

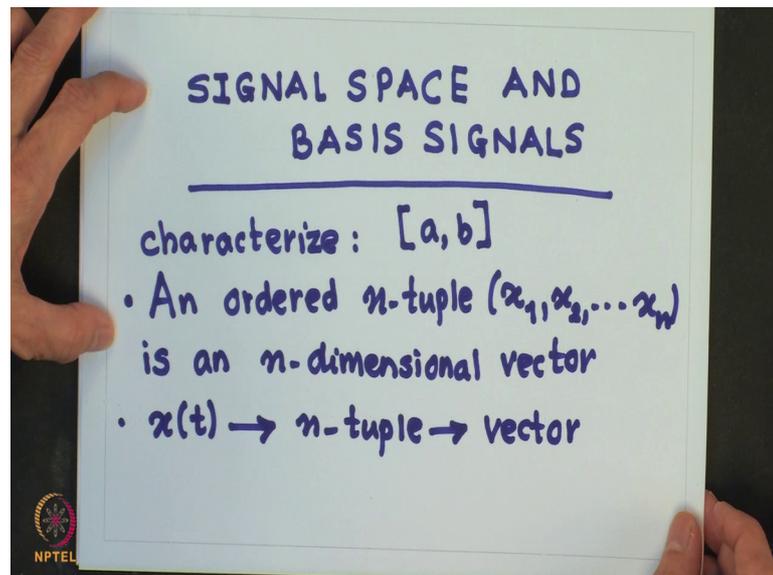
Principles of Digital Communications
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Lecture – 15
Signal Space Representation – I

Over next couple of classes, we will study Signal Space Representations of signals. The motivation for this is to facilitate the analysis of digital signal detection and determination of the optimum receiver.

Signal space representations of the signals is a very useful and effective tool in the analysis of digitally modulated signals. We will study this important approach and show that a set of signals can always be represented by a set of vectors, let us begin with signal space and basis signals.

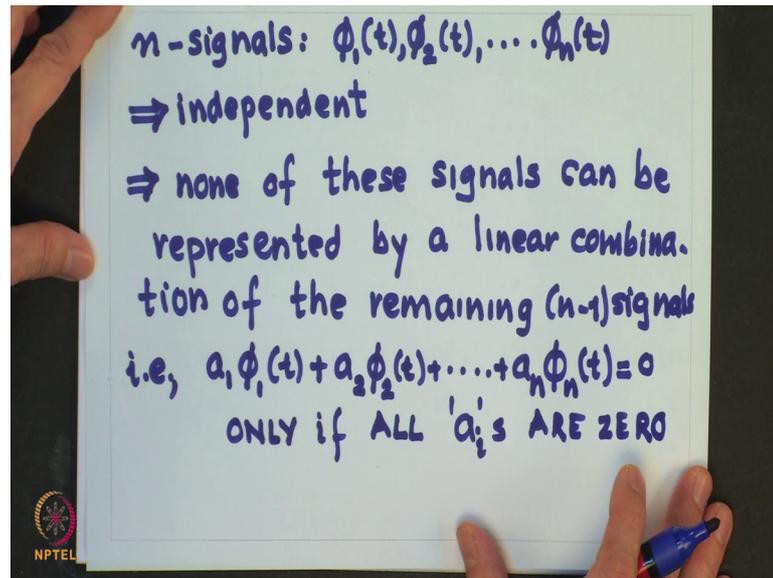
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So, assuming that we are familiar with the vectors and vector spaces; the concept of vector spaces and basis vector will be extended to characterize; continuous time signals defined on some interval say a, b .

Now an ordered n -tuple specified by x_1, x_2 up to x_n is an n -dimensional vector. So, if a signal $x(t)$ can be specified by n -tuple; then it too will be a vector correct. This is basically the principle of representing a signal in terms of vector.

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So, let us assume that we have n-signals $\phi_1(t), \phi_2(t)$ up to $\phi_n(t)$. Now these signals are independent, then it implies that none of these signals can be represented by a linear combination of the remaining n minus 1 signals.

What I mean is that is a_1 times $\phi_1(t)$ plus a_2 times $\phi_2(t)$ plus summation plus a_n times $\phi_n(t)$. If you take the linear combination of all these signals then this will be equal to 0 only if all a_i 's are 0; only if this condition is satisfied then this set from ϕ_1 to ϕ_n will be independent.

So, let us say that we have a signal $x(t)$ and this signal $x(t)$ can be represented by a linear combination of n independent signals.

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$$x(t) = x_1 \phi_1(t) + x_2 \phi_2(t) + \dots + x_n \phi_n(t)$$
$$\{\phi_n(t)\} = \sum_{k=1}^n x_k \phi_k(t)$$

n -dimensional signal space.
 $\{\phi_n(t)\} \rightarrow$ BASIS SIGNALS

This $\phi_n(t)$ will denote this set of n independent signal by this. So, if I can represent this $x(t)$ by linear combination; it means I can write like this, this is some coefficients x_1 ; $\phi_1(t)$, $x_2 \phi_2(t)$ plus $x_n \phi_n(t)$. So, this is your time domain signal and these are some coefficient x_1, x_2, x_n corresponding to this $x(t)$ ok; so, this is equal to.

Now, every single in a certain signal space can be represented by a linear combination of the independent signals given by this set $\phi_n(t)$, then we have what is known as n -dimensional signal space. And this set $\phi_n(t)$ is basically known as basis signals.

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w.l.o.g. $[a, b] \equiv [-\infty, \infty]$

Assumption:

$$\int_{-\infty}^{\infty} \phi_j(t) \phi_k(t) dt = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

$\Rightarrow \{\phi_n(t)\} \equiv$ ORTHONORMAL

Without loss of generality, let us assume this domain a, b to be from minus infinity to plus infinity. And let us assume some constraint on this basis signal as follows; we have $\phi_j(t); \phi_k(t)$ integral of this product equal to 0, if j is not equal to k and is equal to 1 if j is equal to k . For sake of simplicity, we are considering only real signals the same ideas could be extended to complex and signals with little modification, but for this course it is not necessary at this moment ok.

So, if this condition is satisfied then what we say that this set which we have of basis signals is basically orthonormal; if this was not equal to 1; then it would have been known as orthogonal fine. So, for any orthonormal set let us try to determine the coefficients of the linear expansions of a given signal in terms of orthonormal basis signal.

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The image shows a hand holding a whiteboard with the following handwritten mathematical derivations:

$$x(t) = \sum_{k=1}^n x_k \phi_k(t)$$

$$\int_{-\infty}^{\infty} x(t) \phi_j(t) dt = \int_{-\infty}^{\infty} \left\{ \sum_{k=1}^n x_k \phi_k(t) \right\} \phi_j(t) dt$$

$$x_j = \int_{-\infty}^{\infty} x(t) \phi_j(t) dt$$

(x_1, x_2, \dots, x_n)

So, if we know that I can write my signal in this form and if we assume that my $\phi_k(t)$'s are all in an orthonormal set; then to determine x_k it is very simple what you have to do is basically multiply both the sides by say $\phi_n(t)$ and integrate over the domain minus infinity to plus infinity. So, if I do this on the right hand side also, I would get this as follows.

So, now we know that $\phi_k(t)$ and $\phi_n(t)$ are orthogonal except when k is equal to n correct. So, do not get confused with this n and this n ; this is maybe we could change it let us change it to say j correct. So, only when k is equal to j ; this integral of $\phi_k(t) \phi_j(t)$

dt will be equal to 1 correct. So, from this what we will get is that x_j is equal to $\int x(t) \phi_j(t) dt$ correct fine.

So, once this orthonormal basis signal set is specified; $x(t)$ can be represented by this n tuples correct. So, I can get my x_1, x_2 up to x_n correct. So, what it means that alternatively $x(t)$ can be geometrically represented by a point in the space x_1, x_2 up to x_n and this is your n dimensional vector.

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$$\underline{x} = (x_1, x_2, \dots, x_n) \rightarrow n\text{-dimensional vector}$$
$$\underline{x} \rightarrow x(t)$$
$$\phi_1(t) \rightarrow (1, 0, 0, \dots, 0)$$
$$\phi_2(t) \rightarrow (0, 1, 0, \dots, 0)$$
$$\vdots$$
$$\phi_n(t) \rightarrow (0, 0, \dots, 1)$$

So, what it means that I can associate a vector X with the signal $x(t)$. This means that the basis signals x_1 ; sorry the this means that the basis signal $\phi_1(t)$ can be represented by a corresponding basis vector as 1, 0, 0, 0 and $\phi_2(t)$ can be represented as 0, 1, 0, correct and $\phi_n(t)$ would be represented as 0, 0, 1 correct ok fine.

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$$x(t) \rightarrow \text{BL to } B \text{ Hz}$$
$$x(t) = \sum_k x\left(\frac{k}{2B}\right) \text{sinc}(2Bt - k)$$
$$\text{sinc}(x) \triangleq \frac{\sin \pi x}{\pi x}$$
$$\left\{ \text{sinc}(2Bt - k) \right\}$$

An example of this would be a signal $x(t)$ which is Band Limited to say B hertz correct. Now, this signal can be represented by the sampled version of $x(t)$ sampled at the rate of twice B , where B is the bandwidth of the signal. We know that by an Nyquist sampling theorem; I can write this as where your sinc x is defined by $\sin \pi x$ upon πx correct. And it is not very difficult to show that this set is basically an orthogonal set correct.

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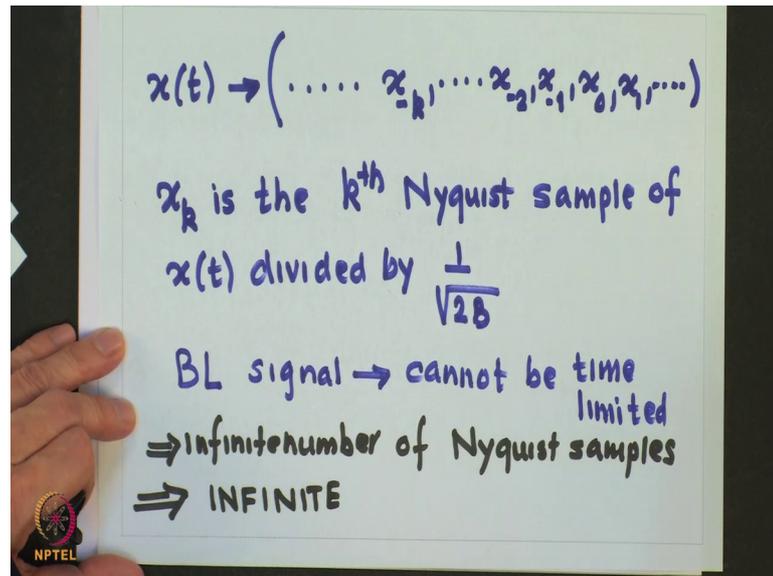
$$\int_{-\infty}^{\infty} \text{sinc}(2Bt - j) \text{sinc}(2Bt - k) dt$$
$$= 0 \quad j \neq k$$
$$= \frac{1}{2B} \quad j = k$$
$$\left\{ \sqrt{2B} \text{sinc}(2Bt - k) \right\}$$
$$x(t) = \sum_k x_k \phi_k(t) \quad x_k = \frac{1}{\sqrt{2B}} x\left(\frac{k}{2B}\right)$$

So, what it means basically is that this is equal to 0, if j is not equal to k and is equal to 1 by $2B$; if j is equal to k . So, in this case this forms an orthogonal set and if you want to

get an orthonormal set from this, we can just normalize by the square root of $2B$ and then this will form an orthonormal set correct.

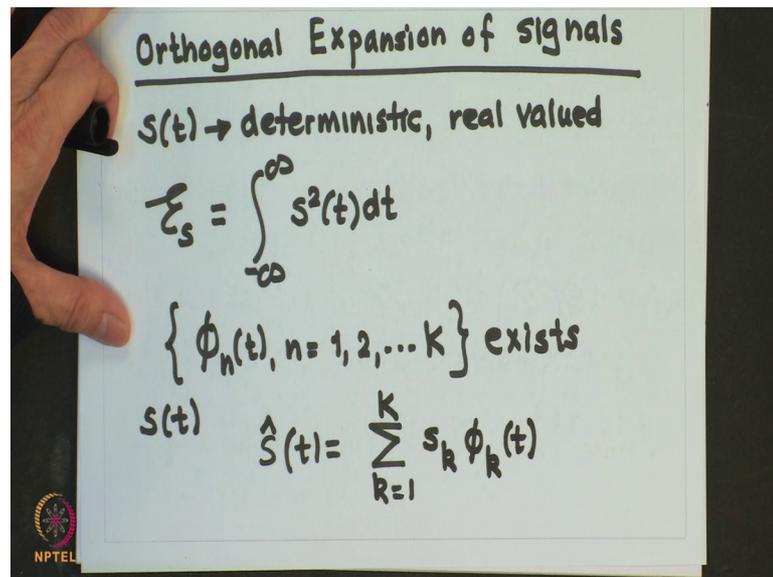
So, I can write now my signal $x(t)$ is equal to where my x_k is equal to $\frac{1}{\sqrt{2B}}$ of these are sample version sample at the rate $2B$.

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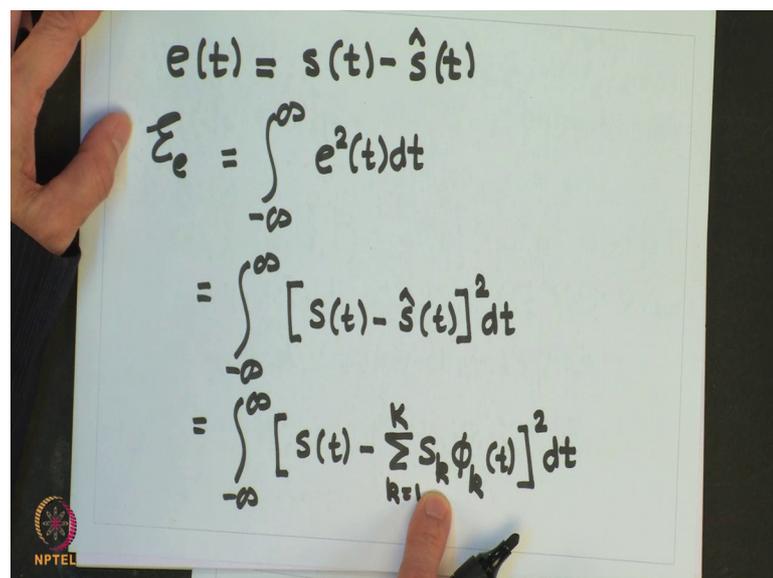
So, any band limited signal $x(t)$ can be given in the form of vector as correct, where your x_k is the k^{th} Nyquist sample of $x(t)$ divided by $\frac{1}{\sqrt{2B}}$. Now, a band limited signal cannot be time limited; therefore, this implies that you will have infinite number of Nyquist samples correct. So, this implies that the dimensionality of this vector space is going to be infinite because this x_k ; k will go from minus infinity to plus infinity.

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So, now let us study another important concept what is known as Orthogonal Expansion of signals ok. So, suppose that we have a signal $S(t)$ which is a deterministic and real valued and it is a finite energy signal. So, what it means that if I calculate this energy, this is finite I can evaluate this integral. Now assume that we have a set of orthonormal functions correct and now my problem is to approximate the original signal $S(t)$ correct; by a weighted linear combinations of this function. So, I want to write $\hat{s}(t)$ is equal to like this. Now, in doing; so, there will be some kind of an approximation error.

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So, let me denote that approximation error as $e(t)$; that would be equal to $s(t) - \hat{s}(t)$. So, now, my problem is to select the coefficient S_k in the representation of the signal $s(t)$ in terms of orthonormal set, in such a way that the energy in this error gets minimized correct.

So, the energy in this error is equal to correct. So, what this implies is I want to choose my coefficients S_k in such a way that this quantity gets minimized, I can rewrite this as. Now, this can be minimized by taking the derivative of this energy with respect to each of this coefficient S_k .

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Minimization of \mathcal{E}_e w.r.t. $\{S_k\}$

$$\Rightarrow \frac{\partial \mathcal{E}_e}{\partial S_k} = 0 \quad \text{for } k=1,2,\dots,K$$

$$\frac{\partial \mathcal{E}_e}{\partial S_j} = \int_{-\infty}^{\infty} 2 \left[s(t) - \sum_{R=1}^K S_R \phi_R(t) \right] [-\phi_j(t)] dt = 0$$

$$S_j = \int_{-\infty}^{\infty} s(t) \phi_j(t) dt \quad j=1,2,\dots,K.$$

So, let us try to do that. So, my problem is to minimization of this error energy with respect to coefficient S_k . So, what I have to do is take the first derivative of this energy with respect to this coefficient S_k and equate to 0; this I have to do for k equal to 1, 2 up to capital K correct.

So, this you can easily take the derivative of this take the derivative of this correct with some respect to some. So, this will S_k will give you I will get a factor of 2 $s(t) - \hat{s}(t)$ correct. And then partial derivative and then I have to take the derivative of this term inside, this will go to 0 and here basically only S_k will come out; so, derivative respect to S_k .

So, this will be minus phi; so, what you will do? Let us take the derivative with otherwise there is a little confusion. So, let us take the derivative respect to S_j correct; so, if we do this basically what I get is $S_j \int \phi_j(t) dt$ is equal to 0 for j is equal to 1, 2 up to capital K ; what we will get from here is S_j is equal to $S \int \phi_j(t) dt$; remember this $\phi_n(t)$ is an orthonormal set ok. So, I will get this ok.

So, what this means basically that the coefficients are obtained by projecting the signal $S(t)$ correct onto each of the functions $\phi_j(t)$.

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$\hat{s}(t) \rightarrow$ projection of $s(t)$
 K-dimensional space
 Spanned by the
 fns. $\{\phi_n(t)\}$
 $e(t) = s(t) - \hat{s}(t)$
 $\langle e(t), \hat{s}(t) \rangle = 0$
 $\equiv \int_{-\infty}^{\infty} e(t) \hat{s}(t) dt = 0$

So, consequently your approximated signal $S(t)$ is the projection of $S(t)$ onto the K -dimensional signal space spanned by the functions $\phi_n(t)$. So, basically it means that now your; this error is orthogonal to this space ok. So, your $e(t)$ which is $S(t)$ minus S hat t correct is orthogonal to approximated signal $S(t)$ correct; this mean this is equivalent to writing $e(t)$ is equal to 0 fine.

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$$\begin{aligned} E_{\min} &= \int_{-\infty}^{\infty} e^2(t) dt \\ &= \int_{-\infty}^{\infty} e(t)[s(t) - \hat{s}(t)] dt \\ &= \int_{-\infty}^{\infty} e(t)s(t) dt \end{aligned}$$

So, let us quickly calculate the minimum energy for this. So, E_{\min} from this will be equal to I have to evaluate this and this can be written as $\int e(t)s(t) dt$. Now we know that $e(t)$ and $\hat{s}(t)$ are orthogonal; so, this turns out to be this quantity.

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$$\begin{aligned} &= \int_{-\infty}^{\infty} [s(t) - \hat{s}(t)]s(t) dt \\ &= \int_{-\infty}^{\infty} s^2(t) dt - \int_{-\infty}^{\infty} \left[\sum_{k=1}^K s_k \phi_k(t) \right] s(t) dt \\ &= E_s - \sum_{k=1}^K s_k^2 \\ \text{When } E_{\min} = 0 &\rightarrow E_s = \sum_{k=1}^K s_k^2 \end{aligned}$$

So, this I can rewrite it as and this is equal to E_s energy in this minus now this can be easily shown to be equal to summation of these coefficients because $\phi_k(t)$ is orthonormal set. So, when E_{\min} is equal to 0, it implies that your E_s will be equal to this quantity.

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$$s(t) = \sum_{R=1}^K s_R \phi_R(t)$$
$$E_{\min} = 0$$
$$\{\phi_n(t)\} \rightarrow \text{"complete"}$$

So, when every finite energy signal can be represented by expansion of the form in the equation $S(t)$ is equal to this and E_{\min} for that is equal to 0; then the set of thus the signal $\phi_n(t)$ is said to be complete.

So, my next problem is that I want to obtain an orthonormal set. Set of signals or basis signals from a given set of signals correct. So, this problem we will discuss in the next class.

Thank you.