Principles of Digital Communications
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Lecture – 23
Probability of Error for M-ary Scheme

Computation of error probability is simplified if we have some knowledge about the; how the decision regions look and this is possible only in simple cases. With the increasing dimension of the message signal space, it becomes increasingly difficult to visualize, this decision regions. So we will derive an analytical expression for error probability in a general M-ary scheme.

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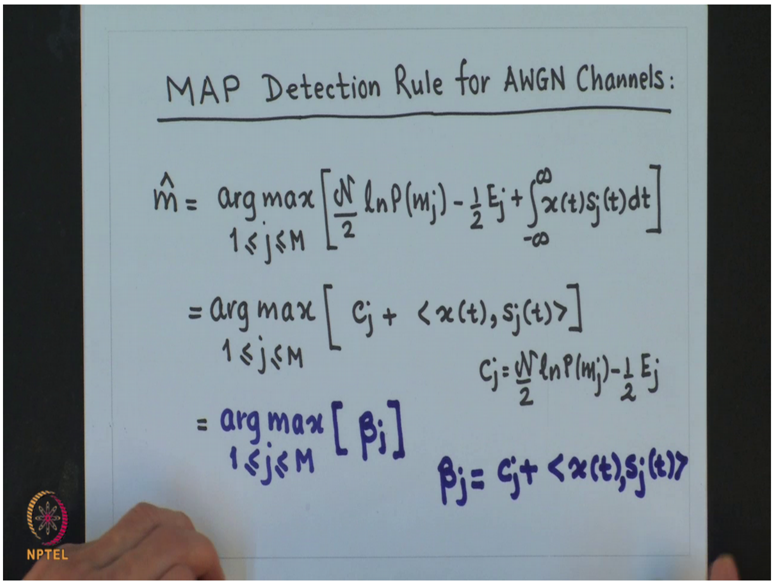

We have seen that map detection rule for AWGN channel is given by this expression which can be rewritten as this where C j is given by this expression.

And now we can go a step ahead and make it little more concise and define what is beta j. So, this beta j is equal to C j plus cross correlation between xt and S j t or the projection of xt on S j t correct.

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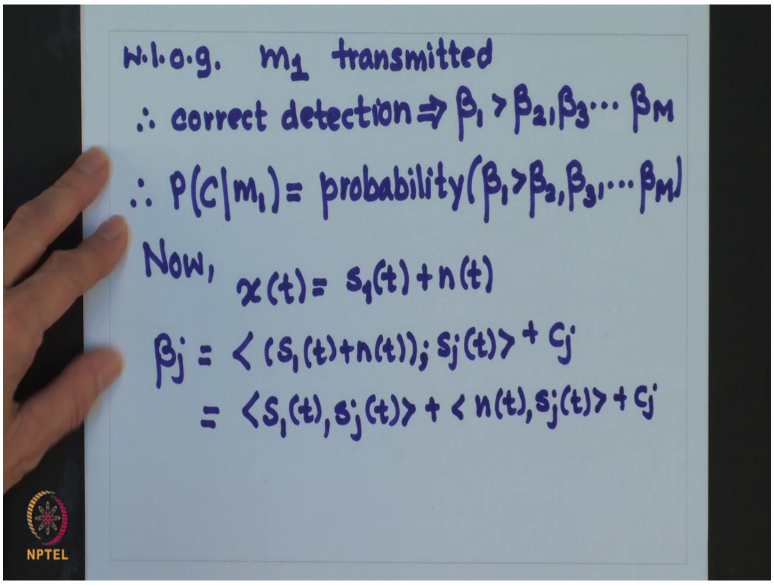

Now without loss of generality, let us assume that we transmit message m 1. Now this implies that for the correct detection beta 1 should be greater than beta 2, beta 3 up to beta M.

So, this implies that probability of correct detection given m 1 should be equal to probability that beta 1 is greater than beta 2, beta 3 and so, on up to beta M. Now we know the received signal xt is equal to S 1 t because we are transmitting message m 1 plus noise nt. So, your beta j is going to be S 1 t plus nt projected over S j t plus constants C j; this I can rewrite as S 1 t, S j t cross correlation plus projection of nt on to S j t plus C j.

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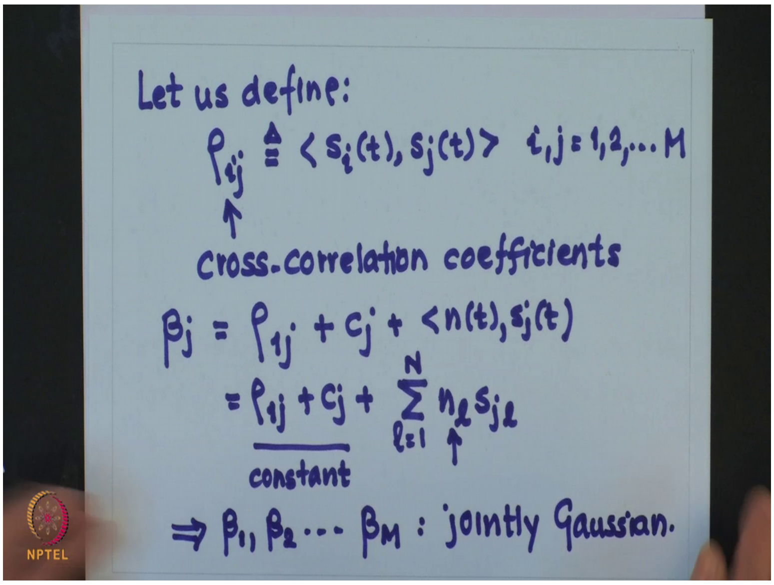

Now, let us define rho i j equal to cross correlation between the signal S i t and S j t correct, for all i j equal to 1, 2 up to M; so, this are known as cross correlation coefficients. So, when m 1 is transmitted your b j will reduced to ok.

Now this can be rewritten as this is the constant; these out here are the components of the noise projected on to the n dimensional message signal space. And we are shown that these are jointly Gaussian random variables each with zero mean and variance equal to italic N by 2 which is the power spectral density of the white noise and being Gaussian, we also shown that these are independent.

So, what this implies that beta j which is a linear combination of jointly Gaussian random variables. So, this implies that your beta 1 beta 2 up to beta M are also jointly Gaussian.

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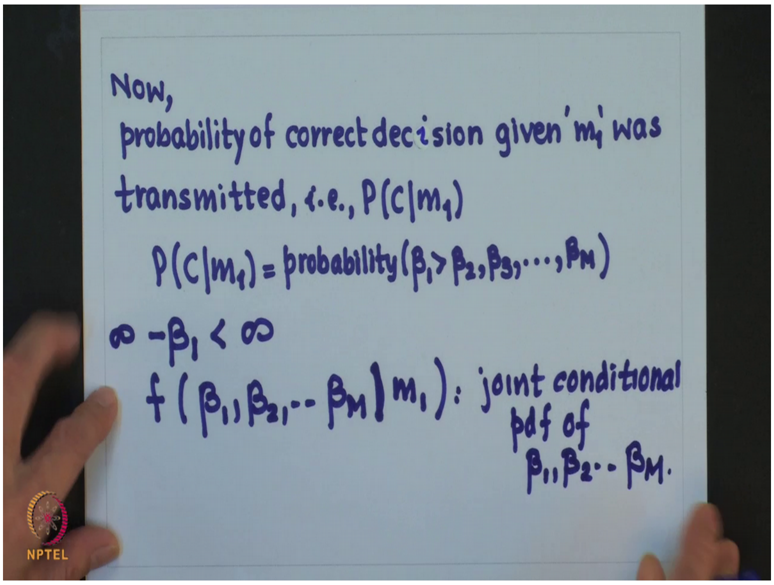

Now we said that probability of correct detection given m 1 is equal to this quantity correct. Now remember beta 1 can lie anywhere between minus infinity to plus infinity correct. And let us denote this pdf joint pdf this is the joint conditional pdf of betas.

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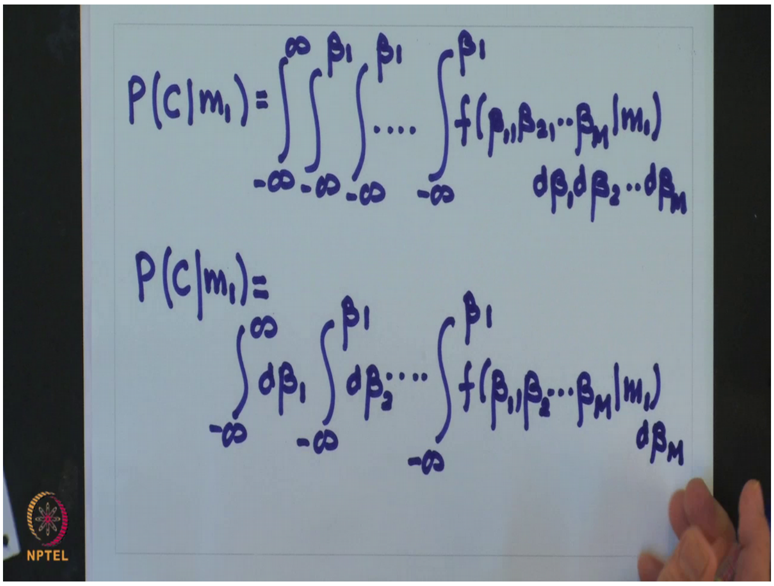

Then we can evaluate probability of correct detection given m 1 by m 4 integration as follows; joint pdf of betas given m 1 and integrated over betas; so, this can be rewritten as.

So, having calculate this we would be required also to calculate the probability of correct detection given m 2 then probability of correct detection given m 3 and so, on.

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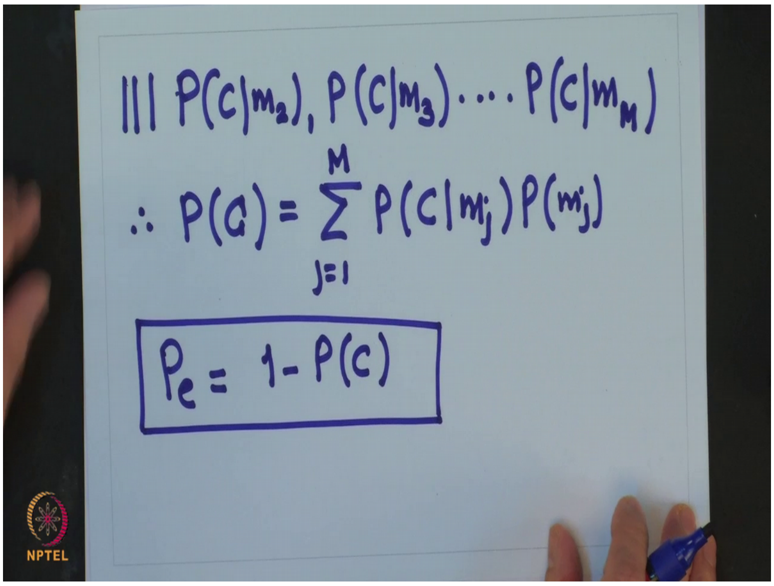

So, we will be required to calculate these quantities once we have this; we can calculate the unconditional probability of correct detection as follows. This is a conditional probability of correct detection given m j multiplied by the probability of m j; j is equal to 1 to m. And once we have the probability of correct detection, we can evaluate the error probability by 1 minus probability of correct detection.

Now, to appreciate what we have done here let us take a simple example.

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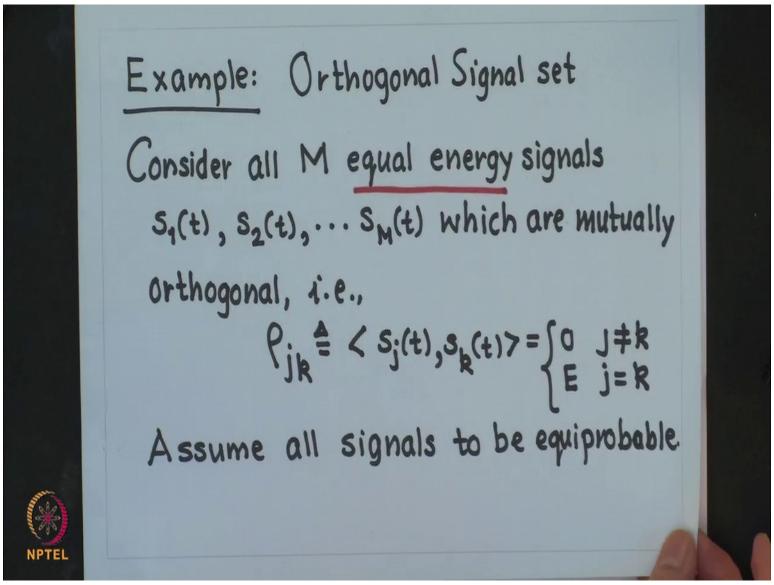

So, let me say that I have an orthogonal signal set consisting of M equals signal energies S 1 t 2 up to S M t. And these are mutually orthogonal it means that cross correlation between S j t and S k t is equal to 0 for j not equal to k and is equal to some constant value E; corresponding to the energy of the signal when j is equal to k. And we have seen that this by definition we could call it as rho j k which is a cross correlation coefficient.

And now let us make one more assumption that all signals to be equi-probable; if we do that let us try to evaluate the probability of error for this case, for the optimum detector.

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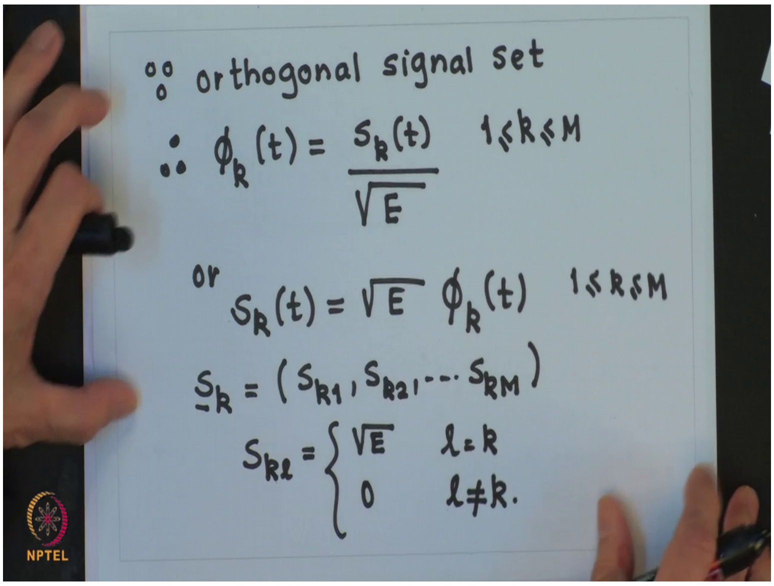

Now because we have orthogonal signal set; therefore your orthonormal basis signals are simply the signals normalized by the energy; which also means I can also write S k t equal to root E times phi k t for k equal to 1 up to n.

So, what this implies that the vector representing the signal S k t, if you have that let me denote it by this. In this vector you will have; in this case n will be equal to M because we will have M orthonormal basis signal and in this you will see that the component of this vector S k l will be equal to root E for l equal to k and is equal to 0 for l not equal to k.

Now, we know that map decision rule is given by this correct.

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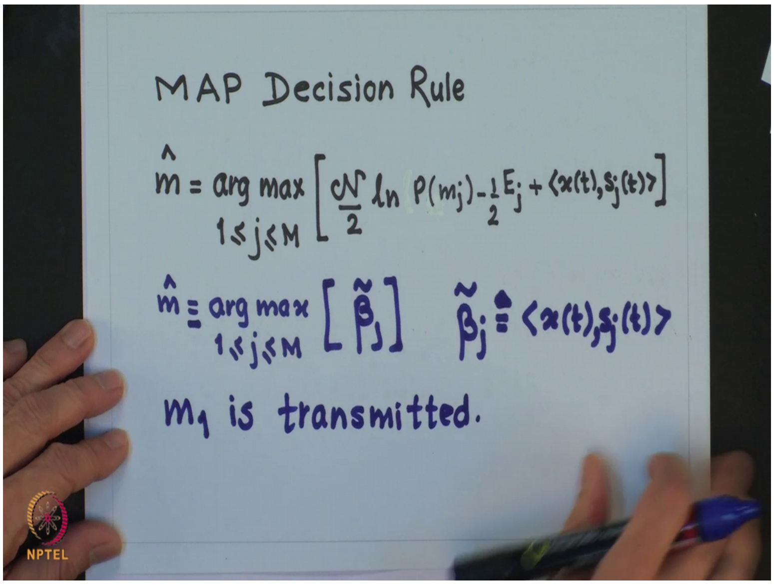

Now in our case the message signals are equiprobable and the energy of each signal is also constant equal to E. So, in this decision rule this term basically becomes a constant and the only variable is this. So, what we could do is basically we could simplify this map decision rule and rewrite it as beta j tilt, we can drop this term where your beta j tilt is by definition equal to projection of x t on to S j t correct.

Now, let us assume that we are transmitting the message m 1 without loss of generality correct.

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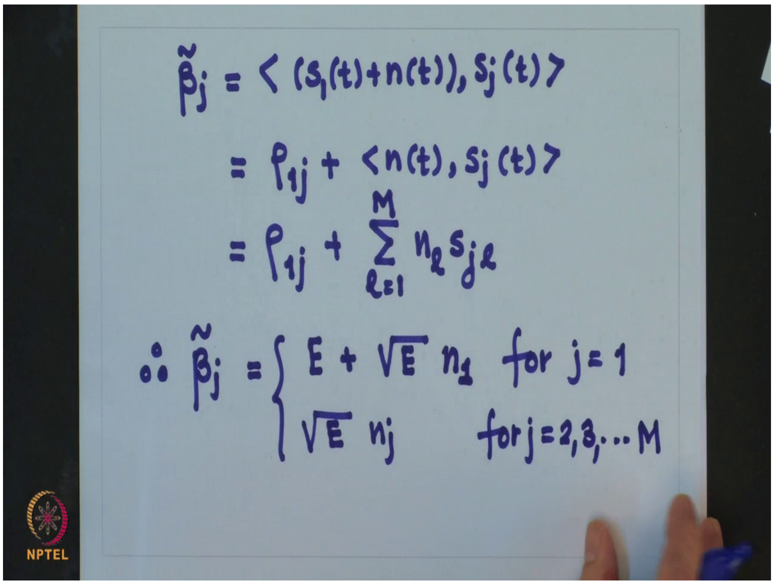

So, I will assume that m 1 is transmitted if I make this assumption then my beta j tilt will be equal to your received vector will be equal to this now. So, this will be equal to rho 1 j cross correlation between this plus projection of nt onto S j t and this I can rewrite it as.

Now, remember your S j l this will be basically equal to root E only when l is equal to j otherwise it is equal to 0. So, what this implies that your beta j tilt is going to be equal to this quantity when j is equal to 1 will be E plus root E that will be when l is equal to j and you will get noise component n 1 for j equal to 1 and for all other case this term will be equal to 0. So, I will get root E nj for j equal to 2 3 up to M.

Now, please note that n 1, n 2 up to nm are all independent Gaussian random variables each with zero mean and variance italic N by 2 that is the power spectral density of the white Gaussian noise. So, what this implies that beta j tilt will be also independent Gaussian random variables because these are jointly Gaussian random variables which are independent. So, beta j hat will be also independent Gaussian random variables with falling properties; the mean value of beta 1 hat would be equal to this value.

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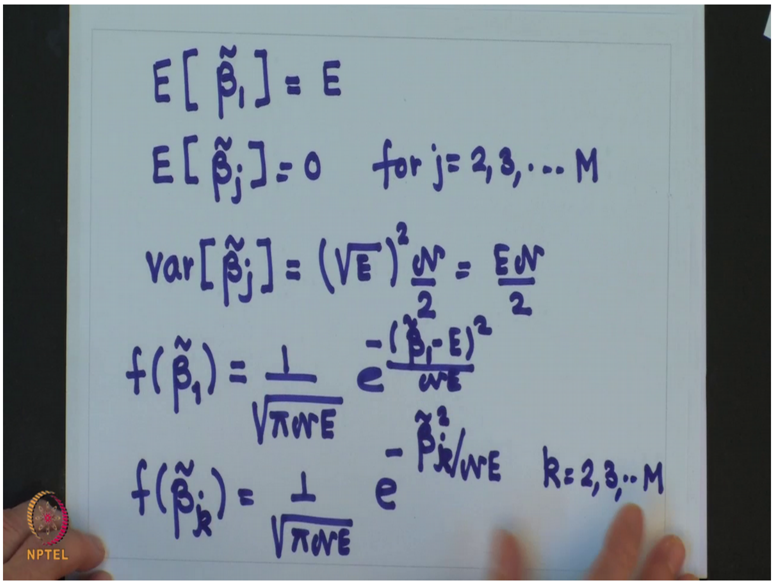

Because the mean of this is equal to 0 and the mean value of all other beta j tilt would be equal to 0. And the variance of beta j tilt will be or equal to root E squared multiplied by the power spectral density of the noise this is equal to E italic N by 2 from here fine.

And now we know that all this beta j tilt are Gaussian random variable. So, we can write the pdf of this as follows this is the variance divided by 2. So, 2 pi 2 2 cancels e raised to minus this is E and this is for beta 1 tilt and beta j tilt; this will be equal to the mean value is equal to 0. So, let us use this k instead of j for k equal to 2, 3 up to M fine.

Now, beta j hat for all j we are just shown that they are all independent Gaussian random variables. So, the joint pdf can be easily written for this as follows would be equal to the product of the individual pdf.

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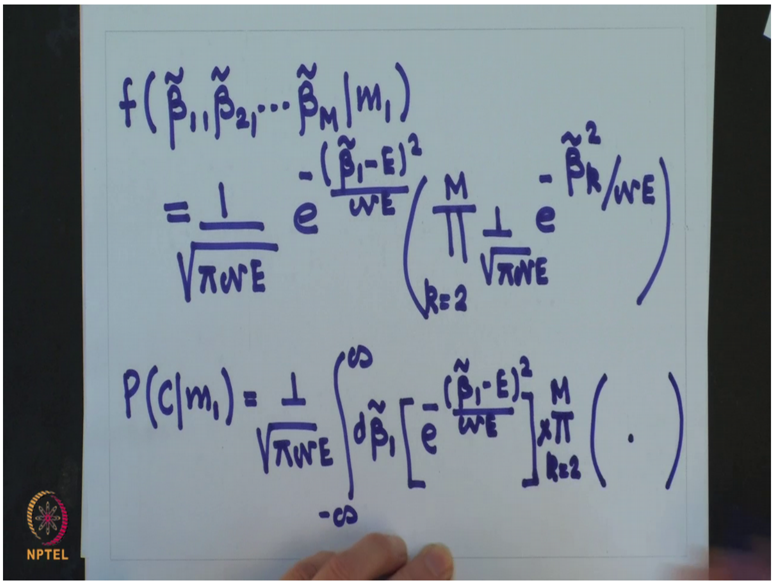

Because they are independent; this is the pdf I am writing for the beta 1 tilt and then product of the other pdf, this is what I get fine.

So, probability of correct detection will be equal to m full integration of this joint pdf beta 1 tilt can take values from minus infinity to plus infinity, I get this quantity. And then I have product this integral out here I will write it separately that integral is B 1 hat minus; so, this term out here just writing it separately on this.

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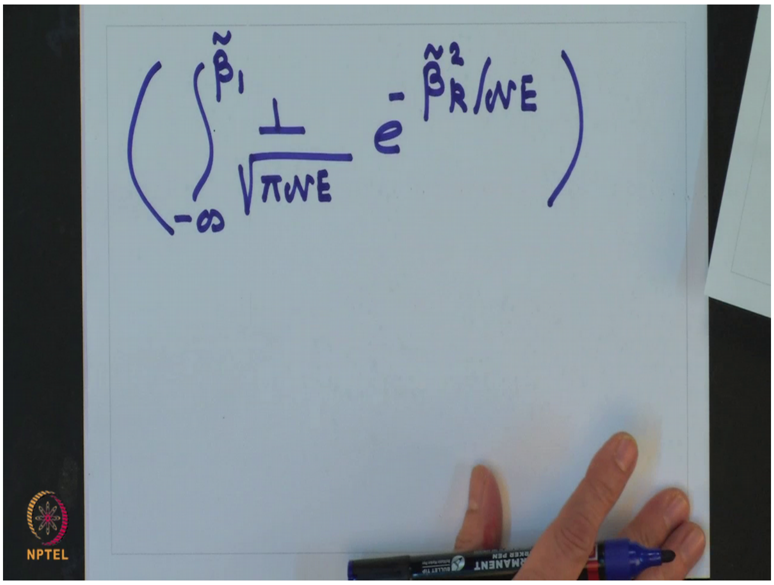

So, this is what I get correct. Now, this term I can just change of variable for convenience; I just beta k tilt I make it equal to x correct. So, this equation which I have here which I have to evaluate can be rewritten as follows simply I can write it like this correct.

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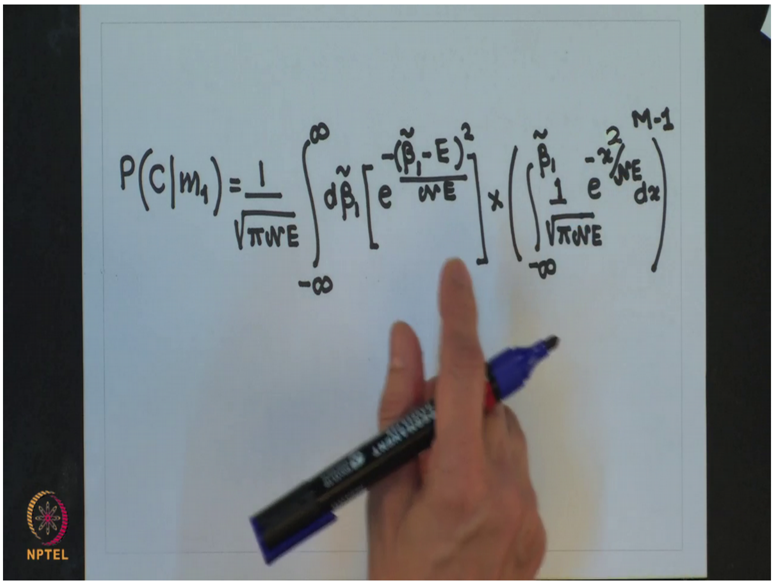

So, this will because M minus 1 terms will be there. So, this gets raised to this term now I can simplify this term and to do that we will use this property of the Q function.

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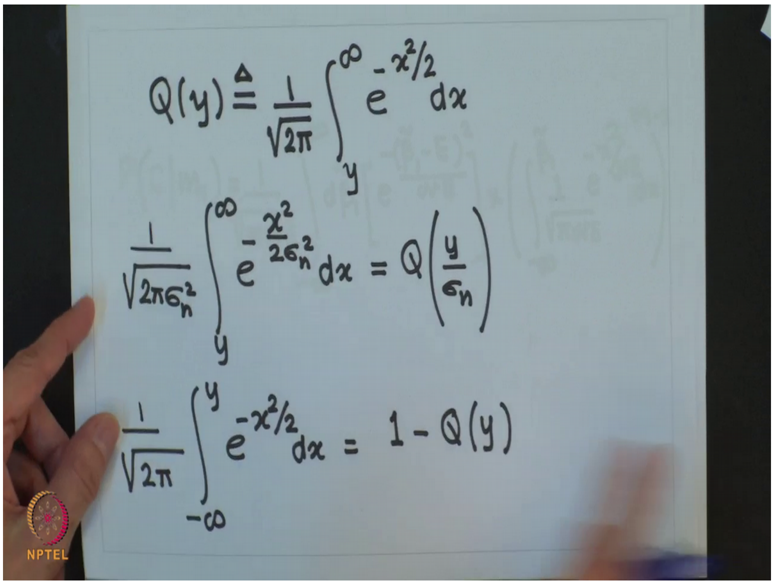

We have defined Q of y to be this quantity this is nothing, but the integration from y to infinity of a Gaussian pdf with variance equal to 1 and mean equal to 0 correct. And if I have in general this Gaussian pdf with the mean equal to 0 and the variance equal to sigma n square, then I can write this from here and I can write this quantity equal to this.

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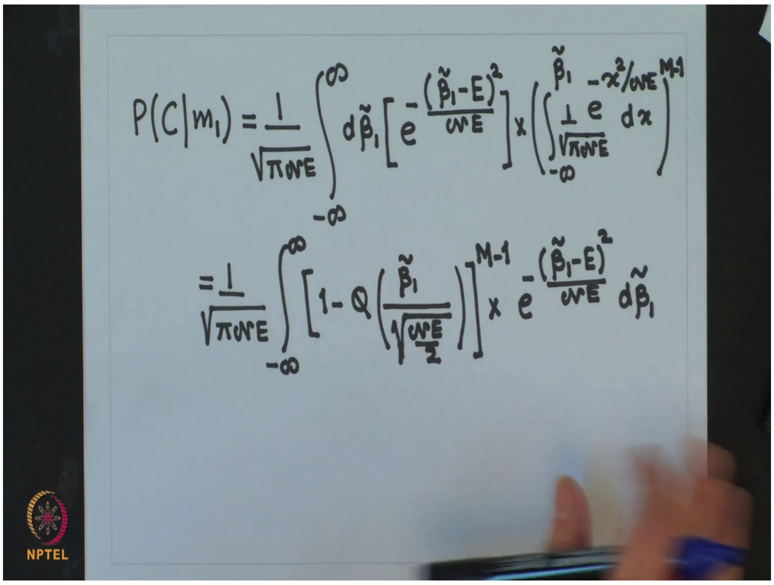

So, given this I can simplify this quantity. So, using this relationship we can get this term and this term basically I can write it in this format.

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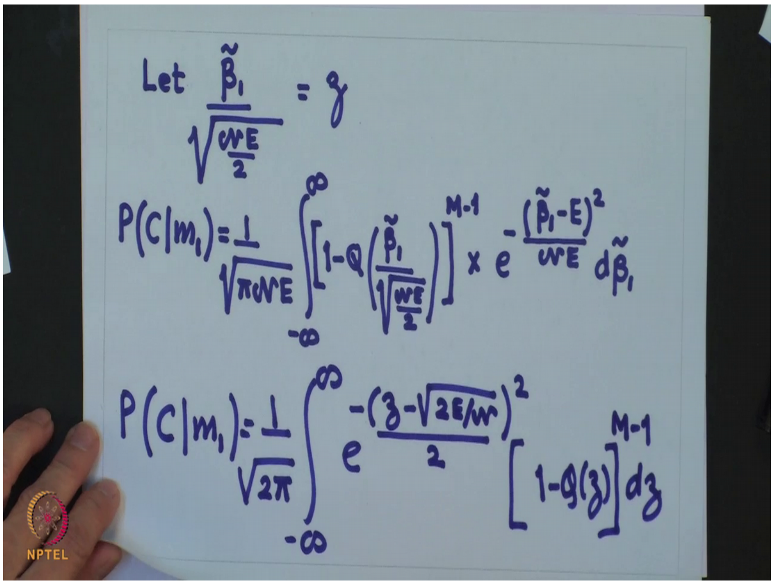

So, this becomes equal to this term out here correct fine and now, let me make one simple substitution this term out here is substituted as z; if I do this basically this expression which I have I can rewrite it as.

Now, you can make one more substitution this energy E I would like to write in terms of bit energy.

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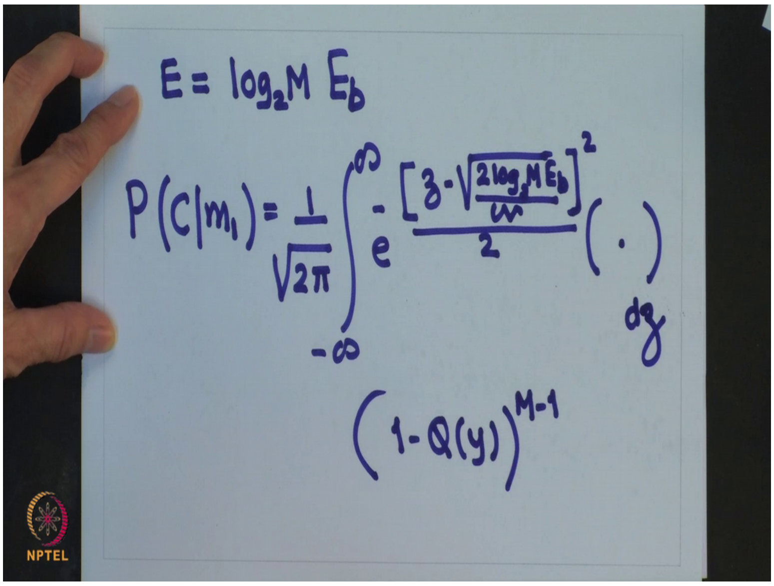

Now bit energy is equal to number of bits which will be given by log to the base 2 of M multiplied by E b. So, if I do this and plug into this equation I get my probability of correct detection given m 1 to be of the form multiplied by this term, this term is equal to 1 minus dz correct. So, this term I am just rewriting it out here.

Now, note that that signal set is geometrically symmetrical; that means, every signal has the same relationship with other signals in the set.

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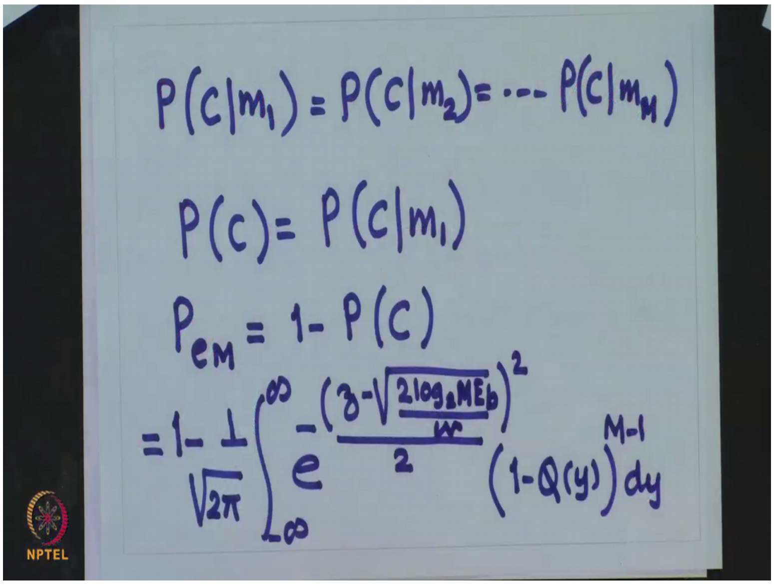

What this implies is that probability of correct detection given m 1 is equal to probability of correct detection given m 2 probability of correct detection given m m. So, the probability of correct detection is equal to probability of correct detection given m 1 correct and probability of error correct for this is a probability of message signal going in the error will be equal to 1 minus probability of correct detection and that would be equal to 1 minus 1 by root 2 pi of this quantity.

Now it is interesting to see that this is difficult expression to evaluate. So, you can no close form solution will be available, but we can plot it. Now without going into any details we can show that as M tends to infinity, the probability of error will tend towards 0 and if you evaluate that you get an interesting result.

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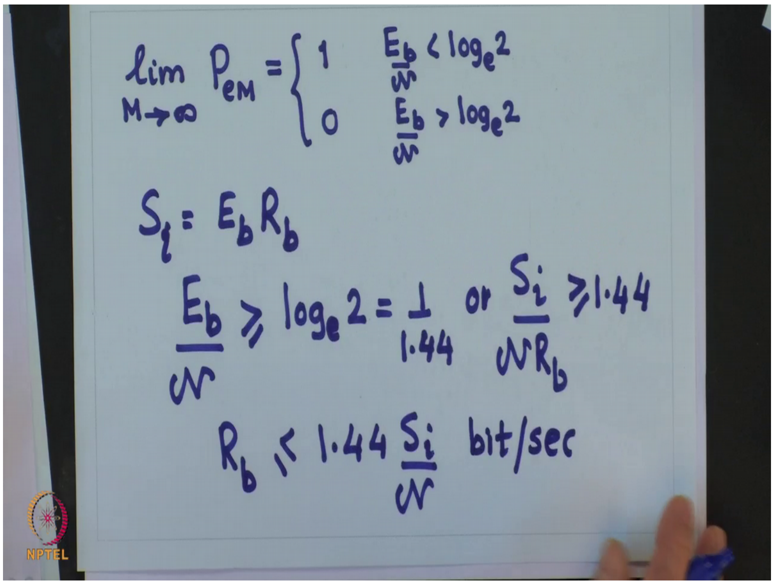

If M equal to tending to infinity we get this, now remember that signal power in is equal to energy in the bit multiplied by the bit rate that is R b. So, what it follows from this that for error free communication; this should be greater than equal to log of to the base e of 2 equal to 1 upon 1.44 or this implies that from this relationship.

So, your rate has to be less than equal to 1.44 times signal power divided by italic N bit per second. Now remember we had derived a capacity of a channel of infinite bandwidth correct as this additive white Gaussian noise and we had shown that channel capacity turns out to be this.

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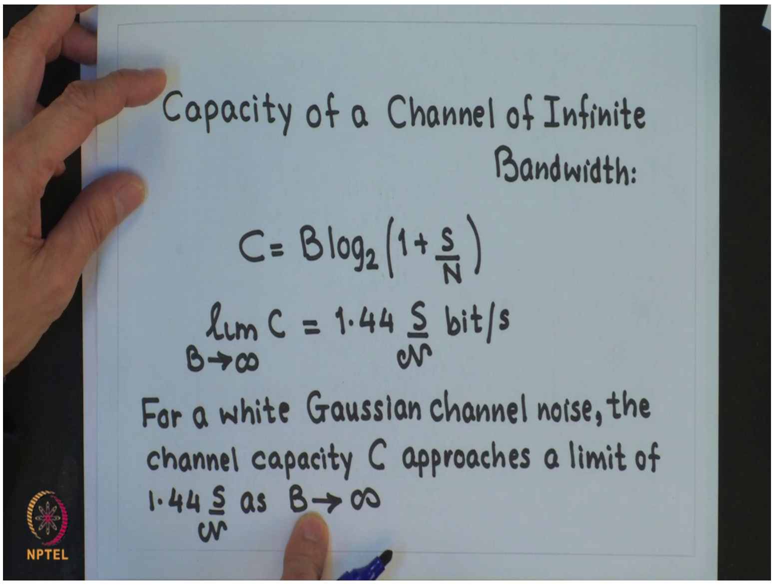

So, for a white Gaussian channel noise; the channel capacity C approaches a limit of this as B tends to infinity.

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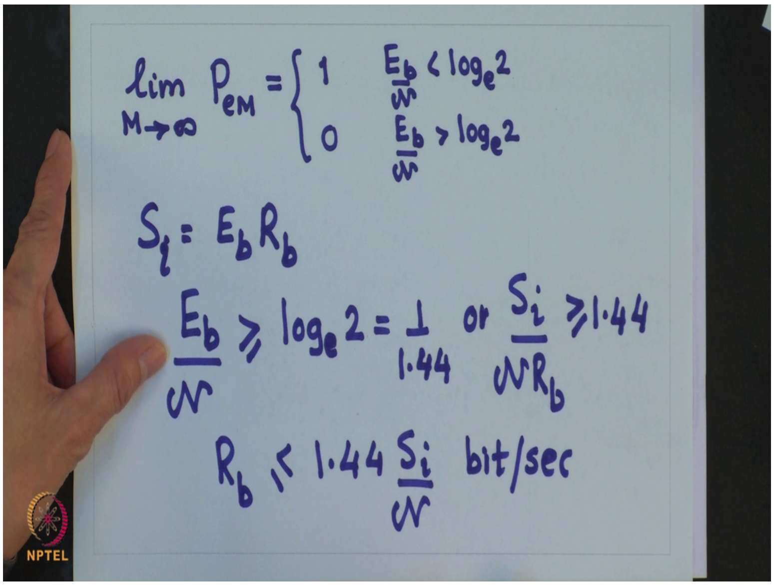

So, for M-ary orthogonal signaling also we can transmit at error free data at a rate up to this quantity correct and this will happen when M tends to infinity.

So, we have shown the relationship of what we had learned earlier with the; this example. So, with this basically we come to the end of our study on M-ary communication in additive white Gaussian noise channel. Quickly let me recapitulate what we have done; first we said that any complete orthonormal basis signals can be used for expansion of zero mean white process.

Next we saw that given a set of m message signals we can always determine a set of n orthonormal basis signals where n could be less than or equal to m using Gram-Schmidt orthogonalization procedure. We extent this is concepts to show that a continuous additive white Gaussian noise channel is in equivalent to a vector additive white Gaussian noise channel. Then we derived for a general vector channel an optimum detector which minimizes the probability of error and showed that it turns out to be what is known as map detector; maximum a posteriori probability detector and this detector reduces to ML detector that is Maximum Likelihood detector when all your message signals are equiprobable.

Then we derived explicit map decision functions and ML decision function for the case of additive white Gaussian noise. We also studied correlation based and match filter based implementation of the optimum detector. And we saw that match filter has an interesting property of maximization of signal to noise ratio at the sampling instant. And finally, we also saw that computation of error probability is simplified if we have some knowledge about how the decision regions look, but this is possible only in simple cases. So, we derive an analytical expression for error probability in a general M-ary scheme.

So, with this we come to the end of the module on M-ary communication in AWGN channel.

Thank you.