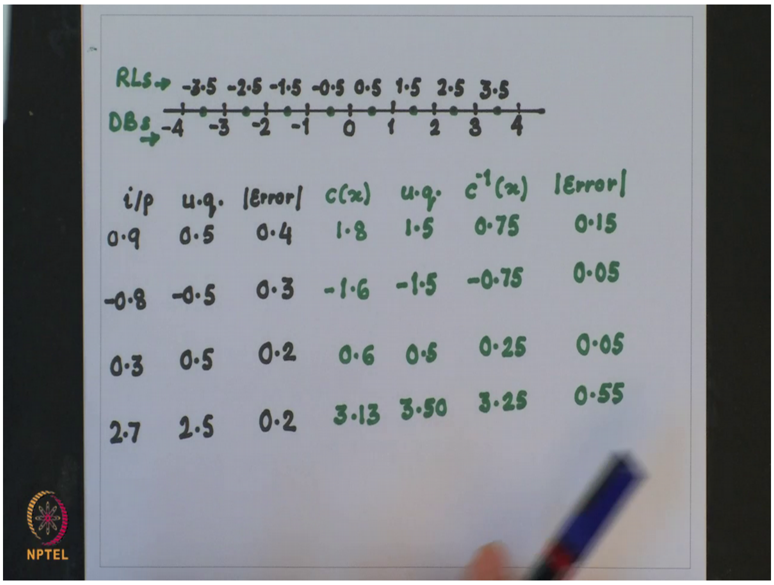

So, I have chosen some representive values for the input and then, shown what happens at the output of a Uniform Quantizer and what happens if I use Companded Quantization correct. So, let us take a value of input to be 0.9 and out here, I have shown you what are a decision boundaries and reconstruction levels for the Uniform Quantizer.

So, if I have a input of 0.9, if I use a Uniform Quantizer. It lies between 0 and 1. So, it is going to be quantized to 0.5 and output I will get it as 0.4.

And if I were to use the Companded Quantization; then, 0.9 would pass through a compressor characteristic which I showed you earlier. The output would be 1.8 and then, it will pass through a Uniform Quantizer, the same Uniform Quantizer which I have showing here and for this basically it will get quantized to value 1.5. So, the notional quantization error is 0.3, but this is not the real quantization error.

Now, after the Uniform Quantization, it passes to the inverse operation and I get the output of 1.5 to 0.75 and if you examine these two values, I get the error of 0.15. In this case we see that we gain about 0.25. Now, you take different values between minus 1 to plus 1, the first 3 rows out here show that and it can be shown that for all these cases there is no increase in the quantization error and for most values, we will get a decrease in the quantization error. Of course, this is not true for the values beyond 1.

So, if I take a value input value to be 2.7, if I were to use Uniform Quantizer, the error would be 0.2. But, if I use the Companded Quantization, I get the error to be 0.55 correct. So, from this table it is very clear that the Companded Quantizer, the Companded Quantizer effectively works like a Non-uniform Quantizer with small quantization intervals; In the interval between minus 1 to plus 1 and larger quantization intervals outside this interval.

So, let us try to find out what is the effective Input-Output Mapping for this quantizer.

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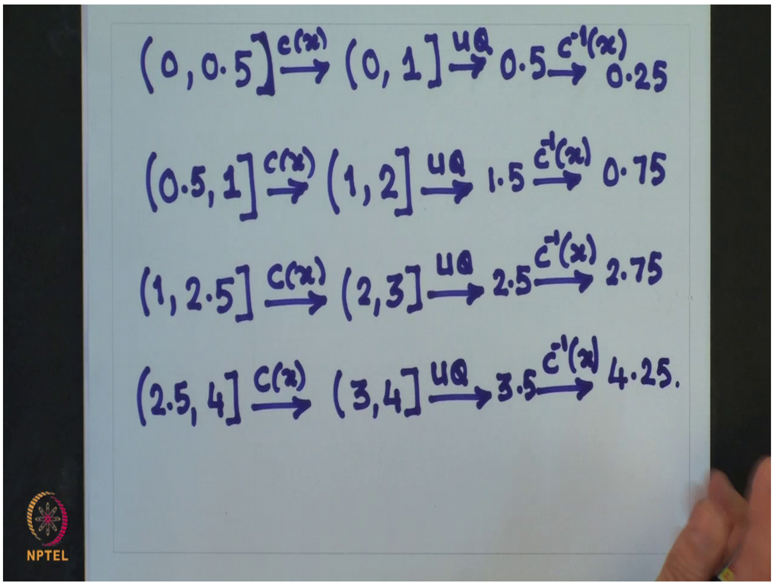

So, when you have the inputs in the interval 0 to 0.5. Then, using the Compressor characteristic, this gets mapped to 0 to 1 and when it passes through a Uniform Quantizer, this gets mapped to value of 0.5 and then, again when it goes to the inverse operation through the expander characteristic, I get it 0.25.

So, similarly for the region between 0.5 to 1, this gets mapped to 1, 2 and then, after passing through the Compressor and using a Uniform Quantizer, this gets mapped to 1.5 and then, using the expander characteristic. It gets mapped to 0.75. Now, 1 to 2.5 at the input gets mapped to 2 to 3 as a input to the Uniform Quantizer and after going through Uniform Quantizer, this goes to 2.5 and then, the inverse of this is equal to 2.75.

And finally, the range between 2.5 and 4 gets mapped to 3, 4 using Uniform Quantizer. This will get mapped to 3.5 and using the inverse operation, I get it as 4.25. So, this output for the particular inputs can be shown in this figure out here.

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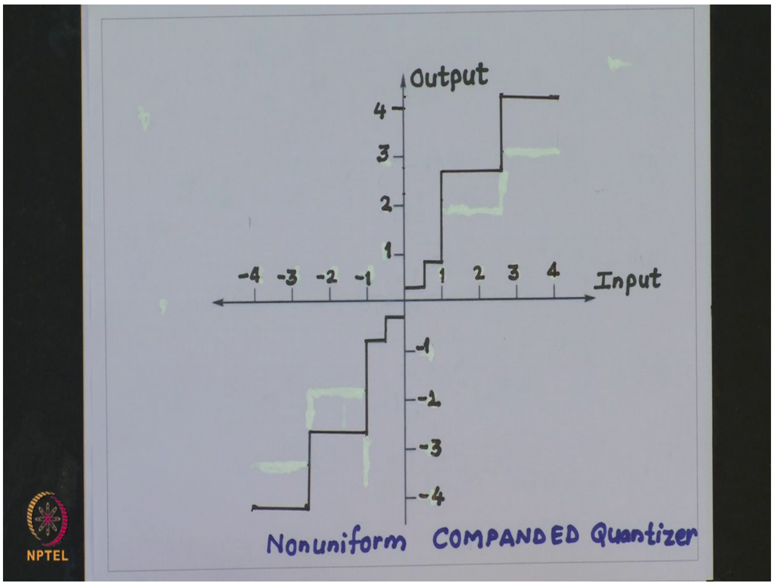

So, this is the Nonuniform Companded Quantizer Input-Output Mapping for the example which we discussed. It is interesting to realize that for the smaller amplitude, we have the smaller quantization intervals; for the larger amplitudes, we have the larger quantization intervals correct and this is acceptable.

So, if you have the input of a smaller amplitude, we would like the quantization error also to be small and if we have implode of larger amplitude, we would like to have or we can tolerate larger quantization error. So, your quantization interval can be larger and this is what is being depicted by this. So, this example shows how you can achieve Non-uniform Quantizer by using this kind of a Compressor followed by Uniform Quantizer followed by Expander.

Now, let us try to generalize what we have studied. So, let us assume that we have a source and its output can be assumed to be bounded to say some value x max.

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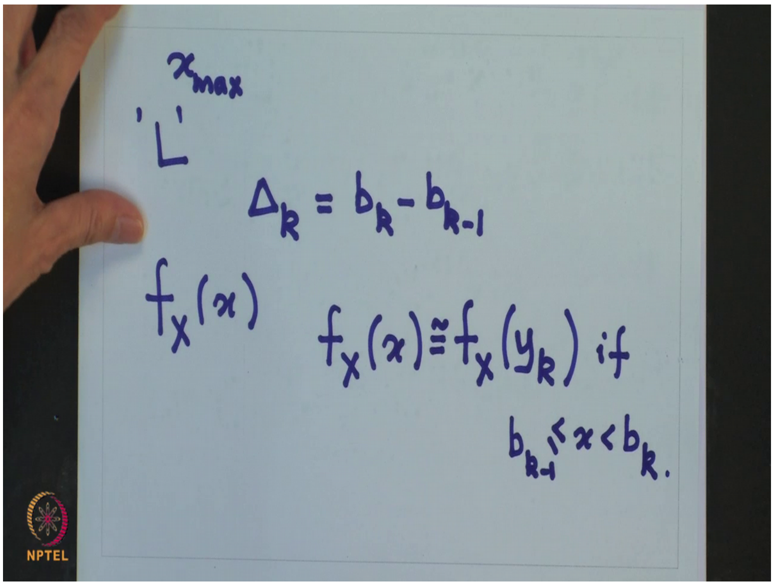

Then, any Non-uniform Quantizer can always be represented as a Companding Quantizer. So, let us see how to do this? And when we do this basically, we can get a quantizer that will be robust to mismatch correct. So, we will show that signal to quantization noise ratio will not change with the change in the pdf or with the wrong choice of the variance.

So, before we do this basically we should understand some of the properties of a high rate quantizer or what I mean by a high rate quantizer? By high rate quantizer, I mean a quantizer with a large number of levels which I indicate by capital L. A nd let me indicate the quantization interval delta k to be equal to b k minus b k minus 1. These are decision boundaries.

Now, if the number of levels L is high; then, the size of each quantization interval will be very small and if we can assume that the pdf of the input is also quite smooth. Then, in this interval this pdf will be essentially constant in the quantization interval. This is my assumption correct. So, for this assumption, I can write the pdf within the interval. This interval equal to approximately this value; where, y k is the reconstruction level of this interval delta k correct; So, this is valid for this. So, it means it satisfies this condition.

Now, for this kind of a model I can immediately write my quantization Noise variance as follows.

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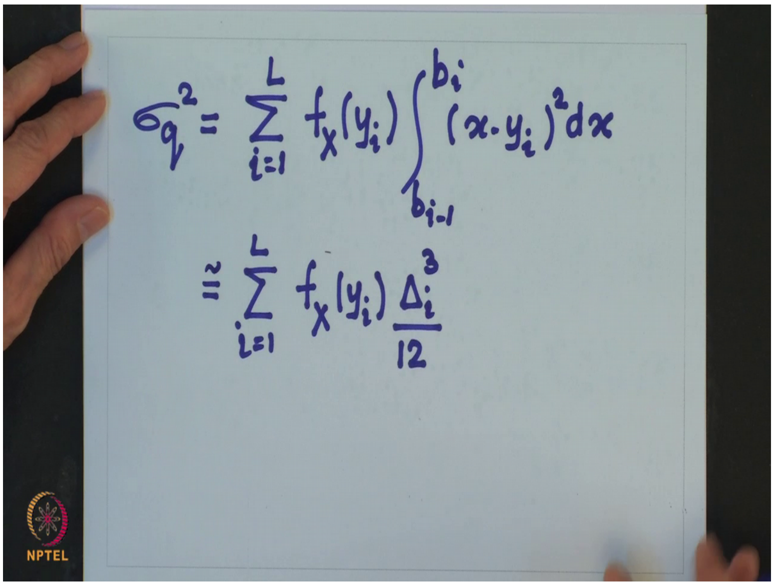

Now, if my interval delta k or delta i is small and if I can assume my pdf is more or less constant in that interval; then, we know that your reconstruction level which is going to be the centroid of the probability mass, in this interval will be the midpoint of b i and b i minus 1; this we have studied earlier.

So, based on this, I can rewrite this equation as follows. So, any particular interval I will get this equation, if I try to evaluate in that interval, the probability will be given by 1 by delta i and then, you have to evaluate this.

So, this will be the answer to this integral correct. You can get this in a different way also; you can argue it out like saying that I know that if I have a free construction level which is the midpoint of the interval; then, the quantization noise which I get from that interval is equal to delta i squared by 12 correct.

And the probability of your input being in that interval is nothing but f x y i multiplied by delta i approximately correct. So, this is approximate. So, both the arguments hold this ok. Now, we will select a compressor function and try to evaluate for that compressor function, this expression. Let me assume that I choose a compressor function cx to be of this form.

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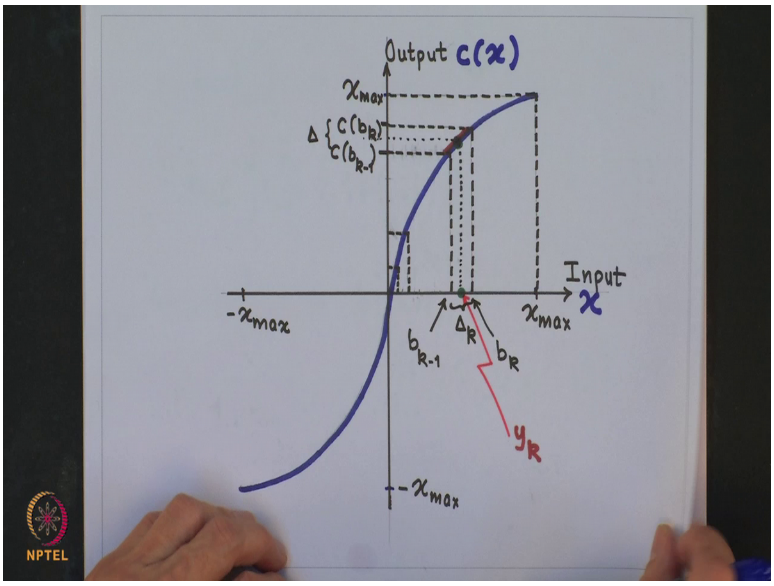

So, this is my input and this is my output without loss of generality, I assume that my output maximum value is also equal to x max and these are the intervals on the x axis that is the input and when I pass it through a Compressor this is the output which I am going to get.

So, this denotes one interval delta k and out here the approximately the y k for this will be located somewhere in middle of this two boundaries. Now, for this point if we try to evaluate, at this point we try to evaluate the differentiation at this point. So, I am interested in calculating and this output, let me denote it as characteristic to be c x. So, I am trying to evaluate the derivative of c x at x is equal to this is my x at x is equal to y k.

Now, if this interval is very small; then, I can approximate this derivative by a straight line in this interval and the slope of this line will be equal to this divided by this. So, this quantity out here is c b k minus c b k minus 1 and that is going to be equal to delta because this quantity out here I want to divide it equally I am going to pass this through a

Uniform Quantizer. So, my decision boundaries on this axis are going to be equally spaced correct.

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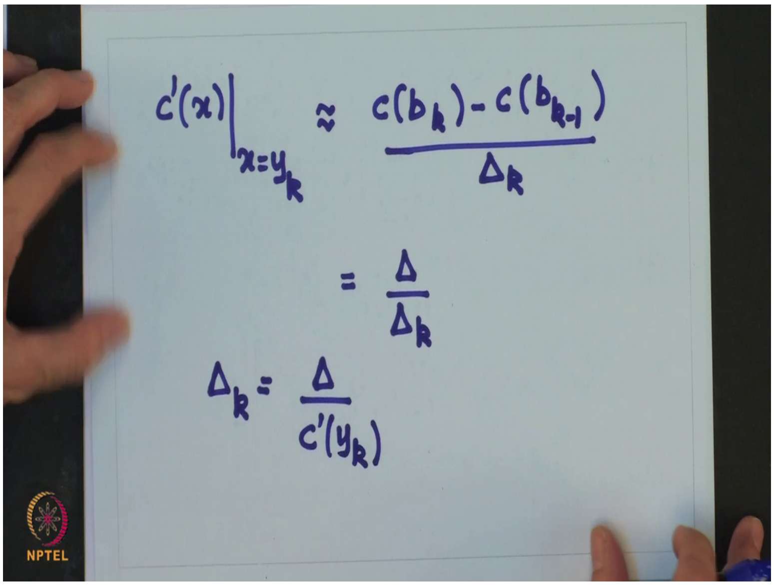

So, let us calculate the derivative of c x at x is equal to y k which is the reconstruction level in the particular input interval. This is approximately equal to c b k minus c times b k minus 1 by delta k. Now this c b k minus c b k minus 1, this quantity out here, we have said is nothing but the quantization step for the Uniform Quantizer and this is equal to delta by delta k.

So, from this quantity I get my delta k to be equal to delta by derivative evaluated at y k, I get this quantity.

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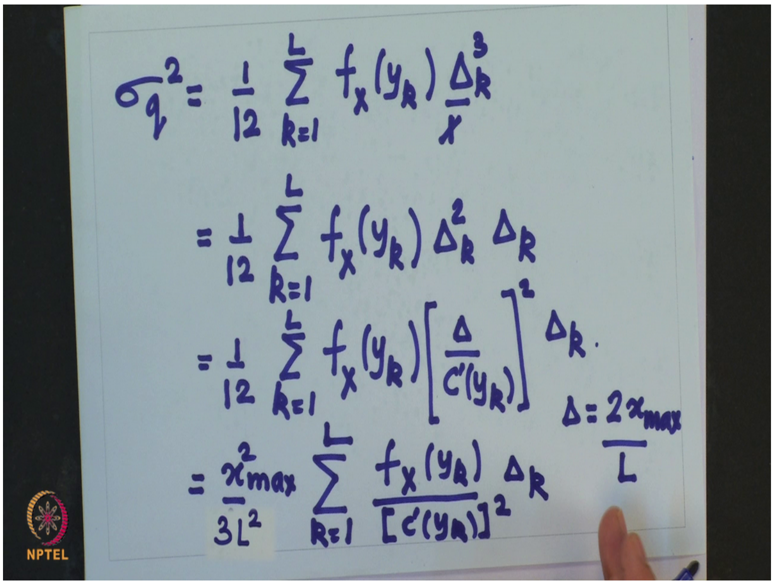

Now, given this we can write our quantization noise power which we had got it equal to 1 by 12 ok. So, I can rewrite this here. So, this expression, I can rewrite it as. This we just derived it. Remember your delta is equal to twice x max by that is the output range of the compressor divided by the number of levels which being which are being used for the Uniform Quantizer.

So, if I use this basically I get the quantity to be equal to. Now, if we make our L to be very large; then, your delta k is going to be very small and in that case what I could do is basically is substitute this summation by an integral.

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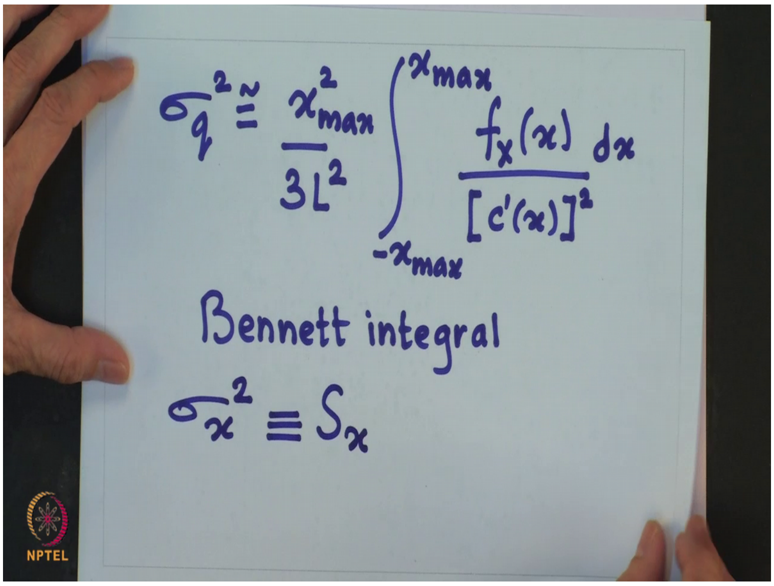

And I get the quantization noise to be equal to approximately. This is known as Bennett integral and it is used in the design of quantizers. The name has been given by the discover all of the integral W R Bennett.

Now, the next question is can we choose that c x such that this quantity out here becomes independent of the shape of f x; is it possible? So, given this is the quantization noise we get. So, for the signal, we have the power in the signal which is denoted by this quantity of S x; both we can use.

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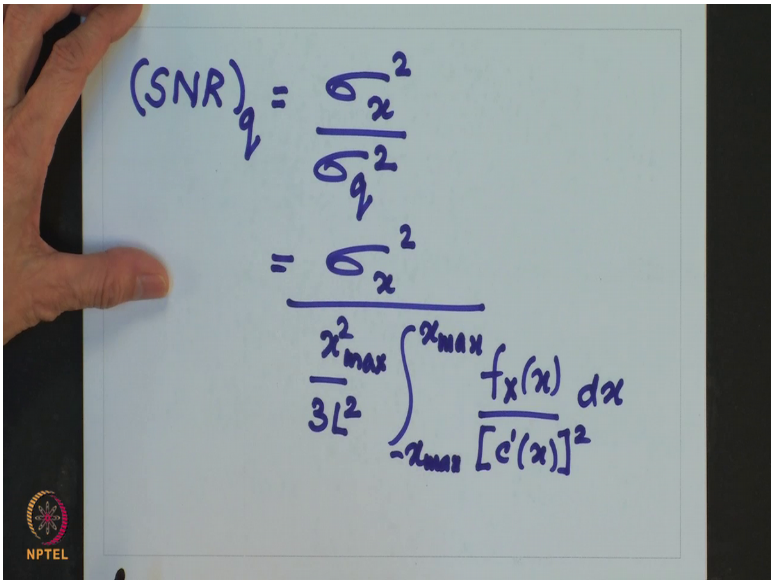

$$(SNR)_q = \frac{\sigma_x^2}{\sigma_q^2} = \frac{\sigma_x^2}{\frac{x_{max}^2}{3L^2}\int_{-x_{max}}^{x_{max}} \frac{f_X(x)}{[c'(x)]^2}\,dx}$$

So, now, if you calculate the signal to quantization noise ratio, we will get it to be. Now, the question is that is it possible for me to choose that c x such that this quantity becomes more or less constant; that means, your quantization noise should become proportional to the signal variance.

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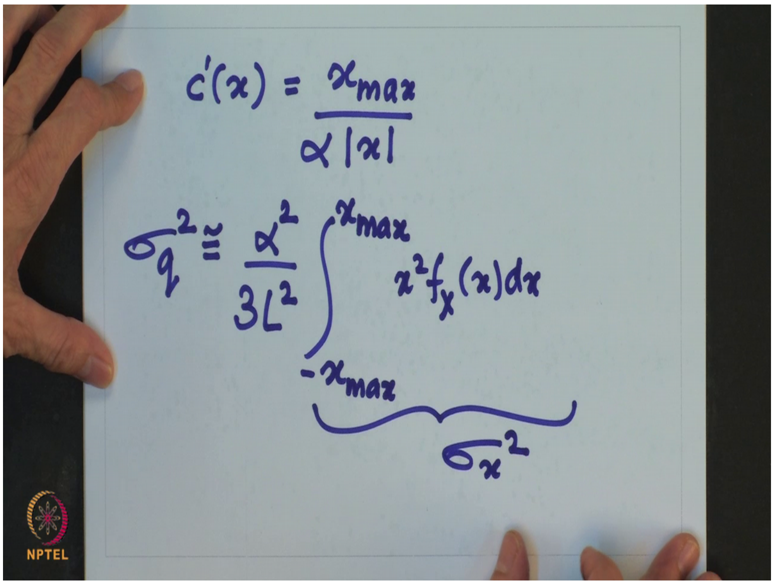

So, let us choose one type of a Compressor characteristic whose derivative satisfies this relationship; where, alpha is a constant. So, if you look at this quantity basically this will tend to 0 as x tends to infinity and it will tend to infinity as x tends to 0.

If we do this then my variance in the quantization noise will be approximately equal to alpha times 3 L squared and this is nothing but the signal variance. So, what we find that the noise variance becomes proportional to the signal variance.

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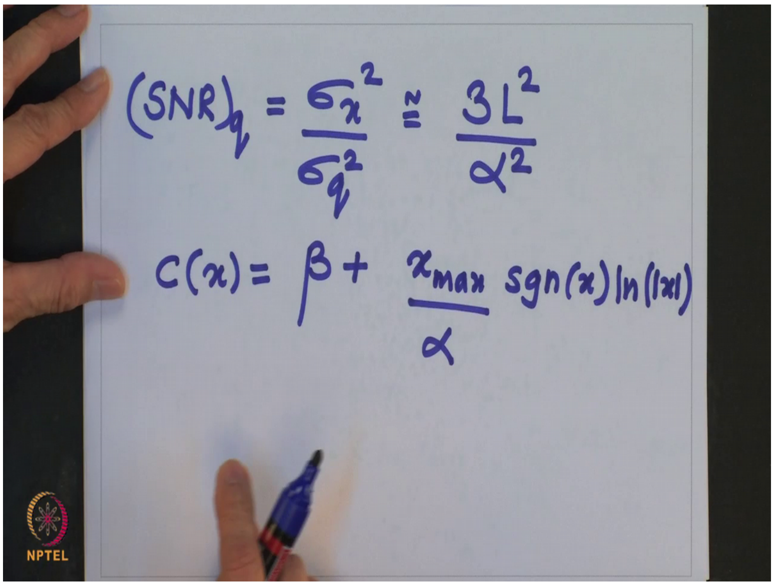

And if you calculate the signal to quantization noise ratio turns out to be equal to approximately 3 L squared by alpha square; where, alpha is a constant remember this relationship is valid under the underlying assumption that we choose large number of levels quantization levels and your pdf is smooth.

Now, if you have the input variance to be low; then, it will imply that your pdf is not smooth and in that case the pdf within a particular interval will not be constant or if you have a large variance; then, your input will not become bounded fine.

Now, given this what is the value of c x. One choice of c x would be equal to beta plus x max by alpha that is the derivative of what we have derived c dash x the value which we have chosen, we can choose beta to be 0. It is a just a constant.

Now, there is a problem with this equation is that for x tending towards 0. This becomes very large. So, this function actually does not pass through the origin. So, how do you design a practical compressor characteristic from this? And this is done by using what is known as Au law and Mu law which will study in the next class.

Thank you.