Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis.
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Week-3.
Lecture-9.3.
Construction of Scaling and Wavelet Functions From Filter Bank.

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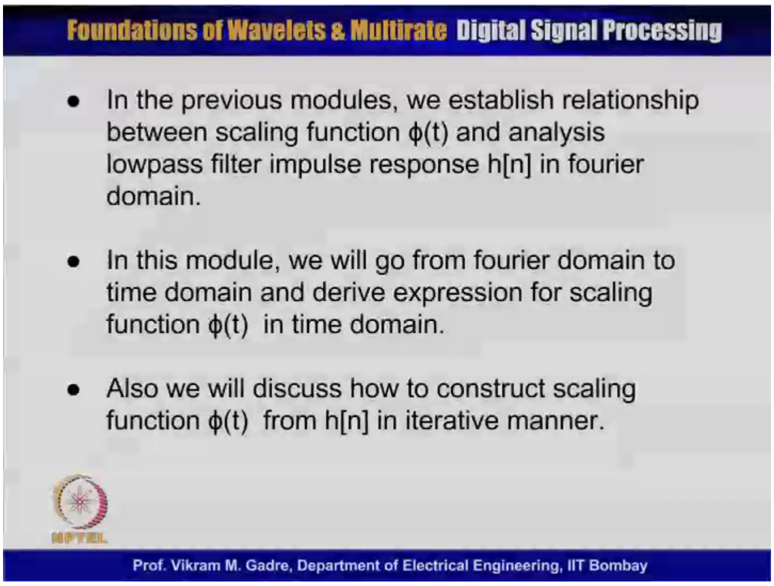

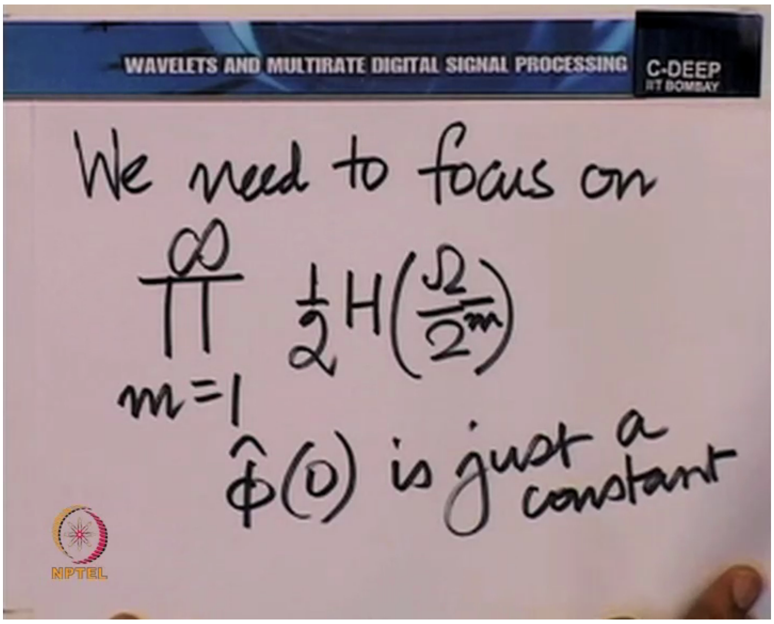

So we need to focus on this infinite product here. Phi cap 0 is just a constant. So let us take just 2 terms in this product instead of infinite terms. In fact, you know now we need to interpret this continuous analog variable a little more carefully here. When we bring in the idea of a continuous analog frequency variable here, then we need to remember that we're taking the Fourier transform of a continuous function. Now, what is the idea of the discrete time Fourier transform which is of course of a sequence becoming the Fourier transform of a continuous function. Well that is simple. Suppose you thought of the sequence as a train of impulses located at the integers.

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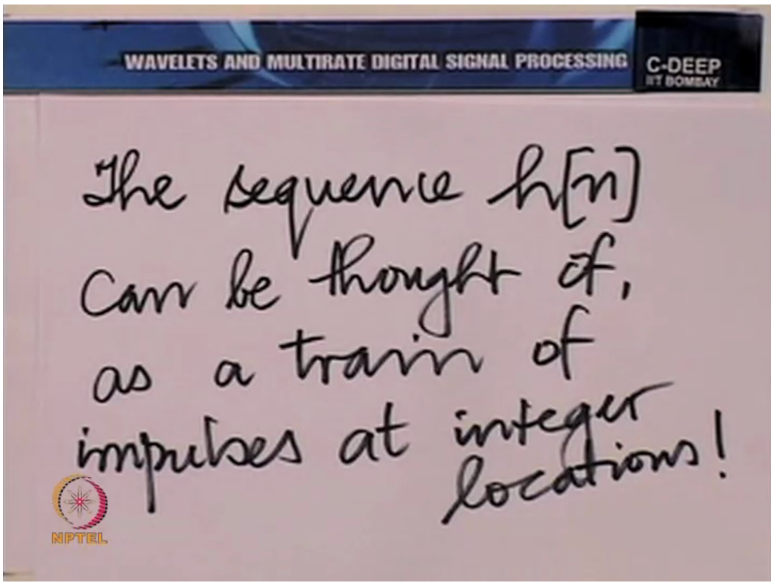

So, the sequence HN can be thought of as a train of impulses at the integer locations. And the train of impulses therefore is of course a continuous function. So you can take its Fourier transform and use the continuous or analog frequency variable. Now, one must interpret capital H of capital omega in that sense.

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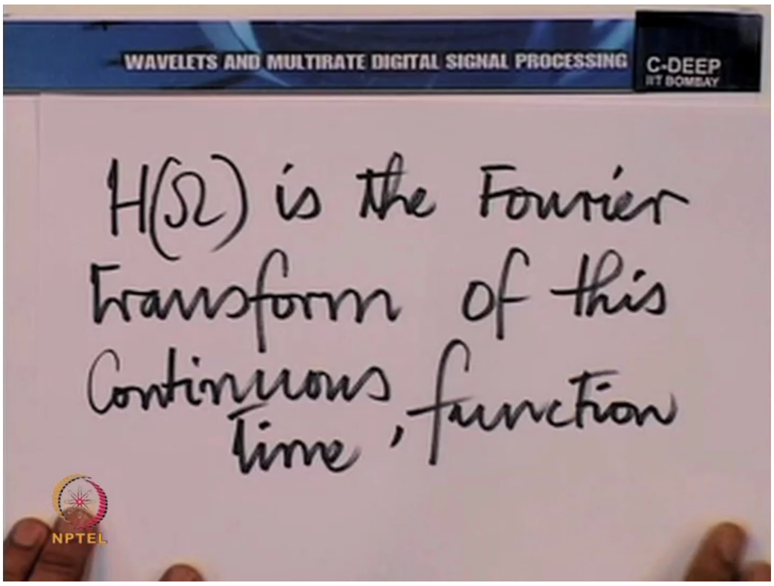

So, capital H of capital omega is the Fourier transform of this analog function or continuous variable function. Maybe I should say continuous time function to be precise. And now, what is half H omega by 2 then?

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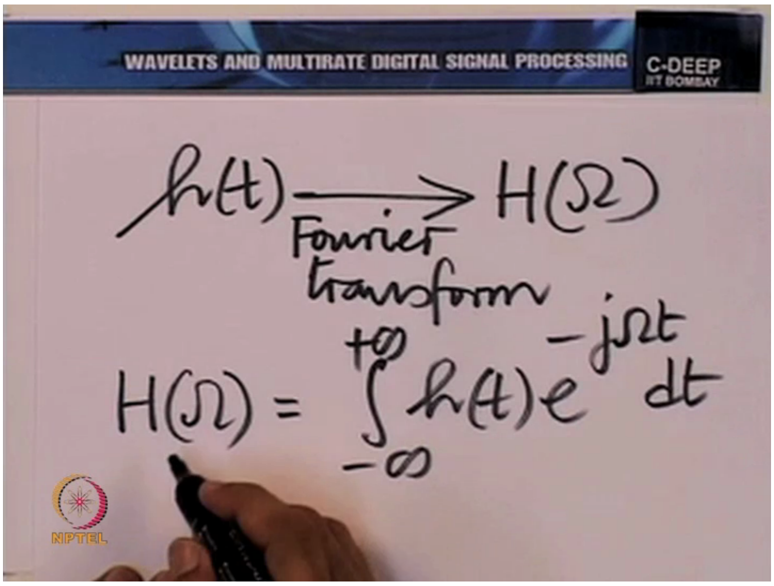

For that purpose, let us assume that we have a function h of t whose Fourier transform is capital H of omega. Of course, we know capital H omega is integral from- to + infinity H t e raised to power –j omega t dt. And if we happen to consider alpha times H omega by alpha with positive alpha.

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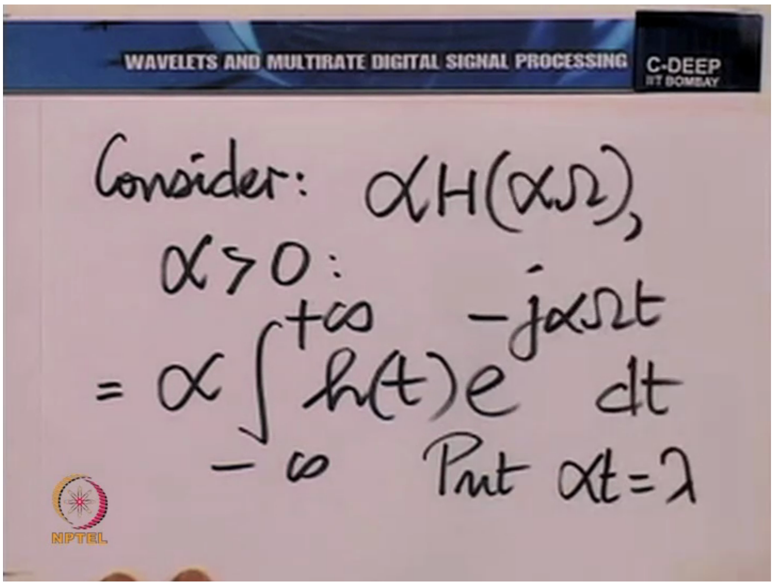

So what I'm saying is consider alpha times H alpha times omega with alpha positive. It is equal to alpha times integral from - to + infinity H t e raised to power – j alpha omega t dt. And now we have a simple step that we can perform. If we simply put alpha t equal to lambda then we would get, you see, alpha t equal to lambda, alpha is strictly positive. So when t runs over all from - to + infinity lambda also runs from- to + infinity.

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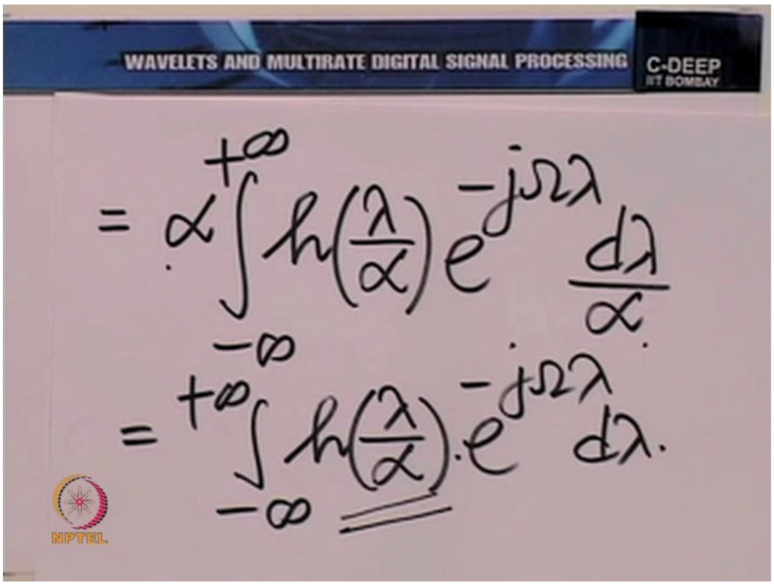

So therefore we would have this is equal to alpha times integral - to + infinity H lambda by alpha making the substitution e raised to power – j omega lambda. Now dt is d lambda by alpha. Now, if we just cancel the alpha here and alpha here, we get integral from- to + infinity

H lambda by alpha e raised to power – j omega d lambda, which is essentially the Fourier transform of H of lambda by alpha as the argument. So we have divided the argument by the positive number alpha.

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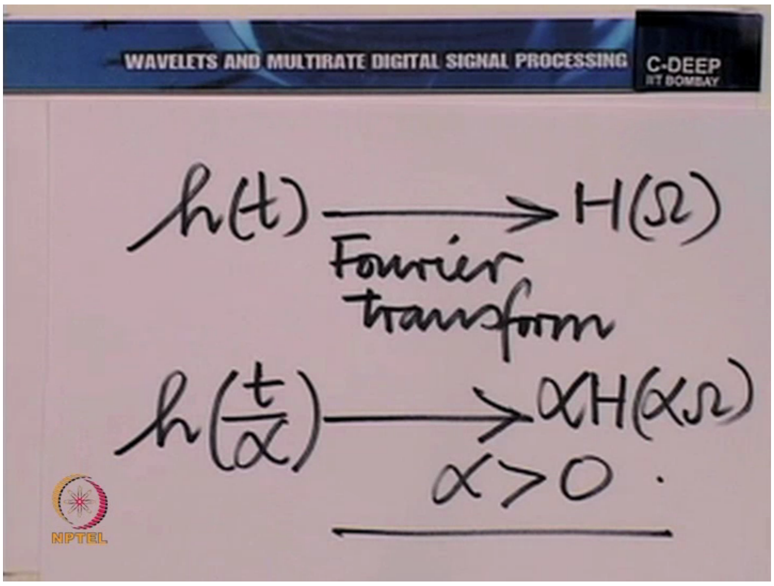

So what we are saying in effect is, if Ht has the Fourier transform H of omega, then H t by alpha has the Fourier transform alpha times H of alpha omega will alpha is of course strictly greater than 0 here. Now of course one can generalise this for alpha real and negative. All that one needs to do is take a modulus outside and no modulus inside but I leave that as an exercise for you. We do not immediately require it, one can easily generalise.

But coming back to the point then, what H omega essentially means? What H omega by 2 essentially means then with a factor of half outside is a dilated version of this train of impulses.

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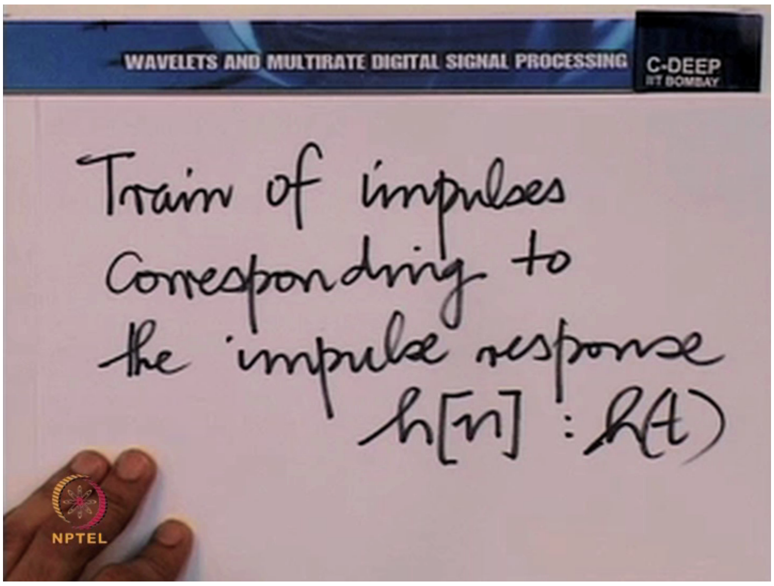

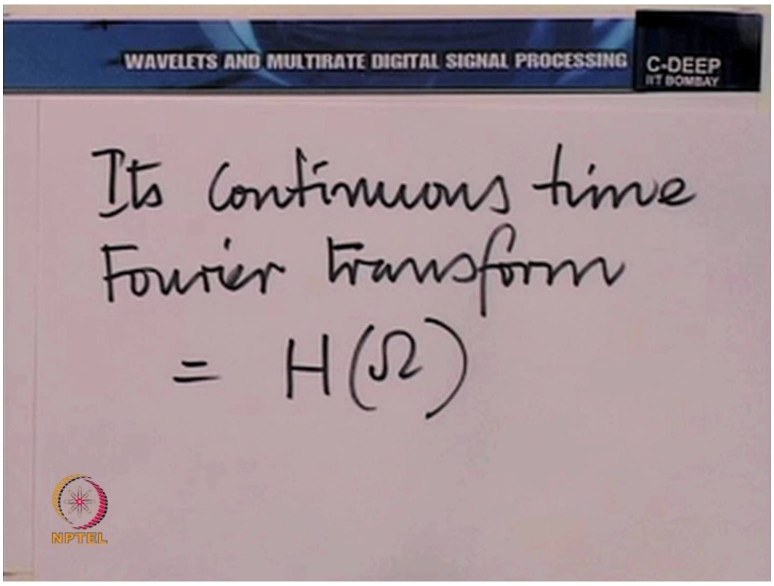

So we have this train of impulses corresponding to the impulse response H of N which we have called H of the continuous variable t, it is continuous time Fourier transform or analog Fourier transform so to speak is capital H of capital omega.

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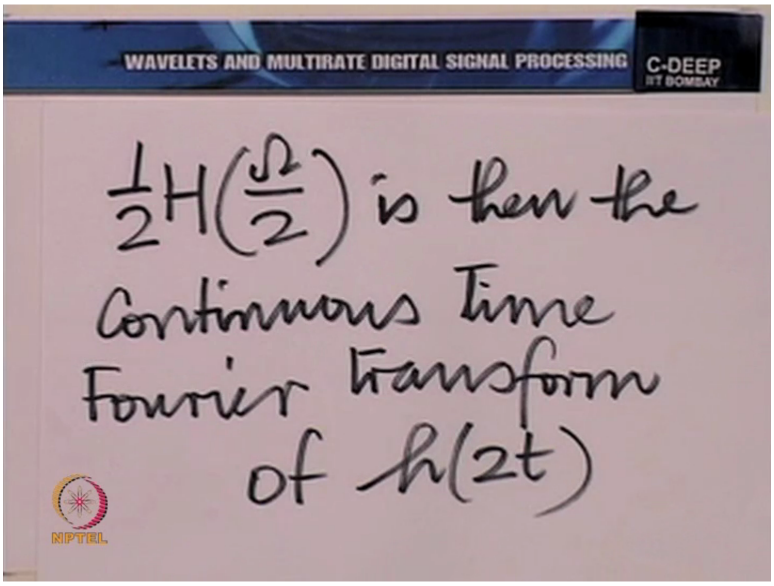

And then half capital H of capital omega by 2 is then the continuous Fourier transform of H of 2t, that is easy to see because you have chosen alpha equal to half there. What do you mean H of 2t? H of 2t means you have squeezed Ht by a factor of 2 on the time axis, on the independent variable.

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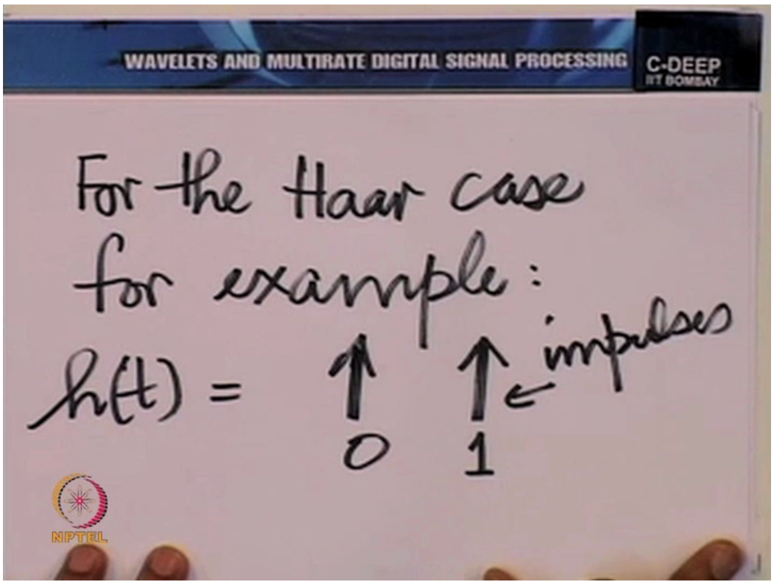

So you have brought the impulses closer. Now when you multiply 2 Fourier transforms, the corresponding continuous functions are convolved. So essentially you may think of H of t here so to speak for the haar case. H of t looks like this, there is an impulse at 0 and an impulse, a continuous impulse remember at 1, these are impulses as understood in continuous time.

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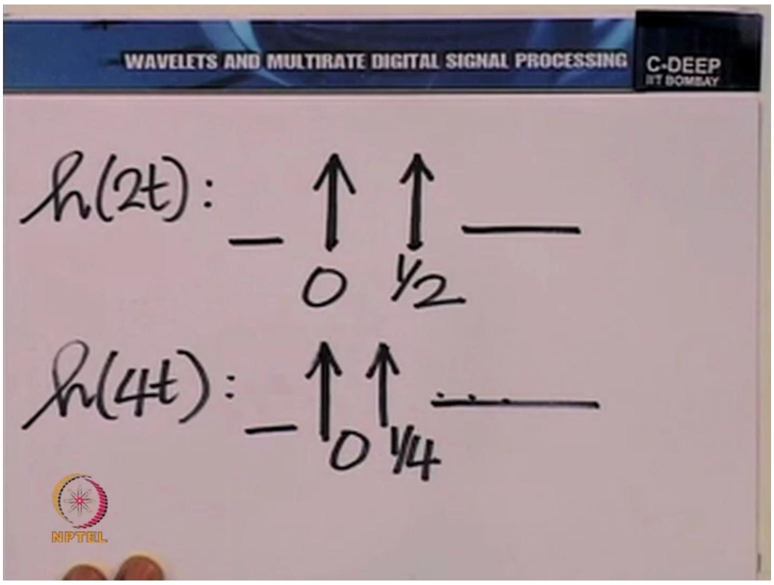

And H of 2t will look like this. You know, if I really wish to be finicky, I should be putting down the strench of the impulses carefully too but let us not get that finicky, this is what H of 2t will look like, there are impulses at 0 and half. H of 4t for example will look like this now. This one is squeezed again by a factor of 2, there will be an impulse at 0 and an impulse at 1 by 4 and so on and so forth. Of course, the rest of it is 0, just 2 impulses. So what do we have now, we have a product. Let us take just 2 terms in that product.

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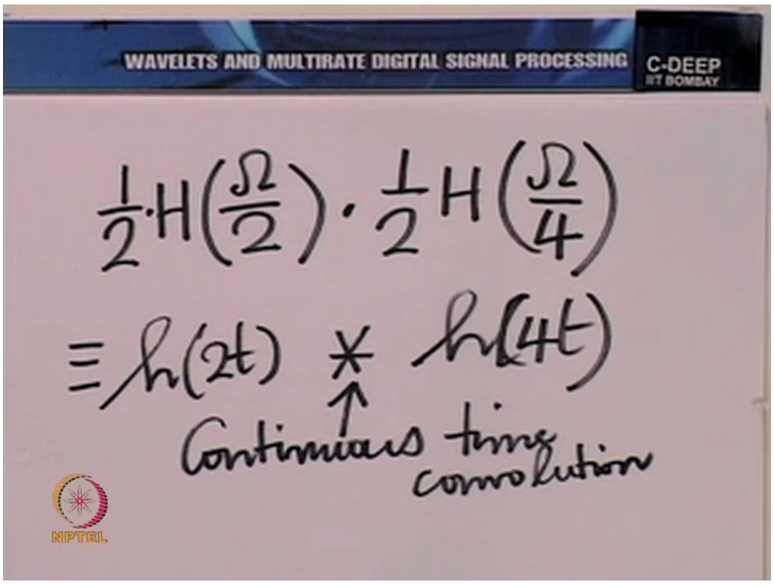

So if we take just the 1st 2 terms, half capital H capital omega by 2 times half capital H capital omega by 4, it corresponds to H of 2t convolved with, this is continuous time convolution here Convolved with H of 4t. Now as I said I'm being a little care less about constants but if you really wish to be finicky you can. I am more interested in getting a feel of the shape of the convolution. I'm not so concerned about the precise heights and so on. Anyway, let me convolve them and show you.

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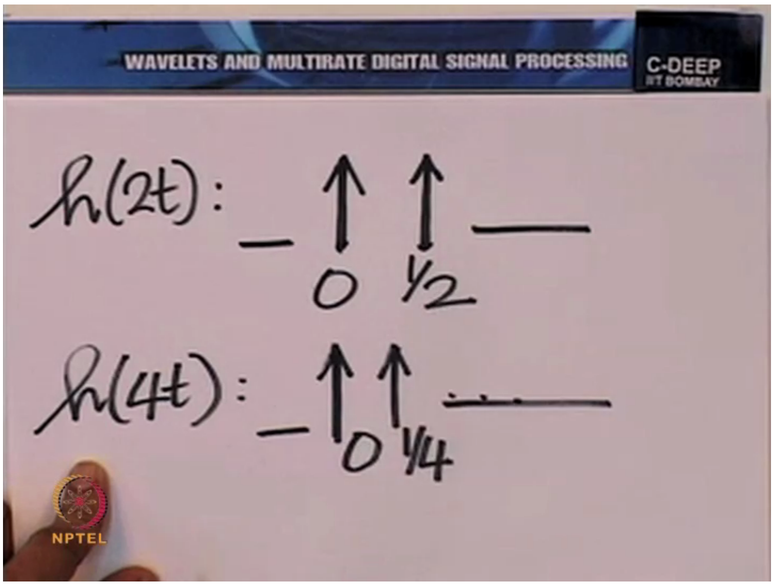

So let us put back H of 2t and H of 4t as we had them here. So we had H of 2t here, essentially 2 impulses located at 0 and half. We have H of 4t 2 impulses located at 0 and one fourth. Now what will happen when you convolve this? You know, when you convolve a continuous time function with an impulse, a unit impulse that gives you back the same continuous time function.

So as I said, if you just ignore the heights and note the heights are equal here then when you convolve this H of 4t with this, you could treat it as a convolution of this with this impulse + the convolution of this with just this impulse and you could sum these 2 independent convolutions. When you convolve this with this impulse, you simply relocate this at the position 0. And in fact, that gives you back, H of 4t. When you convolve H of 4t with this impulse located at half, it simply shifts this function to lie at half.

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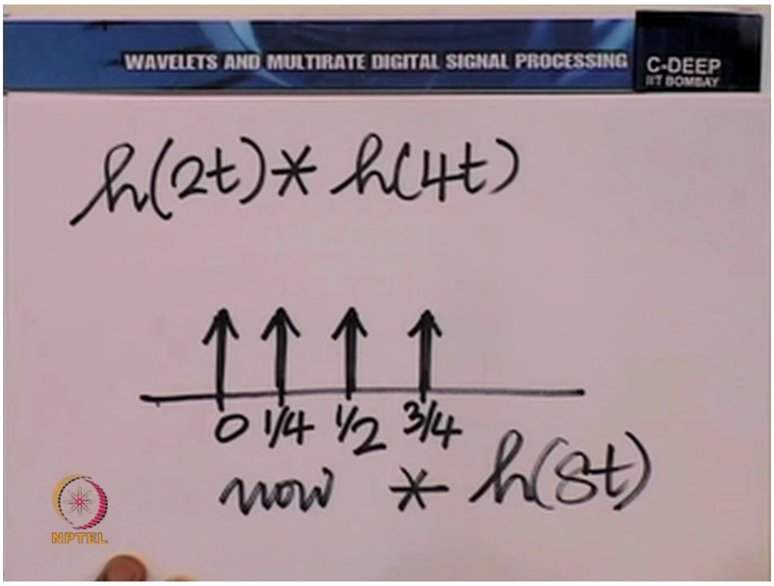

So in effect when you have H of 2t convolved with H of 4t you get something like this. You get an impulse located at 0 ,one at one fourth, 1 at half, one at half + one fourth which is 3 by 4. So you get impulses here. Now convolve this again to take the next term with H of 8t as that infinite product asks you to do. So if you take 3 terms, then you would be now convolving this with H of 8t. How will H of 8t look?

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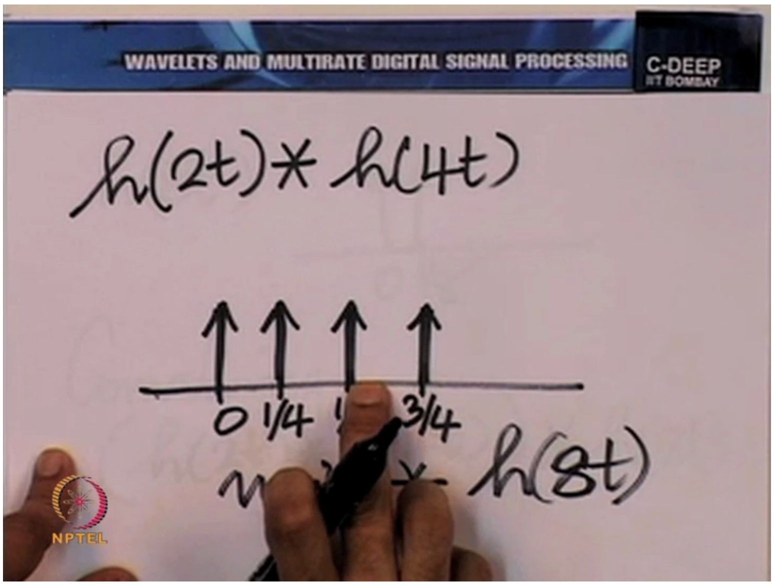

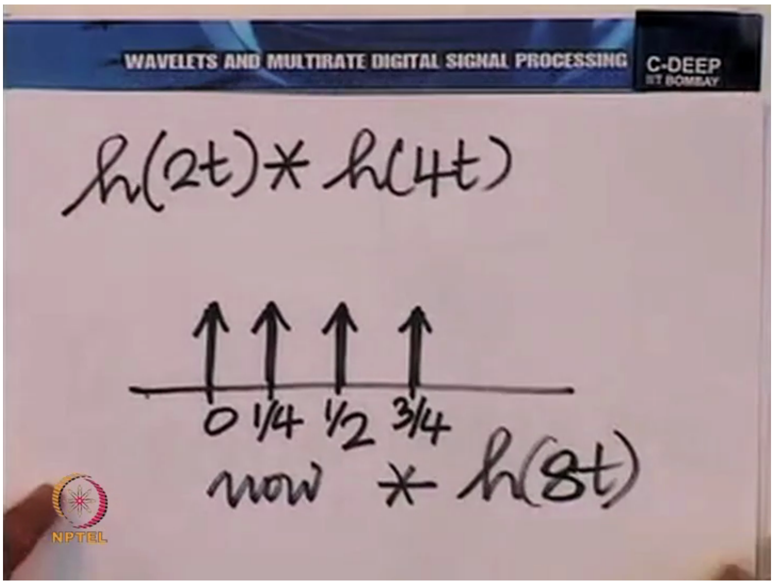

H of 8t looks like this, they come even closer together 0 and 1 by 8. And convolving, H 2t is convolved with H 4t and then the whole convolved with H 8t, what will you have? Essentially this has to be located here, here, here and here. And all these relocated H of 8ts should be added together.

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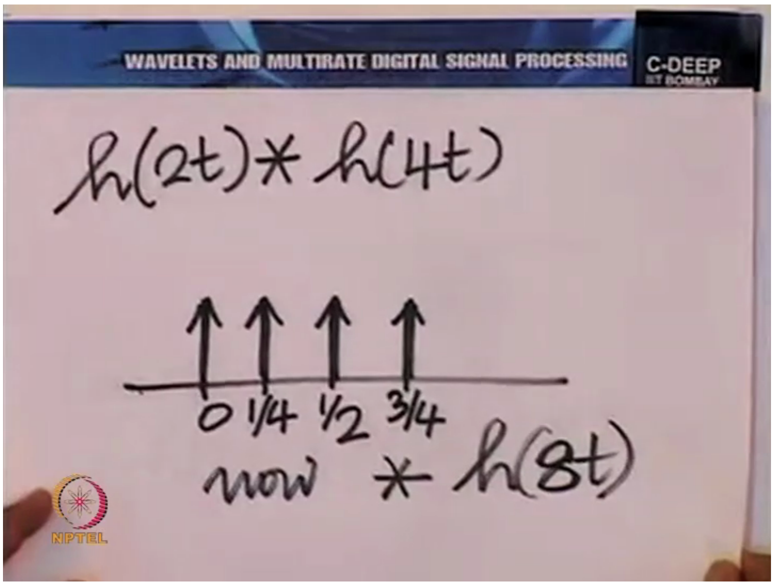

Now, when you locate H of 8t here, you will get an impulse at 0 and 1 by 8 here in the middle. When you relocate H of 8t here, you will get an impulse here and at one fourth + one eighth. That is two eighth + one eighth, three eighth. So let me straightaway now draw, this convolution results in impulses at each of these places.

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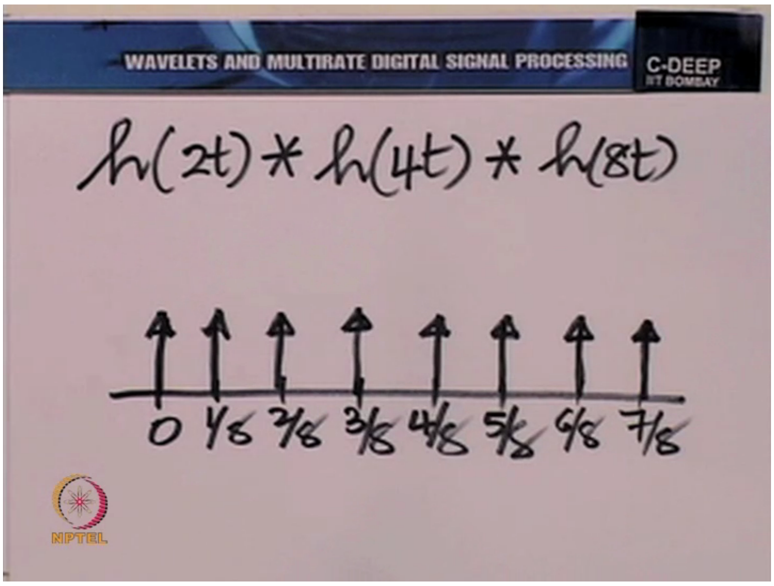

Now you know, we seem to be getting where we want to. What is happening if you think about it? Each time you bring in one more term you are getting a train of impulses where the train has double the size but it lies on the same support. H of t lay on the support, 0 to 1. H of 2t lies on the support 0 to half. Of course, I would not really say 0 to half. You know, there is an impulse at 0 and an impulse at half. But then when you go to H convolved with H 4 t, you get an impulse at 0, at one fourth, at two fourth, and at three fourth.

When you go and bring in one more term, you get 8 impulses. When you bring in one more term next time, you’re going to get 16 impulses, then 32 impulses and the last impulse comes closer and closer to 1.

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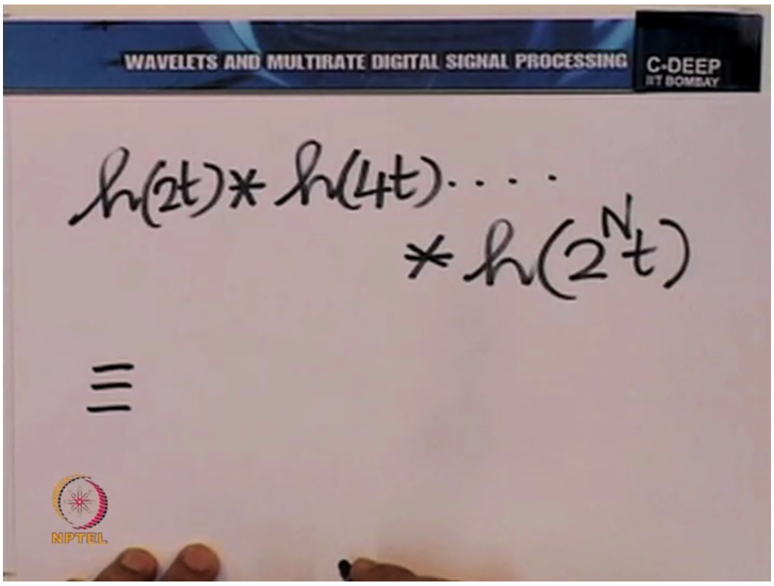

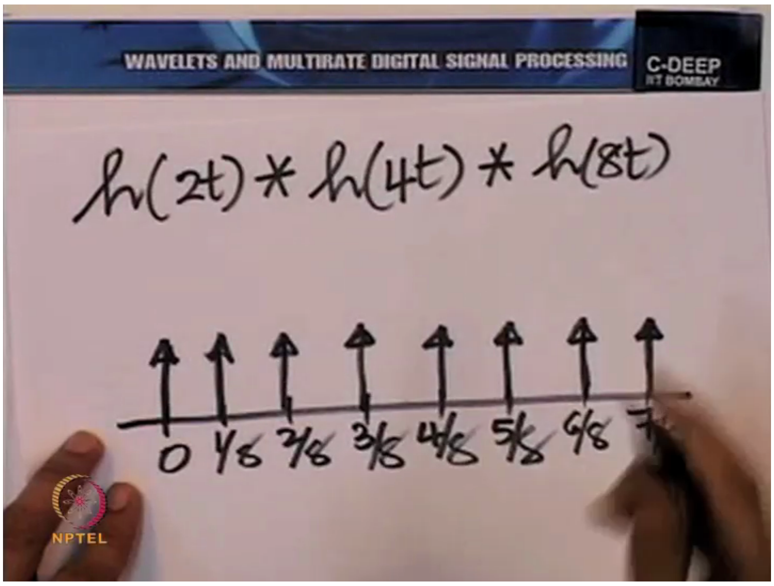

So what we have here effectively is H 2t convolved with H 4t and so on so forth so forth up to H 2 raised to power of Nt is essentially how many impulses? You see, when you reached H of 8t, you had 8 impulses.

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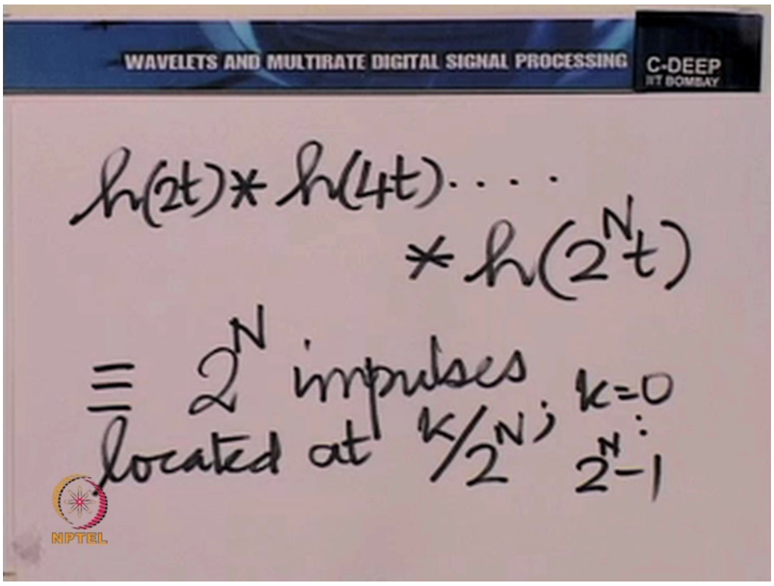

So when you reach H 2 raised to the power of Nt, you have 2 raised to the power N impulses located at K divided by 2 raised to the power of N. K going from 0 to 2 raised to the power of N – 1. So you know, the last impulse as you can see the last impulse is located at 2 raised to the power of N-1 divided by 2 raised to the power of N. So last impulse goes closer and closer and closer to 1.

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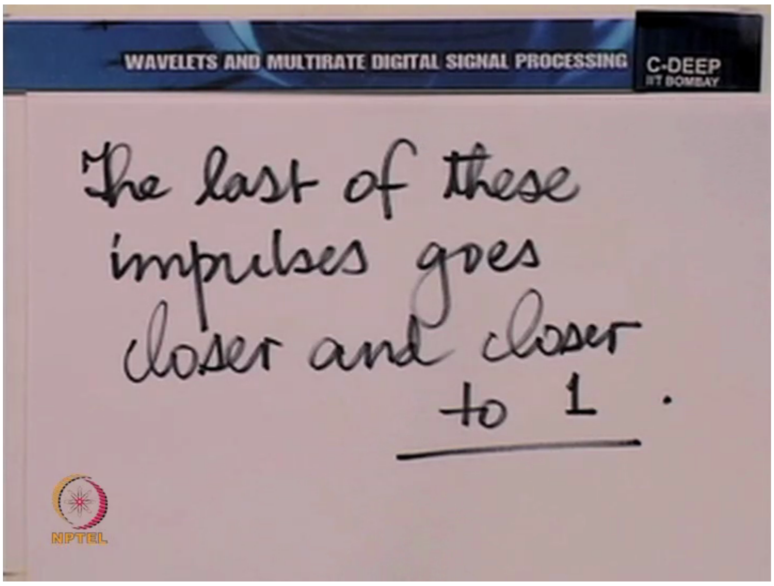

The last of these impulses goes closer and closer to 1. So you know when you have impulses located closer and closer and closer together, you are ultimately coming to a continuous function. Can you remember that idea of expressing a continuous function in terms of impulses essentially captures this? When you say X of t is a Conglomeration of impulses located every point t with strength equal to the value of X at the point t, that is exactly what you are saying.

When you bring impulses closer and closer and closer together, they fuse together to form a continuous function. And, it is very easy to see here what continuous function we are moving towards it is flat and indeed, it is very clear then that we are moving towards which is essentially 5t.

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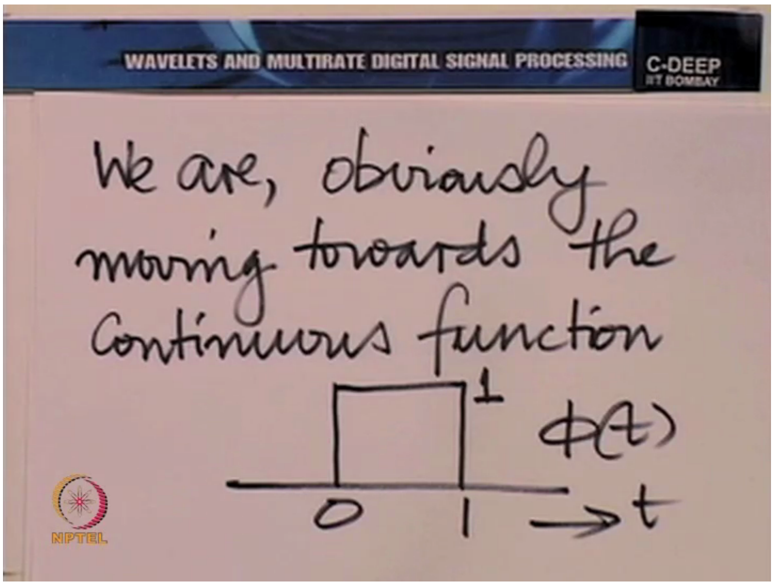

Lo and behold, a very beautiful relationship that we have. We start from the haar low pass filter. We have repeatedly convolved a train of impulses. You know, 1st time it is a train of impulses located at 0 and half, then at 0 and one fourth and you have repeatedly convolved these, iterated the filter bank. Repeatedly convolved these trains of impulses and you are moving towards a continuous time function which is indeed phi t as you can see when you put those impulses closer and closer and closer together.

So now we can see the connection between iterating the filter bank and producing phi. We now need to complete a little detail. How do we get psi? But that is very easy. We already got phi. And then know the dilation equation for psi. So, we have phi now.

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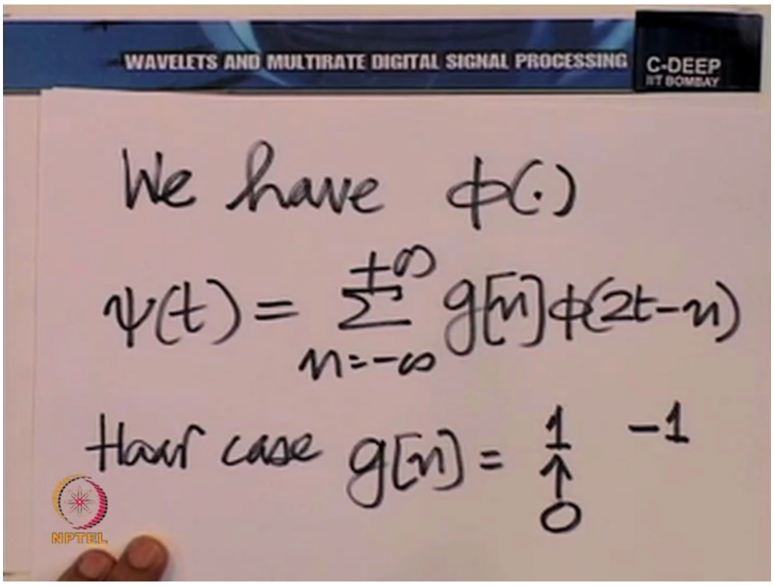

How will psi look? Psi t is essentially summation N going from - to + infinity GN phi of 2t – N. And for the haar case, we know what GN is, GN is essentially 1 and -1. So we can write down psi t in terms of phi 2t and construct from there.

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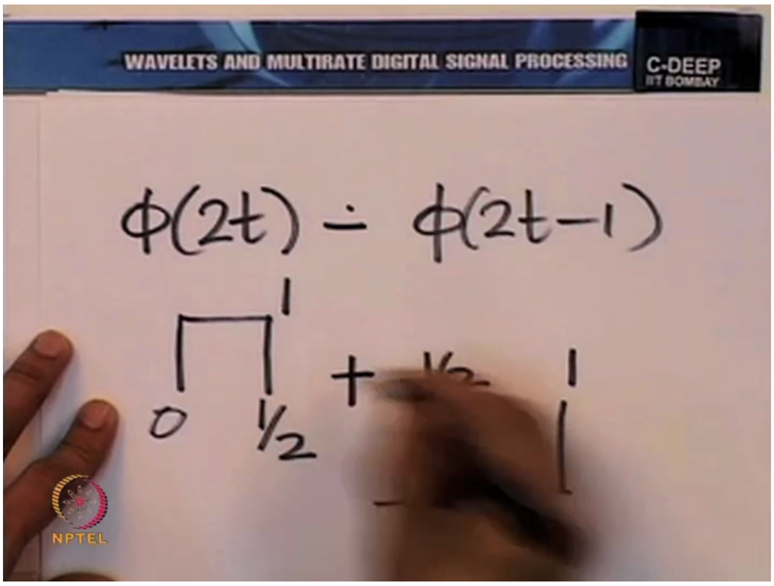

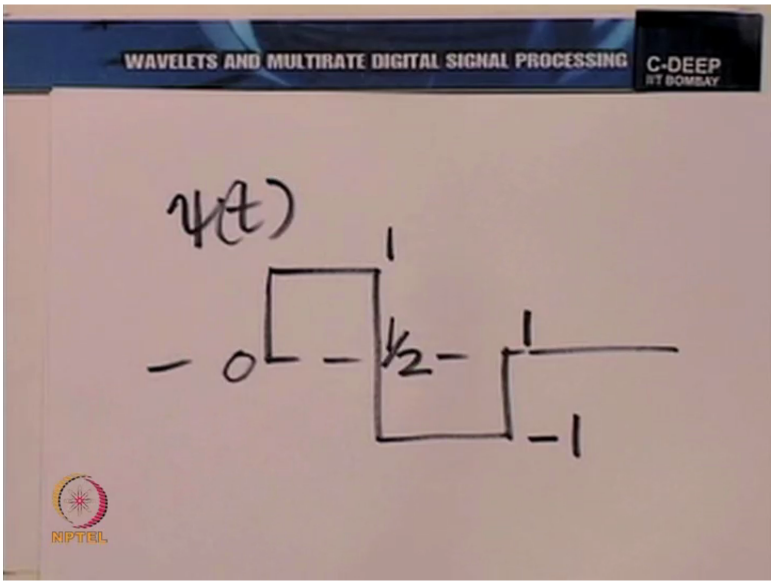

Phi 2t - phi 2t -1, this is phi 2t, this is - phi 2t -1 and when we put these 2 together, we get psi t, there we are. We have completed this iteration and building phi t and psi t starting from HN and GN. Now we have a convincing reason to conclude that there is an intimate relation between the low pass filter and the high pass filter in the 2 band filter bank and the scaling function and the wavelet function in the multiresolution analysis. In fact, we have constructively established that relation.

We have shown a procedure by which we can construct phi t and psi t from these impulse responses. And therefore, we are now convinced that if we understand how to design 2 band filter banks and if this iteration is going to converge each time we design a properly designed 2 band filter bank which allows this iteration, we get a new multiresolution analysis. With that background, we shall conclude the lecture today and proceed in the next lecture therefore to explore the two-band filter bank more deeply, thank you.