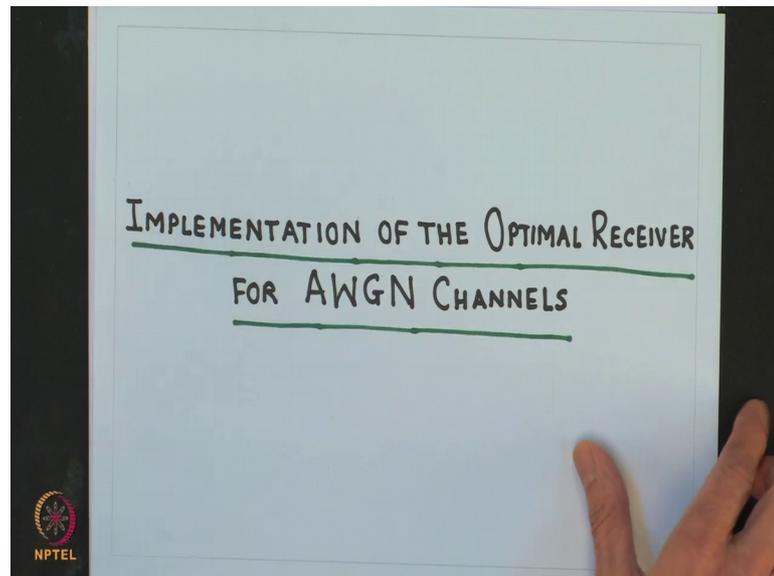


Principles of Digital Communications
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Lecture – 21
Optimal Receiver : Matched Filter

The Optimal Receiver for an AWGN Channel uses map detection rule.

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Now, we look at the implementation of this optimal receiver for the AWGN channels.

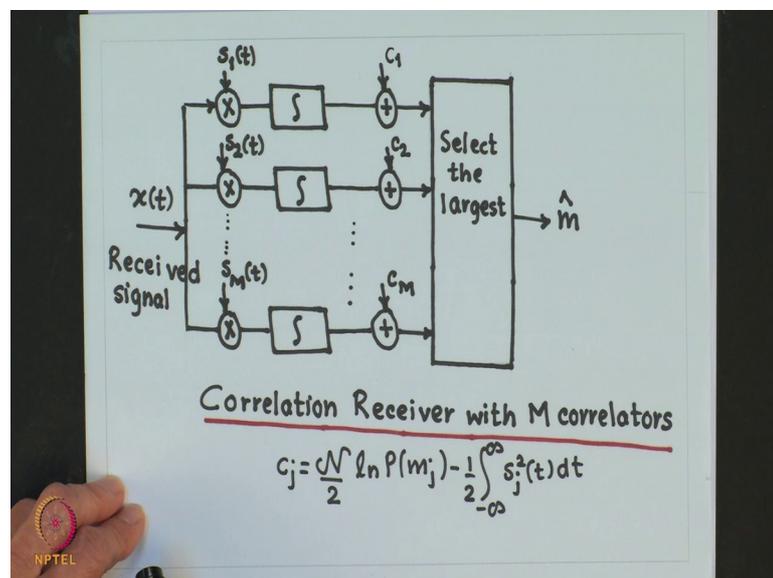
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MAP Detection rule in an AWGN Channel:

$$\hat{m} = \arg \max_{1 \leq j \leq M} \left[\int_{-\infty}^{\infty} x(t) s_j(t) dt + C_j \right]$$
$$C_j = \frac{\sqrt{N}}{2} \ln P(m_j) - \frac{1}{2} \int_{-\infty}^{\infty} s_j^2(t) dt$$

We have studied one form of map detection rule, which is given here and there C_j is the bias term. Now, this can be easily implemented as shown in this figure.

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This basically corresponds to the correlation. the receive signal basically gets correlated with $s_1(t)$ here, it gets correlated $s_2(t)$ and $s_M(t)$ correct. And then to this outputs, you add the bias terms and select the largest value and decide the message transmitted.

Now, because of this block out here, which is cross correlation between the signal $X(t)$ and $S_j(t)$ this is known as a correlation receiver and, since there are m number of signals corresponding to M number of messages we have M correlators out here.

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MAP detection rule in an AWGN Channel:

$$\hat{m} = \arg \max_{1 \leq j \leq M} \left[\underline{x} \cdot \underline{s}_j + c_j \right]$$

$$\underline{x} = [x_1, x_2, \dots, x_j, \dots, x_N]$$

$$x_j = \langle x(t), \phi_j(t) \rangle, \quad 1 \leq j \leq N$$

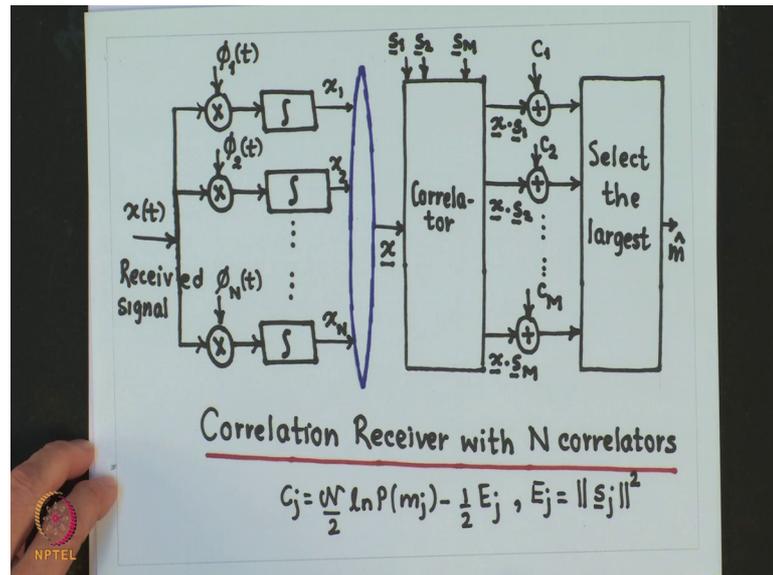
$$c_j = \frac{\sqrt{N}}{2} \ln P(m_j) - \frac{1}{2} E_j$$

$$E_j = \langle s_j, s_j \rangle = \|s_j\|^2$$

We have also studied another form of map detection rule in AWGN channel and it is given by this expression here, your vector x is dependent on the received signal. So, any element of this vector say x_j is obtained by taking the projection of the received signal $x(t)$ onto ϕ_j .

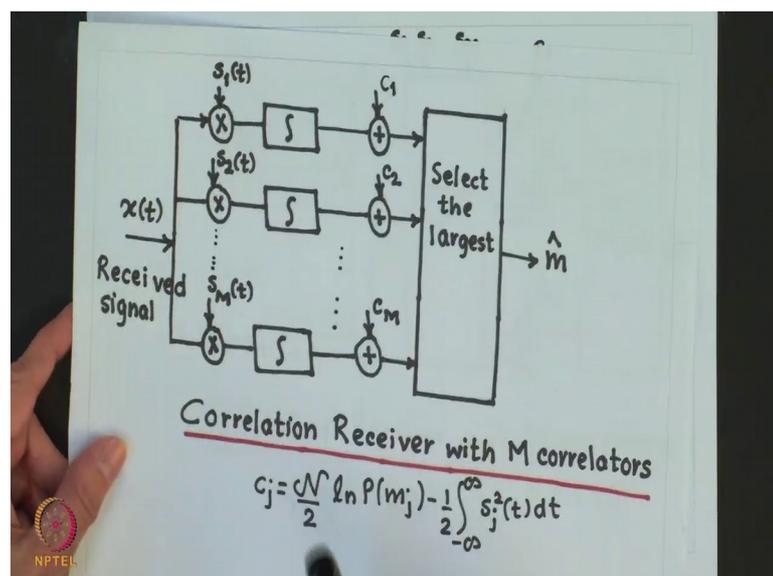
Now, to implement this we will first have to get this vector x and, then take the dot product with S_j . And this can be implemented using the block schematic diagram receiver as shown in this figure.

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So, this gives you the projection of $X(t)$ on different basis signals, you get your vector x each of the this vector gets, correlated or you take the dot product with M copies of the M vectors corresponding to M message signals and, then you add the bias terms and select the largest output. Again this basically of this is known as correlation receiver because, of cross correlating $X(t)$ with different ϕ_j 's and integrating it. And here you require m number of correlators corresponding to N basis signals.

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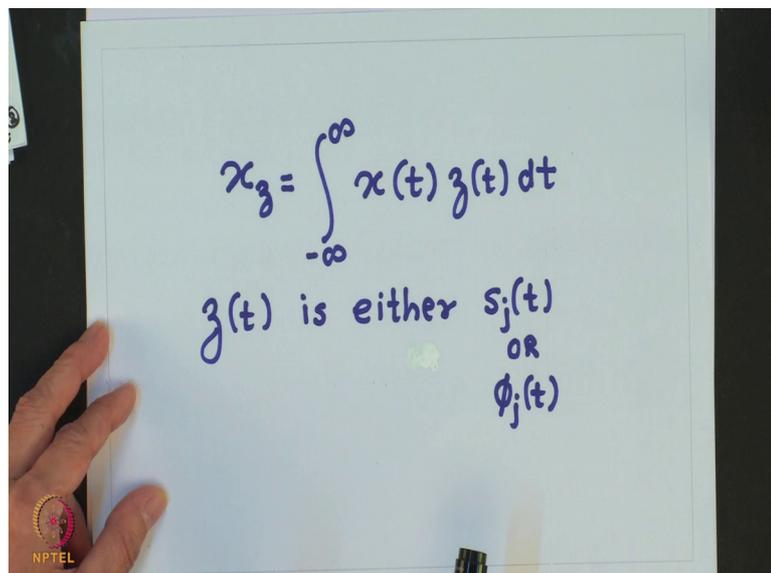
Now, it might appear that this is more complex than the earlier block diagram, but in a practical situations this may turn out to be easier to implement because, the value of N would be much lesser than M and then in that case you will require only N correlators whereas, the here you require m correlators. Now, both this correlation receivers can also be implemented using what is known as matched filter.

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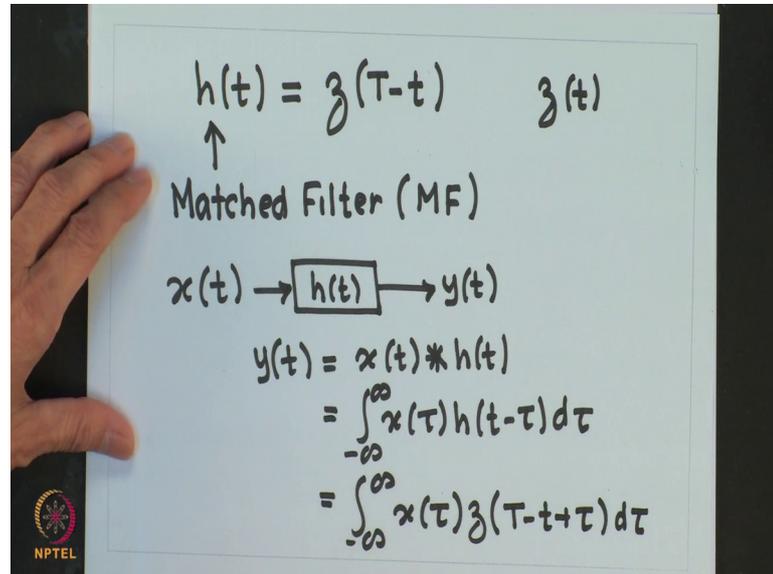
So, we will study now the concept of matched filter. In both the correlation receiver implementation which we just saw.

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We compute quantities of the form $x(t)$ multiplied by $z(t)$, let me call this as $x(t)z(t)$ and $z(t)$ could be either $S_j(t)$, or it could be $\phi_j(t)$.

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$$h(t) = z(T-t) \quad z(t)$$

$$\uparrow$$
 Matched Filter (MF)

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

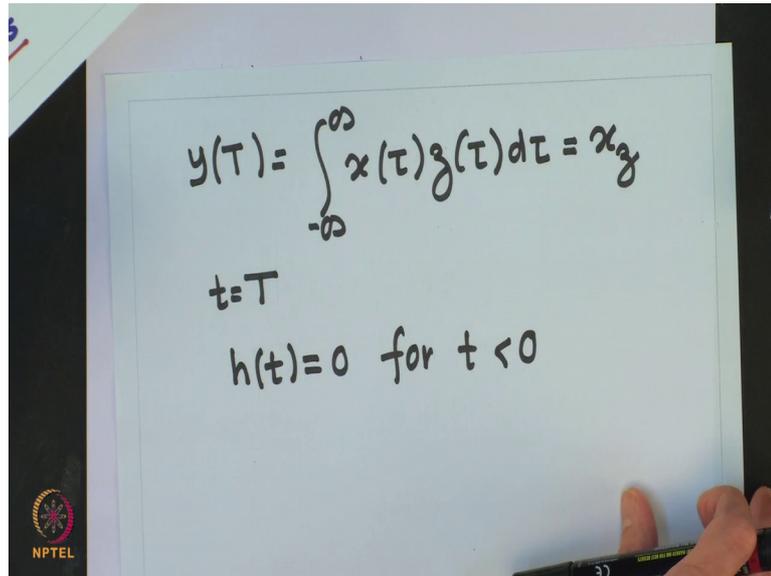
$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) z(T-t+\tau) d\tau$$

So, let us define, a filter with an impulse response $h(t)$ and let it be equal to $z(T-t)$. Then this filter is called a matched filter because, it is a filter which is matched to the signal $z(t)$ using this relationship. So, now let us assume that you have the input $x(t)$ to this filter $h(t)$.

So, let us compute the output for this $y(t)$ is equal to convolution of $x(t)$ with $h(t)$, this is equal to this is equal to substituting the value for $h(t)$.

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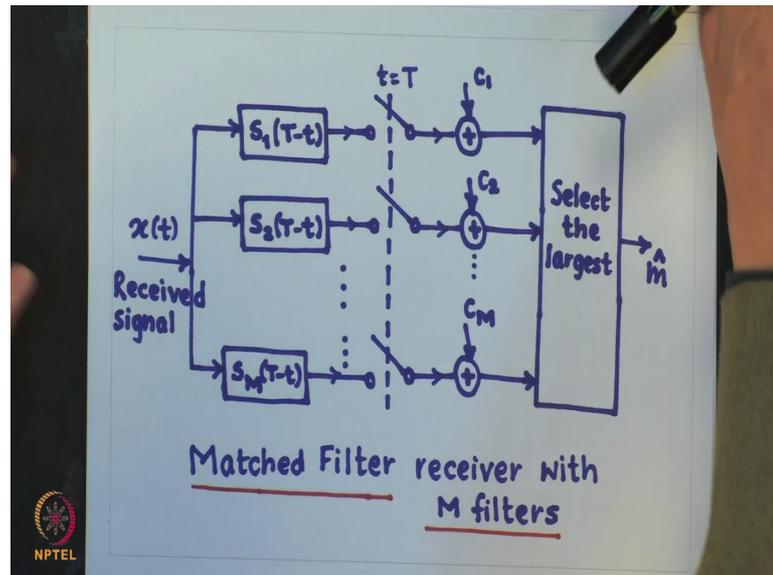
The image shows a whiteboard with handwritten mathematical equations. The first equation is $y(T) = \int_{-\infty}^{\infty} x(\tau)z(\tau)d\tau = xz$. Below it, the text $t=T$ is written. The second equation is $h(t) = 0$ for $t < 0$. In the bottom left corner of the whiteboard, there is a small circular logo with the text 'NPTEL' below it.

Now, if you evaluate this output $y(t)$ at t equal to capital T , then we get $y(T)$ is equal to $\int_{-\infty}^{\infty} x(\tau)z(\tau)d\tau$, which is xz which we had defined earlier.

So, in other words the output of the correlator xz can be obtained by sampling the output of the matched filter at time t equal to capital T . Note that this sampling has to be done, exactly at t equal to capital T , where this capital T is the arbitrary value used in the design of the matched filter. Now, as long as this condition is satisfied the choice of capital T is irrelevant however, from a practical point of view T has to be selected in such a way that the resulting filters are causal. So, what we mean that we must have $h(t)$ should be equal to 0 for t less than 0.

This puts a practical limit on possible values of capital T . And so, using this concept a matched filter implementation of the optimal receiver, which we had studied earlier let us see that how does it look. So, if we use for $z(t)$ to be equal to $S^*(t)$ correct then this is the matched filter receiver which I get.

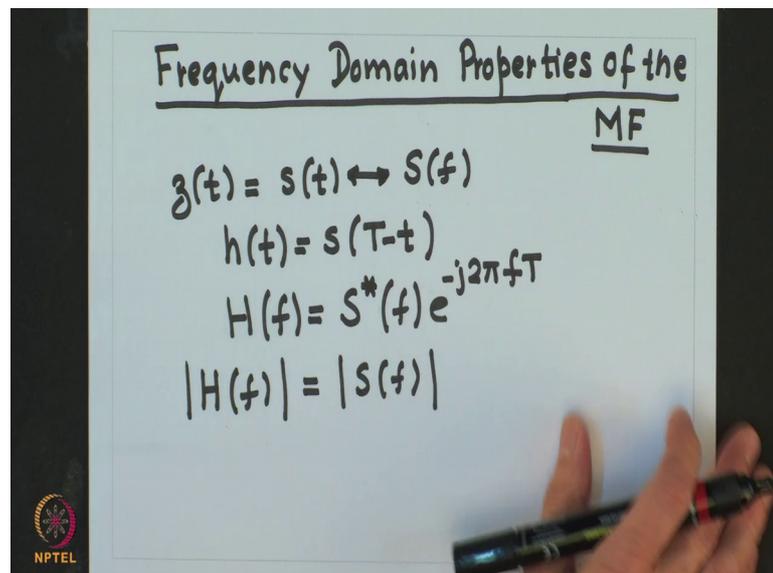
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And this is equivalent to the correlation receiver with M correlators correct. So, this filter can be implemented using the matched filter. So, in place of these 2 blocks they have been substituted by just one single filters correct.

So, similarly we could implement the matched filter for this case. So, we can substitute these two blocks by its matched filter. So, $z T$ gets substituted by $\phi_j t$. And if you do that we get this block correct it is important. Now, these are known as matched filter with N filters.

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Because corresponding to N basis signals and this have to be sampled at exactly t equal to capital T , if we do that kind of a sampling, then the output from all the four receivers which we have seen will be same fine. Now, let us look at some of the properties of this matched filters so, we look at the frequency domain properties of the matched filter.

So, in our case what we will do is we will choose $z t$ to be $S t$. So, you are matched filter now, $h t$ is going to be $S t$ minus capital T minus small t your $S t$ could be one of your message signal. So, it could be either $S_1 t$, $S_2 t$ or up to $S_m t$ any one of this ok. So, now, taking take the Fourier transform of this, if you take the Fourier transform we get this equal to so, the matched filter has a frequency response that is the complex conjugate of the transmitted signal spectrum. To this has spectrum which is given by $S f$ correct. So, this matched filter frequency response is the complex conjugated of the transmitted signal spectrum multiplied by the face factor e raise to minus $j 2 \pi f T$, which represents the sampling delay of capital T .

So, in other words the magnitude response of the matched filter is identical to the magnitude response of the signal $S t$ and the phase of $H f$ is the negative of the phase of $S f$ shifted by $2 \pi f T$ ok. Another interesting property of the matched filter is the signal to noise maximizing property.

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$$x(t) = s(t) + n(t)$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

$$= \underbrace{s(t) * h(t)}_{y_s(t)} + \underbrace{n(t) * h(t)}_{v(t)}$$

$$H(f)S(f)$$

So, let us assume that we have a signal $x t$ which is equal to desired signal $S t$ plus noise $n t$ and this signal $x t$ is passed through a filter with impulse response $h t$ to get the output

$y(t)$. So, your $y(t)$ will be equal to $x(t)$ convolve with $h(t)$ this will give me $S(t)$ convolved with $h(t)$ plus noise convolve with $h(t)$. Now, I call this component of the due to the signal as $y_S(t)$ in the output. And this is the component due to the noise. So, let me call it as $n(t)$, we assume that this noise $n(t)$ is a 0 mean Gaussian process ok.

So, this your $n(t)$ is also going to be Gaussian process, we also assume that this is a white noise. So, then; obviously, this will not be a white noise, but still it will be a Gaussian, it will satisfy the Gaussian characteristic ok. So, now, the Fourier transform of your output component corresponding to the signal basically is equal to $H(f)$ multiplied by $S(f)$. So, this is the output due to the signal part and noises $n(t)$. So, if you want to compute the output due to the signal part, we could use the inverse Fourier transform of this and if we do that we would get it as follows.

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$$y_s(t) = \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi ft} df$$

$$y_s(T) = \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT} df$$

$$\text{Var}[v(\tau)] = \int_{-\infty}^{\infty} \frac{N}{2} |H(f)|^2 df = \frac{N}{2} E_h$$

And if the signal is sampled at T equal to capital T , then I would get the output to be equal to this expression. And as far as the noise is concerned, the output noise will have 0 mean because, we are assuming that the input noise has 0 mean and, it will have variance which will be given by this expression.

So, variance of my noise is going to be power spectral density is going to be squared, multiplied by input power spectral density and I have to integrate this over the frequency range. So, this is the value which I will get this is equal to nothing, but italic N by 2, I am assuming white noise and E_h , where E_h is the energy in the impulse response $h(t)$.

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The image shows a hand holding a piece of paper with handwritten mathematical equations. The equations are as follows:

$$(SNR)_o \triangleq \frac{y_s^2(\tau)}{\text{Var}[y(\tau)]}$$
$$y_s^2(\tau) = \left| \int_{-\infty}^{\infty} H(f) S(f) e^{j2\pi f\tau} df \right|^2$$
$$\leq \int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} |S(f) e^{j2\pi f\tau}|^2 df$$
$$= \epsilon_h \epsilon_s \quad H(f) = \alpha S^*(f) e^{-j2\pi f\tau}$$

An NPTEL logo is visible in the bottom left corner of the paper.

So, now with this let us define the signal to noise ratio at the output of the filter $H(f)$ to be this quantity, output signal squared divided by the variance of the noise that is $\epsilon_h \epsilon_s$.

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The image shows a hand holding a piece of paper with handwritten text and an equation. The text and equation are as follows:

Cauchy-Schwarz inequality for finite energy, complex-valued functions $X(\omega)$ and $Y(\omega)$ is given by

$$\left| \int_{-\infty}^{\infty} X(\omega) Y(\omega) d\omega \right|^2 \leq \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega$$

with equality only if $Y(\omega) = c X^*(\omega)$ where c is an arbitrary constant

An NPTEL logo is visible in the bottom left corner of the paper.

Now, we know from the Cauchy Schwarz inequality for finite energy complex valued function $X(\omega)$ and $Y(\omega)$ is given by this expression correct. And this is equal to only if $Y(\omega)$ is some constant arbitrary constant multiplied by $X^*(\omega)$. So, if you use this relationship.

Then, we can write this $|y_S|^2$ which is equal to $|H(f) S(f) E|^2$ raised to $j 2\pi f T$ squared. Now, this will be always less than or equal to this quantity by Cauchy Schwarz inequality. So, this is equal to energy impulse response $H(f)$. This is the energy in $S(f)$ and this will be equal only if $H(f)$ is equal to of the form some constant α multiplied by x^* conjugate $e^{-j 2\pi f T}$.

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The image shows a whiteboard with the following handwritten equations:

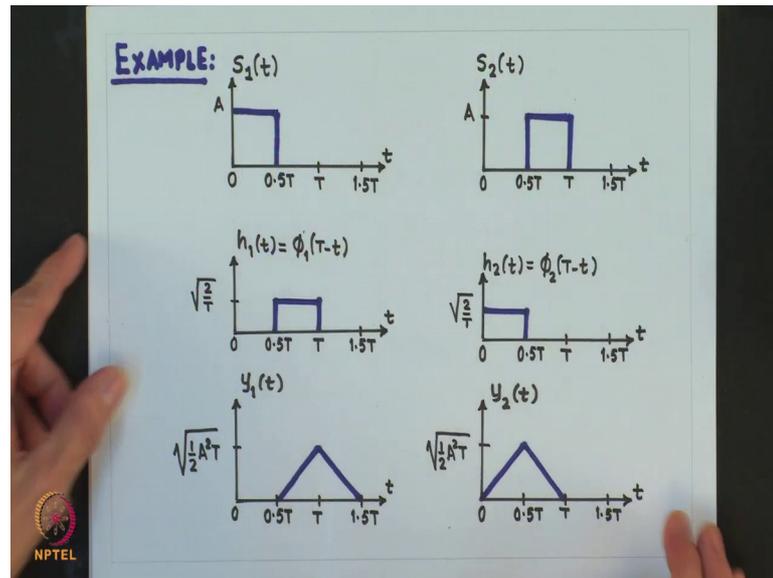
$$(SNR)_0 = \frac{|y_S|^2(T)}{\text{Var}[y(T)]}$$

$$\leq \frac{E_s E_h}{\frac{N}{2} E_h} = \frac{2 E_s}{N}$$

A hand is visible on the left side of the whiteboard, and an NPTEL logo is in the bottom left corner.

So, from this we get signal to noise ratio at the output of the matched filter, which is equal to this quantity always less than equal to so, this comes out to be twice E_s by N ok. So, what this shows that the filter $H(f)$ that maximize the signal to noise ratio at a output, must satisfy the relationship $H(f)$ is equal to this correct. So, we could choose α equal to 1 correct. So, and the maximum value which we can get is $2 E_s$ by N correct ok. So, quickly to appreciate what we have done let us take one example.

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So let us say I have 2 signals $S_1(t)$ and $S_2(t)$ correct. Now, looking at this signal it is very easy to see what are the orthogonal basis signal to represent these 2 signals, there will be similar signals except for the normalizing factor.

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The figure shows a hand-drawn diagram on a whiteboard with the word "EXAMPLE:" written at the top left. It contains two equations for orthogonal basis functions. The first equation is $\phi_1(t) = \begin{cases} \sqrt{\frac{2}{T}} & 0 \leq t \leq 0.5T \\ 0 & \text{otherwise} \end{cases}$. The second equation is $\phi_2(t) = \begin{cases} \sqrt{\frac{2}{T}} & 0.5T \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$. A hand is pointing to the equations with a black marker. An NPTEL logo is visible in the bottom left corner of the whiteboard.

So, what I will get is here, $\phi_1(t)$ will be equal to this signal and $\phi_2(t)$ will be equal to this signal, which is similar to this signal except for the heights fine; having done this, if you were required to implement this using the matched filter implementation, given by

this block diagram. In our case here, now N will be equal to 2 so I will get x_1 and x_2 . So, I need to compute $\phi_1(\tau - t)$ and $\phi_2(\tau - t)$.

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$$h_1(t) = \phi_1(\tau - t) = \begin{cases} \sqrt{\frac{2}{T}} & 0.5T \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

$$h_2(t) = \phi_2(\tau - t) = \begin{cases} \sqrt{\frac{2}{T}} & 0 \leq t \leq 0.5T \\ 0 & \text{otherwise} \end{cases}$$

So, if you do this basically; so, by this is this and from here, I can get my $h_1(t)$ simple is this. And looking at this figure also it is very simple look here my $\phi_1(t)$ and $\phi_2(t)$ are this, $\phi_1(\tau - t)$ means, I first reflect by ϕ_1 about y axis that means, I take the mirror image and then shift that by capital T to get my $h_1(t)$.

So, here I take the mirror image and shifted by capital T if I do it, basically I will get this quantity that is my $h_1(t)$. And similarly here I reflecting about this point and, then shifted by capital T I will get this for my $h_2(t)$ correct. So, I have my 2 filters. And now we need to compute x_1 and x_2 . So, let me assume for this case that I have transmitted my signal $s(t)$.

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The image shows a whiteboard with handwritten mathematical expressions. At the top, it says 'S₁(t) is transmitted'. Below that are two equations: 'y₁(t) = S₁(t) * h₁(t)' and 'y₂(t) = S₁(t) * h₂(t)'. The next line states 'y₁(t) and y₂(t) are sampled at t=T'. This is followed by two equations for sampling: 'y_{1s}(T) = y₁(t)|_{t=T}' and 'y_{2s}(T) = y₂(t)|_{t=T}'. The final line shows the results: 'y_{1s}(T) = √(1/2) A T = ε_s' and 'y_{2s}(T) = 0'. An NPTEL logo is visible in the bottom left corner of the whiteboard.

$$S_1(t) \text{ is transmitted}$$
$$y_1(t) = S_1(t) * h_1(t)$$
$$y_2(t) = S_1(t) * h_2(t)$$

$y_1(t)$ and $y_2(t)$ are sampled at $t=T$

$$y_{1s}(T) = y_1(t)|_{t=T} \quad y_{2s}(T) = y_2(t)|_{t=T}$$
$$y_{1s}(T) = \sqrt{\frac{1}{2}} A T = \epsilon_s \quad y_{2s}(T) = 0$$

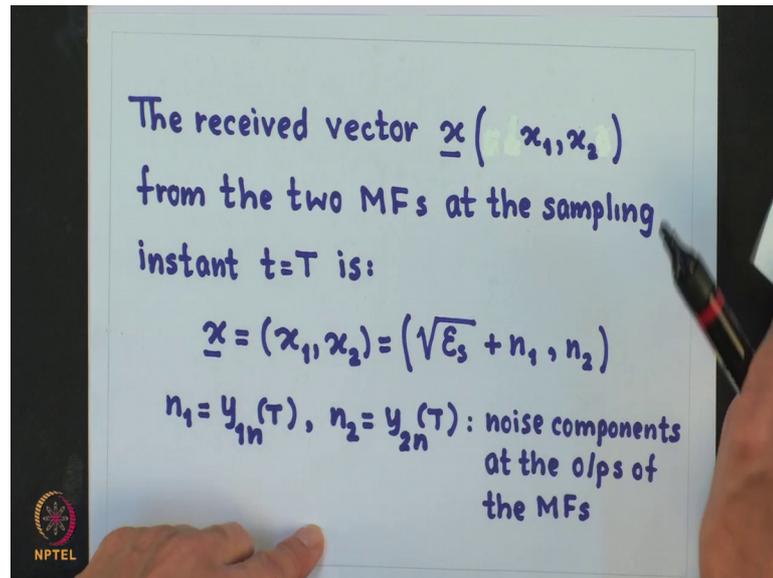
So, if I transmit my signal $S_1(t)$ the outputs from the 2 filters are going to be $S_1(t)$ multiplied by $h_1(t)$. The other matched filter output would be $h_1(t) S_1(t)$ multiplied by $h_2(t)$ at the receiver basically we receive along with this signal $S_1(t)$ the noise also correct.

So, we look at a noise component later on, we are just looking at what happens at the output with the signal components. Let me denote $y_1(t)$ and $y_2(t)$ at the outputs this. And sample them at t equal to capital T , these are the values which I sample them. Now it is very simple to see that that this is going to be output at capital T and this is going to be output from the other filter at capital T .

The graphically the outputs have been shown here if you have to convolve this with this. So, if you convolve with this with this you will start getting output only from $0.5 T$ capital T , it will go to the maximum value and decrease, same input has to be applied to this filter $h_2(t)$, if you do this basically you will start getting the output like this.

Now, you sample this both this output at t equal to capital T . If I do this basically so, from the first channel, I will get this value and from the second channel I will get this value, corresponding to the signal component of the output fine. So, your received vector your what we are saying is this vector x_1, x_2 that will be your receive vector.

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So, that is simple I denote it $x_1 \times x_2$. So, at t will be equal to so, the first one I will have the signal component. And this value you see here basically when this and this coincides, the maximum value here is going to be given by what I have written out here. It is A multiplied by root of 2 by T , which is equivalent to root of half $A^2 T$ correct fine ok.

So, I will get this is equal to the signal energy, this is exactly equal to signal energy in this $A^2 T$ square root of that sorry, this will be the square root of the signal energy at this point ok. So, I get the square root of the signal energy plus, there will be a noise component. And the second x_2 , as far as the signal is concerned it is 0 because, we have sample at t equal to capital T . So, there is only noise component. So, I call it is n_2 so, this is the noise component $y_{1n}(T)$, this $y_{2n}(T)$ into noise component ok. And what are these noise components, how do I get them simple my $n(t)$ is getting projected on the basis signals. You have 2 basis signal ϕ_1 and ϕ_2 .

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$$y_{kn}(T) = \int_0^T n(t)\phi_k(t)dt, \quad k=1,2$$
$$\text{Clearly, } E[n_R] = E[y_{kn}(T)] = 0$$
$$\text{Var}[n_R] = \frac{N}{2} \int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N}{2}$$
$$\text{(SNR)}_0 \text{ for the first MF: } (\text{SNR})_0 = \frac{(\sqrt{E_s})^2}{\frac{N}{2}} = \frac{2E_s}{N}$$

So, I will get y_1 sampled at t equal to T and y_2 sampled at t equal to T by evaluating this. Now obviously, it is very simple to see here because, we are assuming to be 0 mean noise process, this is 0 variance. Again it is going to be the integration of the power spectral density. Output power spectral density is this squared $H(f)$ filter squared multiplied by input power spectral density we are assuming it to be white noise. So, it is N by 2. And then you have to integrate that over the all frequencies., if you do this basically this is equal to N by 2, remember this quantity out here is the same as because, we are assuming this filter $H(f)$ is equal to this filter or this filter.

So, this is the energy basically in the filter, you could use directly in the frequency domain, or you could compute in the time domain. We know that energy of the orthonormal signals are going to be 1 and that is how you get this to be equal to 1, this is N by 2. And so, if you calculate the signal to noise ratio at the output for the first matched filter, you will get the signal component which is equal to $\sqrt{E_s}$ squared, that is same as E_s and the noise variance which is equal to this. So, if you take the ratio I get this quantity, which is the same quantity which we derived for the matched filter.

So, we have said that the maximum value of the output of the matched filter, as for the signal to noise ratio is concerned is twice multiplied by the energy of the signal divided by N . So, this agrees with what we have derived earlier for the maximum value of

signal to noise ratio, at the output of the matched filter when sampled at t equal to capital T ok.

Now, in our study of M-ary communication, we have design the optimal receiver. Now what is required is basically we are supposed to also analyse the performance of this receiver, in terms of error probability. And to do that basically we require to understand the concept of decision regions in the signal space. And then using that concept we will have to evaluate the probability of error of a particular receiver. And this will do next.

Thank you.