Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-7.
Lecture-18.2.
Time-Bandwidth product and its properties.

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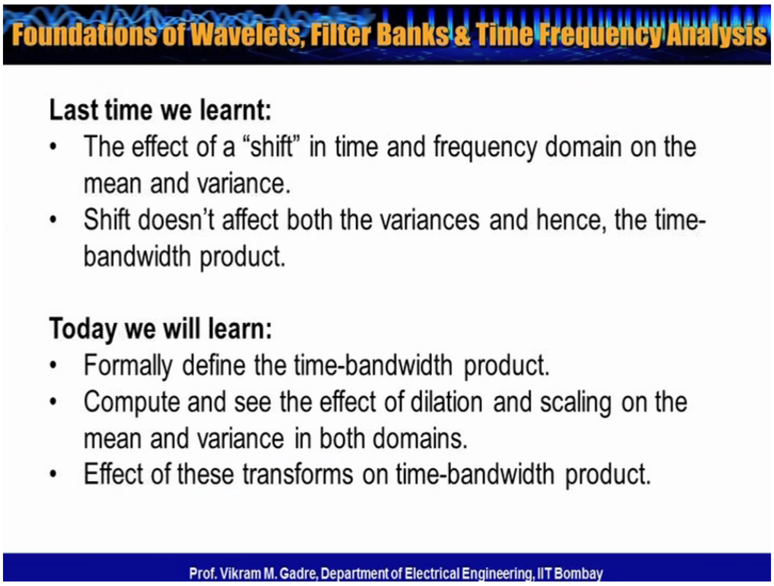

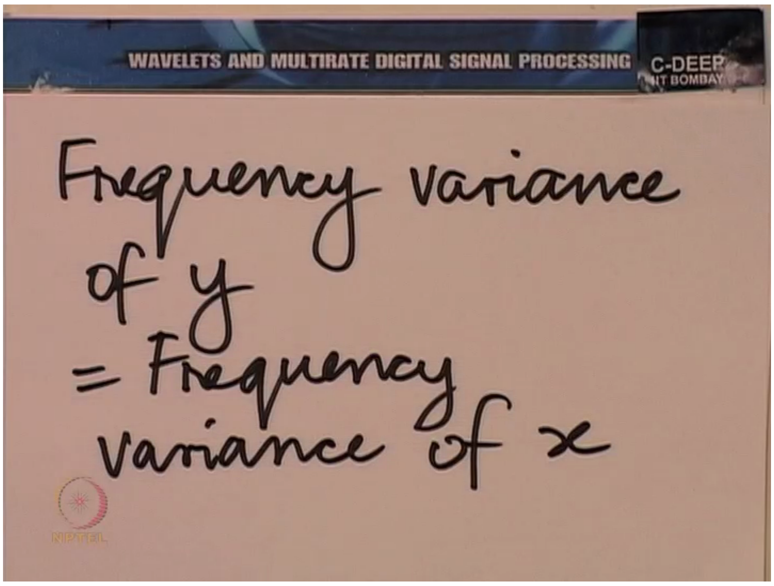

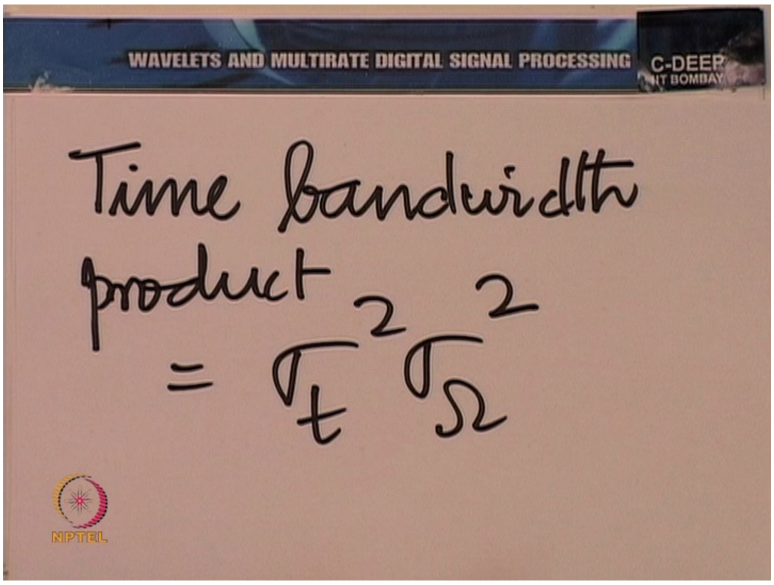

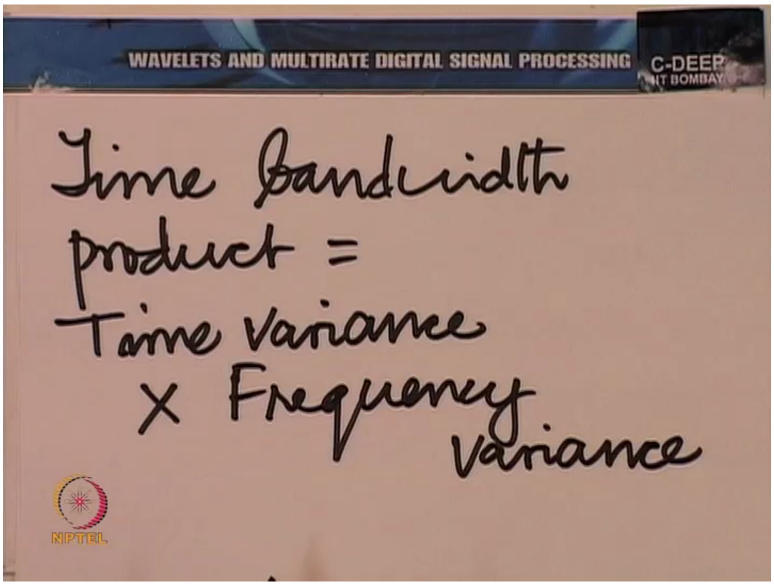

So far so good, so we have taken 2 dual operations, the timeshift and the frequency shift. And in both of them we have had no change in the variances. And obviously when there is no change in the variances, there is going to be no change in the product of the variances. So the Sigma t squared Sigma omega square product which we shall now give a name, we call it the time bandwidth product. So we introduce the term for time bandwidth product given by the product Sigma t squared times Sigma omega square.

Essentially the time bandwidth product is the product of the time variance and the frequency variance. And this product is very important from a number of perspectives. As you can see there is a strong invariance that this product exhibits. When you shift a function in time or when you shift it in frequency, in other words modulate it in time, this product is unchanged, let us make a note of that.

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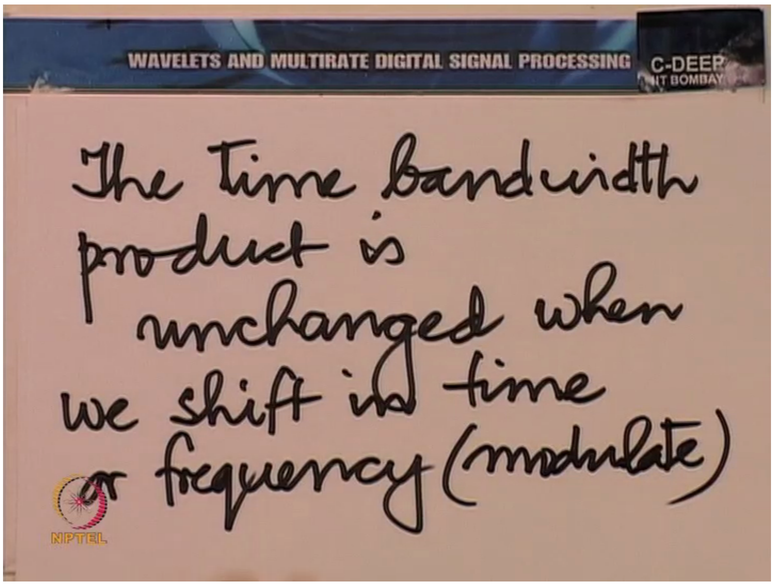

the time bandwidth product is unchanged when we shift in time or frequency. And remember shifting in frequency is modulation in time. Now we would like to look at this product in a little greater depth. What happens to this product when the stretch or compress? you see, in the wavelet transform or for that matter in all the discussions that we have had in some of the previous lectures, we have talked about moving along the frequency axis by a process of dilation.

We saw that when we stretched or compressed a function, a bandpass function, it was equivalent to moving that function on the frequency axis, moving in a constant quality factor fashion, keeping the ratio of the Centre frequency to the bandwidth a constant. Now in this process of compression or expansion together called dilation, what is happening to this fundamental quantity, the time bandwidth product is a question that you would now like to ask and answer.

So the 1st thing that we observe is that you know, if you look at the scaling, let us, let us look at that process, suppose you have this function, you multiply it by a constant, that is a trivial thing, we will settle that matter 1st. So we have, you see here we are talking about scaling the independent variable, when we stretch or compress, we are talking about scaling the independent variable. What happens when you scale the dependent variable, so if you multiply the function by a constant.

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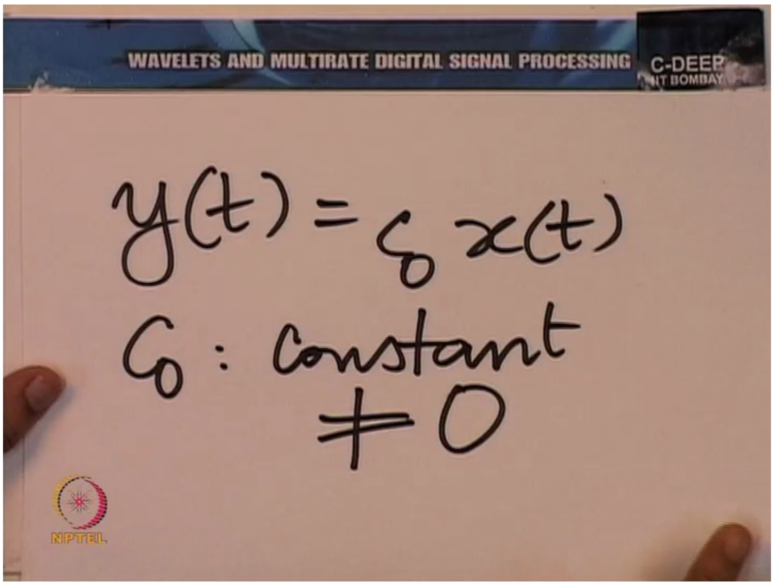

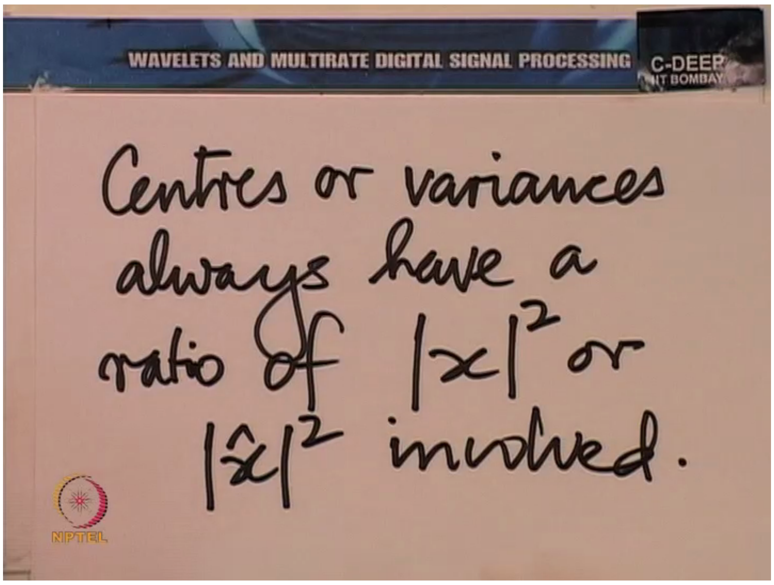

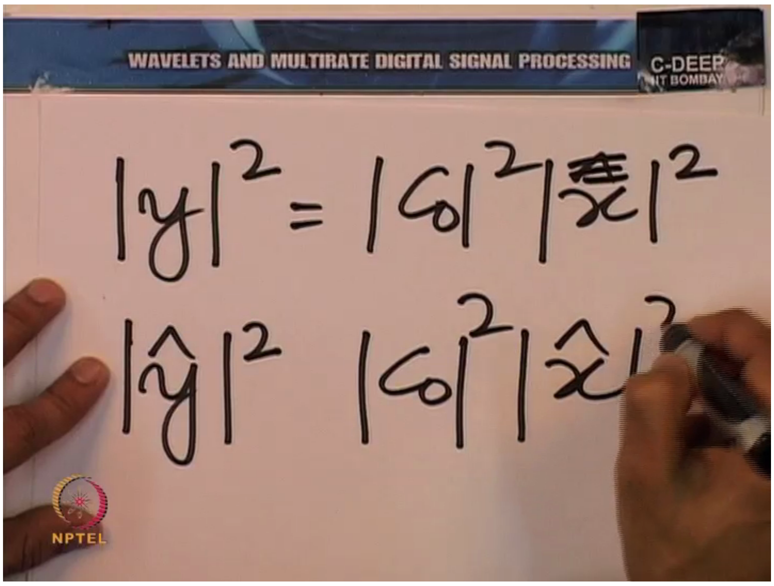

We must answer that question 1st. So, again it is very easy to show that if yt is equal to some constant, let us say C0 times xt, C0 is a constant, of course unequal to 0, that is straightforward. So if C0 is a constant, it is very easy to see that in the process of defining the Centre or the variance, the centres or variances always have a ratio of mod x squared or mod x cap squared involved. Now, you know if yt is C0 times xt mod y squared or for that matter mod y cap squared is going to be mod C0 squared times mod x square, I am sorry, here it is just x.

So here to, it is mod C0 squared mod x cap squared. Now, when we take a ratio, this mod C0 square being nonzero is going to cancel from the numerator and denominator and therefore when we scale the dependent variable, there is no effect on the time centre or the frequency Centre. Let us make a note of that.

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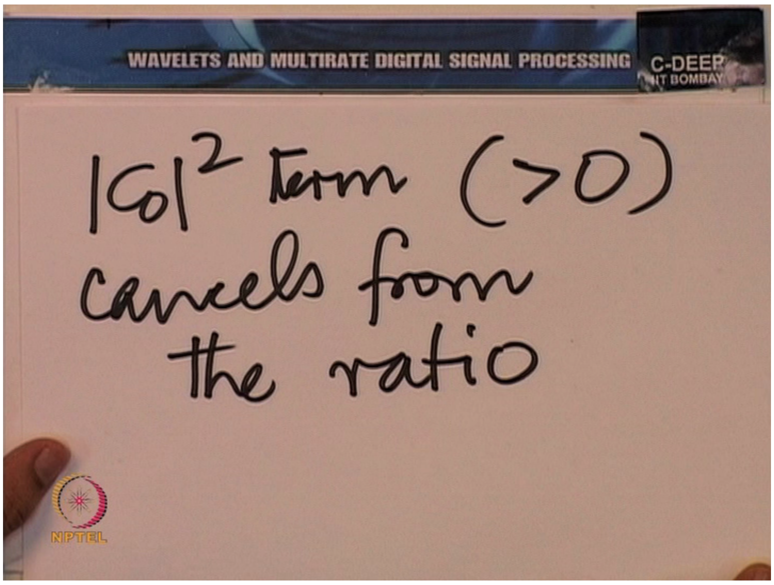

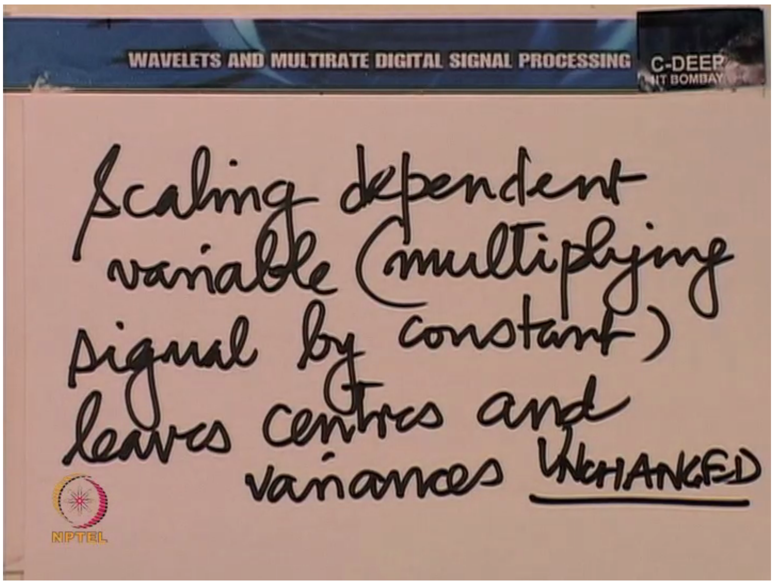

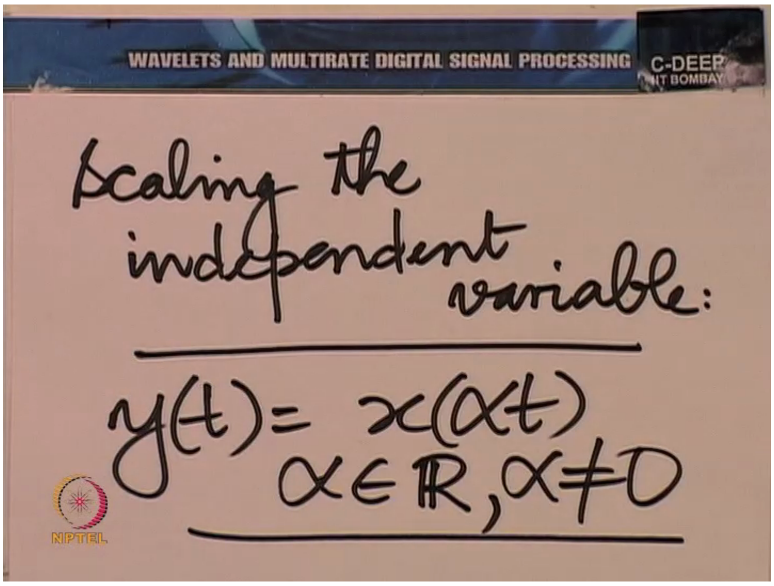

The mod C0 squared term of course greater than 0 strictly cancels from the ratio and therefore scaling the dependent variable or essentially multiplication by a constant, multiplying the signal by a constant leaves the centres and variances unchanged. This is important, scaling the dependent variable leaves the centres in the variances unchanged. And now the most difficult of them all and yet elegant and beautiful, what happens when we scale the independent variable?

In other words, let us consider yt equal to x of Alpha t and Alpha is a real number, Alpha not equal to 0. Now we know what happens to the Fourier transform in this. So you will recall that when we scale the t variable by Alpha, the Fourier variable capital omega is scaled by one by Alpha but there is also a 1 by mod Alpha factor that emerges outside.

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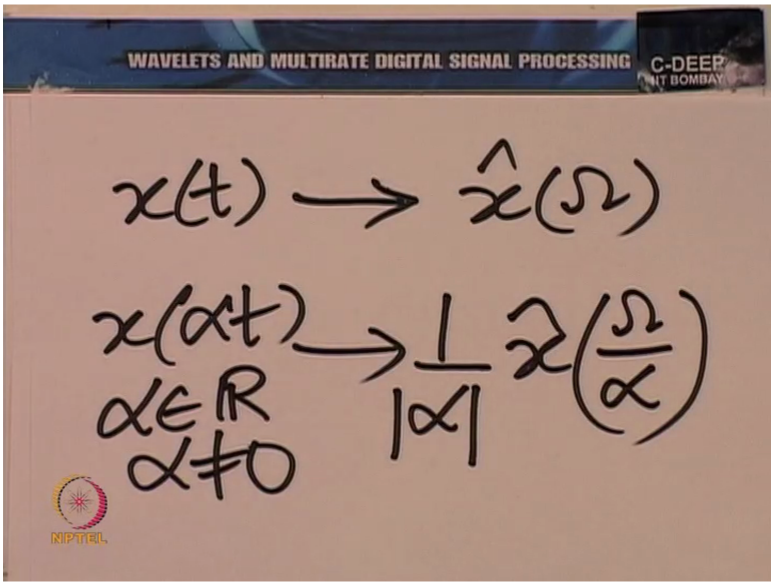

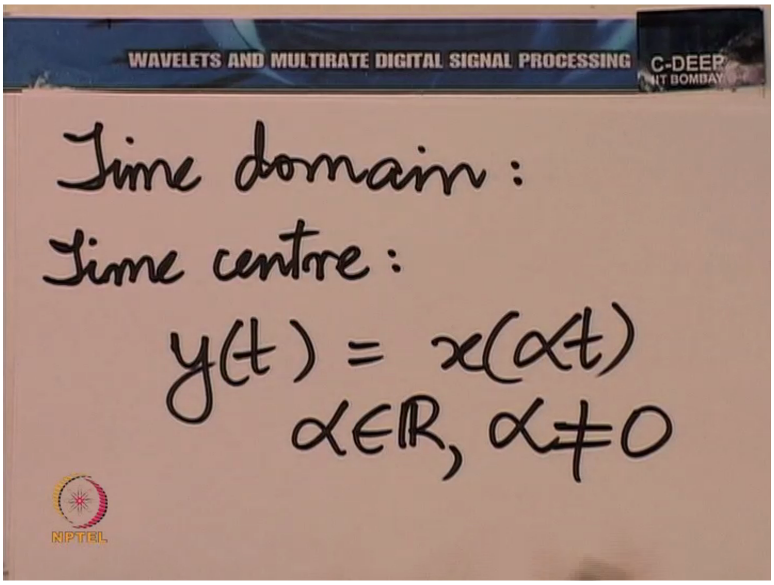

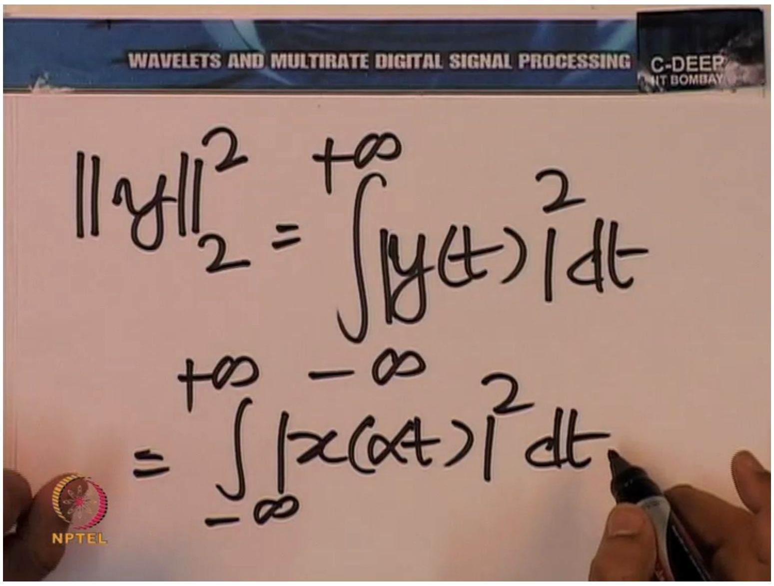

So what we are saying is if xt has the Fourier transform x cap omega, then x of Alpha t with Alpha, real number Alpha not equal to 0 has the Fourier transform 1 by mod Alpha x cap omega by Alpha, here it is Alpha, here it is mod Alpha. So, in fact it is adequate for us to see what happens to one of them, the time or the frequency domain, and the other one can be interpreted. Let us take the time domain for simplicity.

So, consider the time domain, let us see what happens to the time centre. Let yt be equal to x of alpha t, as usual alpha real, Alpha not equal to 0. And let us ask what is the norm of yt, Norm squared of yt in the L2R sense against the norm of x. So the L2 norm of y is this, the L2 norm square I mean. And of course this is easily seen to be mod x alpha t square dt. Now it is an easy integral to evaluate, all that we need to do is to make a transformation of variable.

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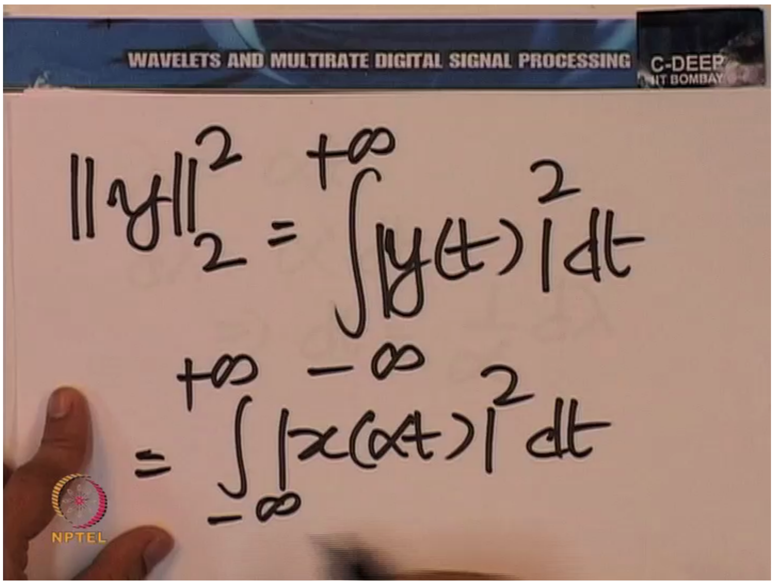

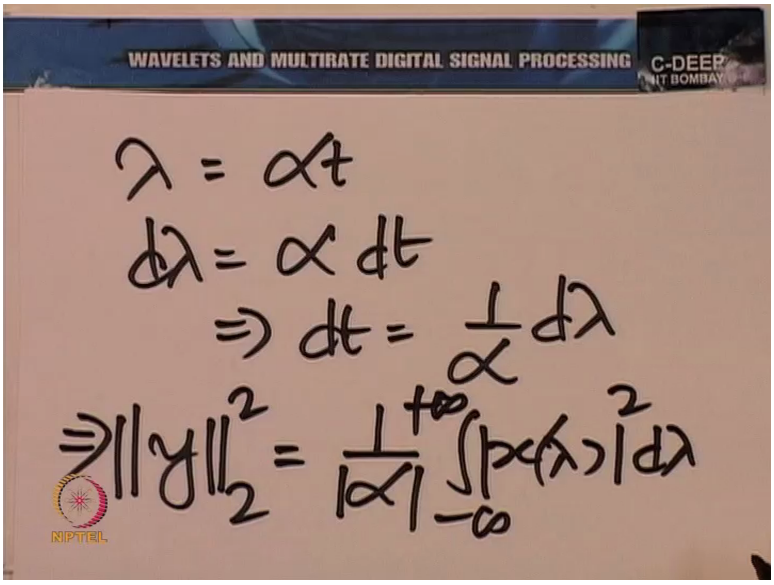

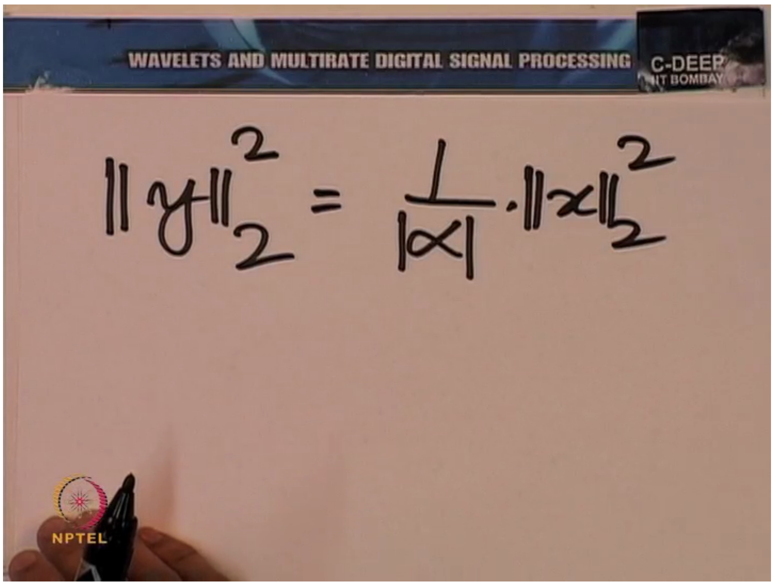

So we have lambda is Alpha t, whereupon d lambda is Alpha dt and therefore dt is 1 by Alpha d lambda. Now here in the integral, you see let me just bring the integral back here, the relevant integral. So we are going to replace dt by d lambda by alpha and depending on whether Alpha is positive or negative, if Alpha is positive, the limit still remains from - to + infinity, if Alpha is negative, the limits go from + infinity to - infinity but then there is also a 1 by alpha negative there.

So 1 by Alpha is negative and therefore you take the negative in the reversal of integral together, all in all we always have the following. The norm of y in the L2R sense is always 1 by mod Alpha times integral from - to + infinity x lambda mod square d lambda. Whereupon what we have concluded is norm y in the L2R squared sense squared is 1 by mod Alpha times the norm of x in the L2R sense squared.

Now, of course we can use the same reasoning, I have illustrated the thought behind the reasoning and if we write down the time centre and the frequency Centre expressions, let us write down the time centre, as we said we are going to focus on time, so let us write down the time centre expressions.

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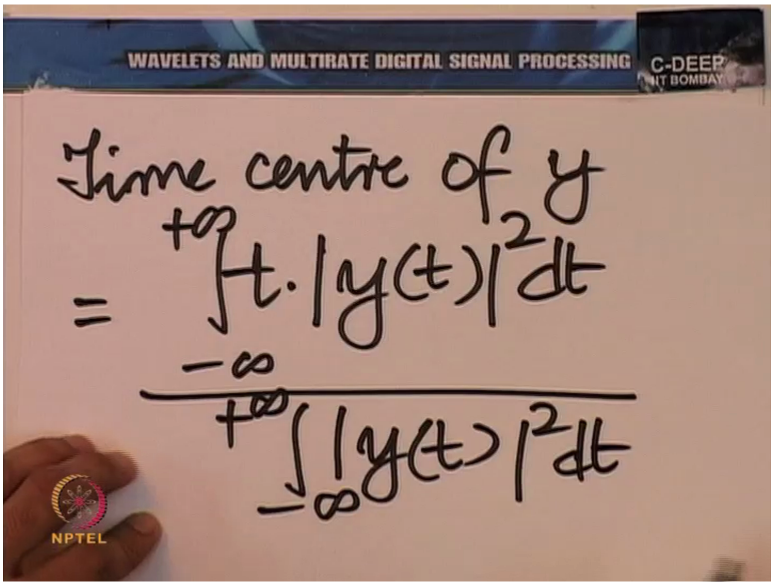

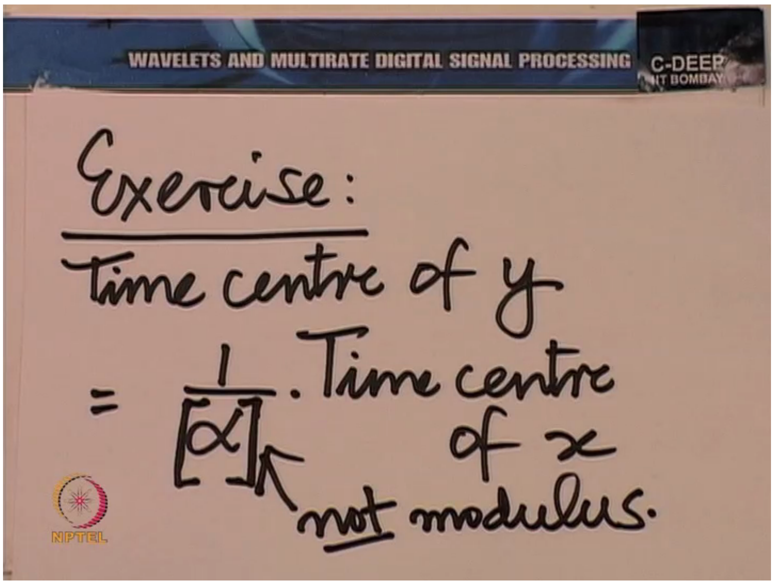

The time centre of y is going to be given by t times mod yt squared dt integrated divided by the integral of mod yt square dt. Now again I shall not repeat all the working that I have done to relate the norms of y and x. Essentially the central idea there is a transformation of variable, put lambda equal to Alpha t and do this all throughout the integral. And one can straightaway write down the final result.

It is very easy to show by making that substitution that the time centre of y is equal to 1 by mod Alpha times the time centre of x. Actually not mod Alpha, I am sorry, it should be just 1 by alpha because we must take into account the sign as well, not modulus. Y#ou must know where it is modulus and where it is not, here it is not modulus. So for example, I will take the example of Alpha equal to -1. If Alpha is -1, then the time centre is also reflected.

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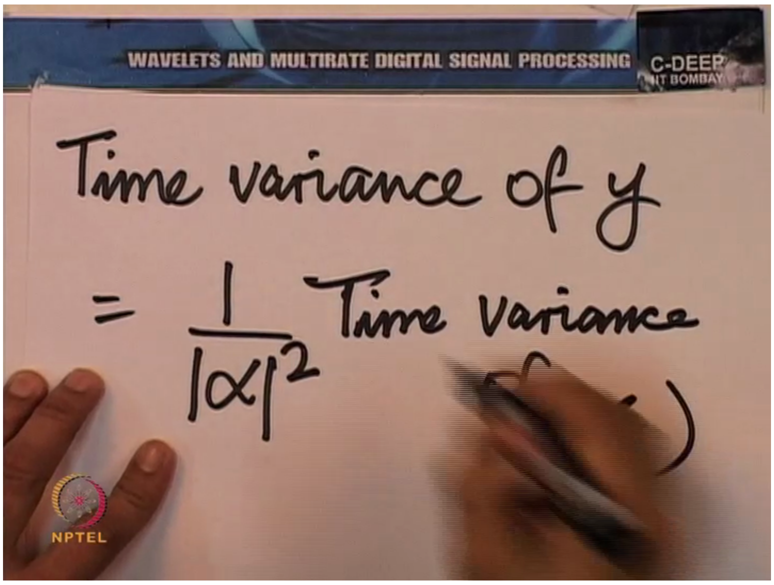

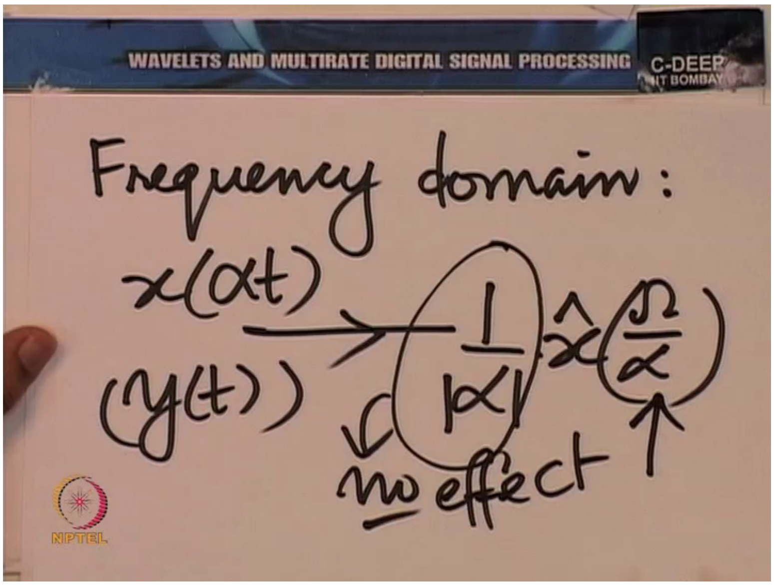

So if the time centre of x was t0, the time centre of y becomes - t0. So it is not mod alpha, it is just alpha. Sometimes simple intuitive reasoning can also clarify certain points for us, anyway here it is not modulus of alpha. Similarly one could use the same transformation and show the frequency, sorry the time variance of y. Now here it is modulus, so it will be 1 by modulus squared of the time variance of x.

And in fact if we now note that what we are doing in time is reversed in frequency, so in time we have t being replaced by alpha t, in frequency capital omega gets replaced by capital omega divided by Alpha with a scaling of 1 by mod Alpha. But please remember, the mod alpha scaling is not going to affect either the time centre or the time variance or the frequency Centre or the frequency variance. So scaling the independent variable, scaling the dependent variable has no effect, we saw that a few minutes ago.

Scaling the independent variable has an effect, let us make a note of this. So we have in the frequency domain x of alpha t which is essentially yt corresponds to 1 by mod alpha x cap omega by alpha and this has no effect, no effect on the variances or the Centre as we saw a minute ago. It is essentially this that we need to look at carefully, this one, the 1 by alpha scaling. And one can use a set of steps very similar to what we did in the time domain and arrive at the following conclusions.

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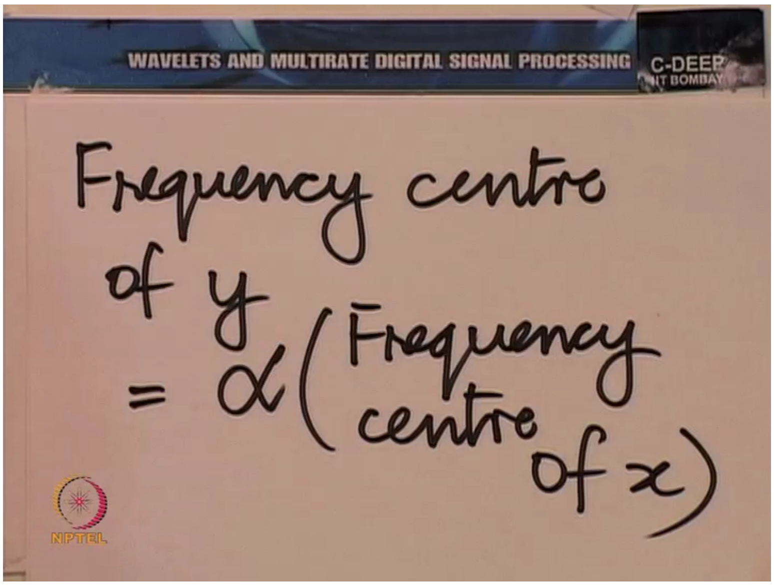

Frequency variance of y

$$= |\alpha|^2 \begin{pmatrix} \text{Frequency variance} \\ \text{of } x \end{pmatrix}$$

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Time bandwidth product of y

$$= \begin{pmatrix} \text{Time variance} \\ \text{of } y \end{pmatrix} \begin{pmatrix} \text{Frequency variance} \\ \text{of } y \end{pmatrix}$$

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WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING C-DEEP IIT BOMBAY

$$= \frac{1}{|\alpha|^2} \begin{pmatrix} \text{Time variance} \\ \text{of } x \end{pmatrix} \cdot |\alpha|^2 \cdot \begin{pmatrix} \text{Frequency variance} \\ \text{of } x \end{pmatrix}$$

cancel

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So the conclusions are, the frequency Centre of y is equal to Alpha times the frequency Centre of x. And as far as the frequency variance goes, the frequency variance of y is mod Alpha times the frequency variance of x. I leave it to you as an exercise to prove this, it is easy to do by making the same kind of substitution of variance. Anyway, now let us look at the time bandwidth product. The time bandwidth product of y is essentially the time variance of y multiplied by the frequency variance of y.

But the time variance of y by this argument is clearly seen to be 1 by mod Alpha square times the time variance of x multiplied by, so I am continuing this as a product here, multiplied by the frequency variance of y is mod Alpha square times the frequency variance of x, that is very interesting. The time variance is multiplied by 1 by mod Alpha square and the frequency variance is multiplied by mod Alpha square.

So when you take a product, these 2 cancel and because they cancel, you get just the time variance of x times the frequency variance of x which is essentially the time bandwidth product of x.

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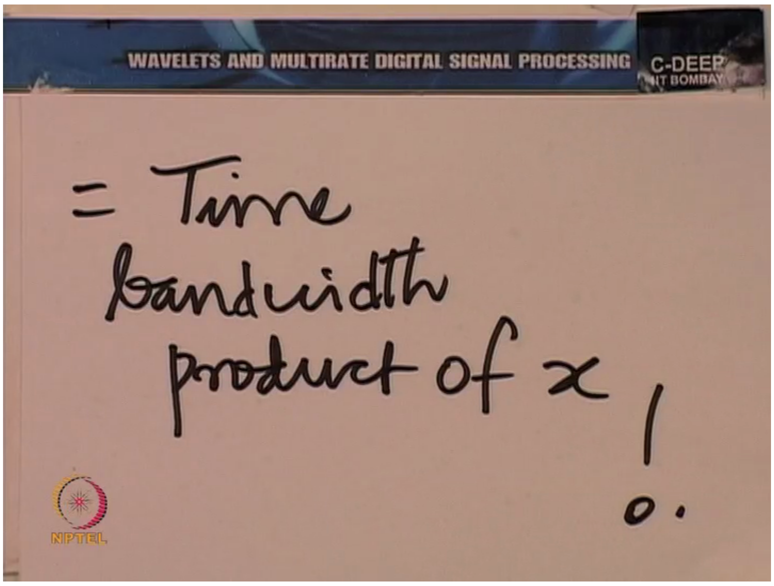

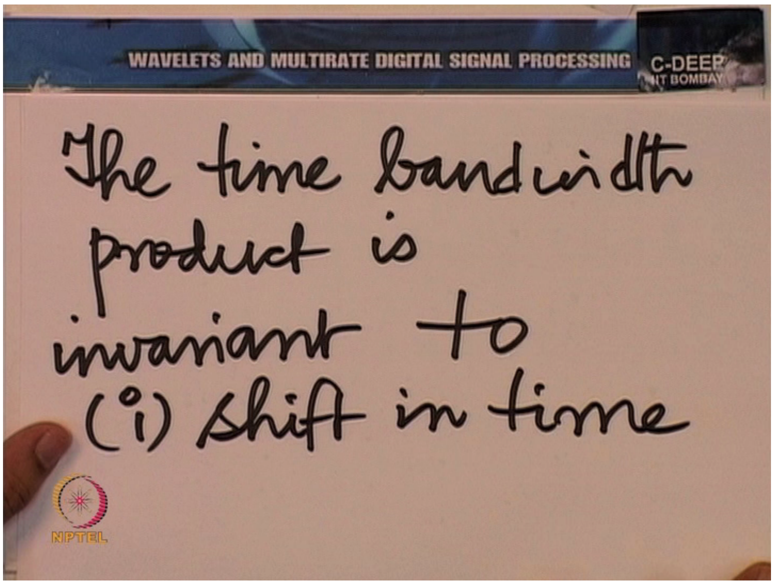

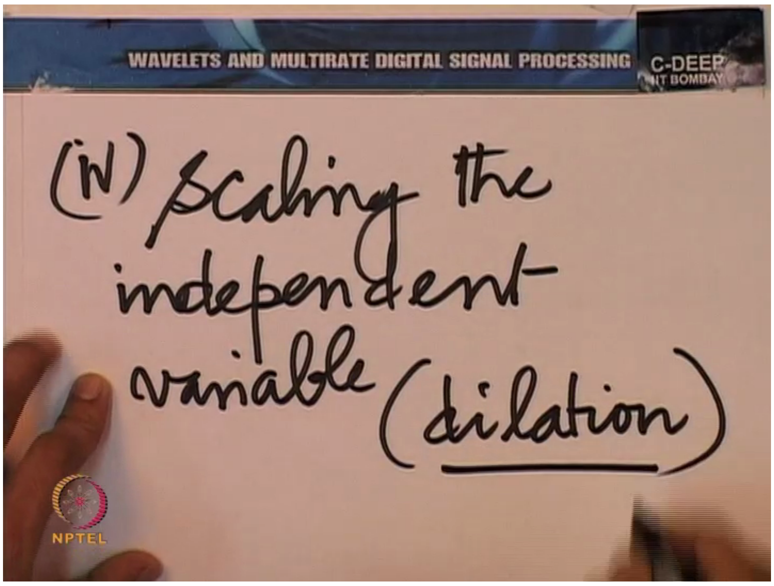

This is a very significant statement that we have made. What we have just shown is that the time bandwidth product is unaffected by a scaling of the independent variable as well. It is very strongly invariant, as we can see, it is a very serious statement, let us write it down. The time bandwidth product is invariant to number 1 shift in time, number 2 shift in frequency or modulation in time, number 3 multiplication by a constant, multiplying the function by a constant.

And finally number 4, the most interesting of them all, scaling the independent variable. Essentially what we have been calling dilation all this while, dilation by alpha, so it is invariant to dilation.