Foundations of Cryptography
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Lecture – 58
Zero Knowledge Protocols Part II

Hello everyone, welcome to this lecture just a quick recap. In the last lecture we had started our discussion on zero knowledge protocols. So in this lecture we will continue our discussion on zero knowledge protocols specifically we will introduce zero knowledge protocols for NP complete class.

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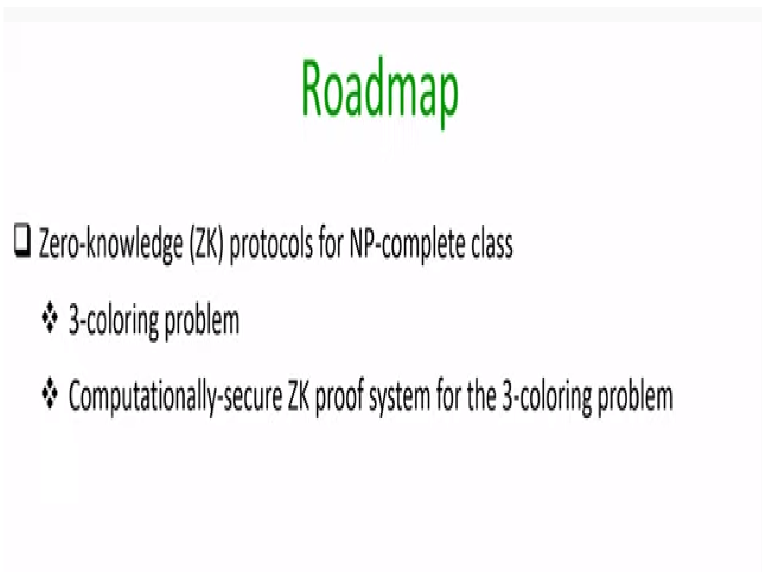

For that we will introduce the 3 coloring problem and we will see a computationally secure zero knowledge proof system for 3 coloring problem.

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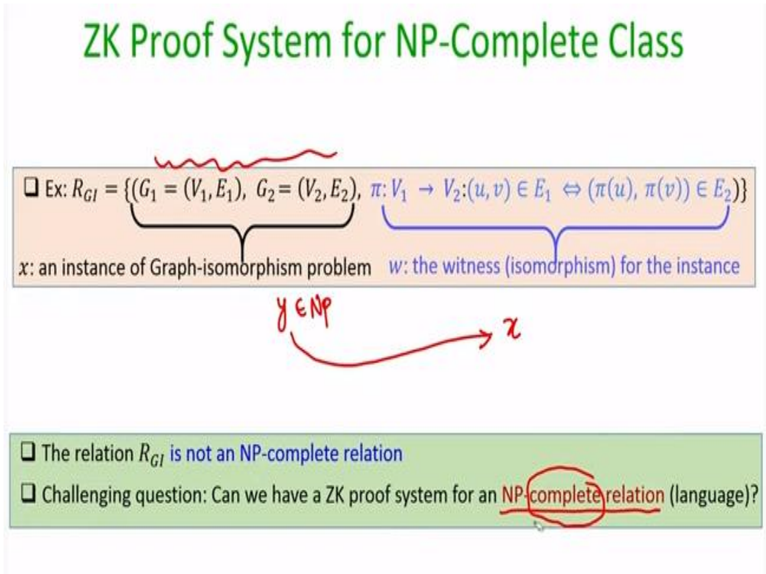

So recall that in the last lecture we have seen that how exactly we can specify a relationship. For instance, if you recall the relationship for graph-isomorphism problem then the relationship will consist of x, w pairs where x is a problem instance. So in this example if we consider a graph-isomorphism problem than the x instance is basically the publicly known description of 2 graphs and the witness component corresponding to this x instance will be the isomorphism mapping isomorphism between the vertex set of the first graph to the vertex set of the second graph.

And indeed if we have an x or problem instance where the 2 graphs are isomorphic then we should have a corresponding witness w and in the last lecture we have seen that how we can come up with a zero knowledge proof system which allows the prover to show whether an instance x has a corresponding witness w available with the prover and not without revealing anything about the witness w.

But it turns out that the relation graph-isomorphism is not an NP complete relation and next challenging question is can we have a zero knowledge proofs system for any NP complete relation? So you might be for people who might be wondering what exactly is NP complete problem or NP complete relation. So an problem instance x is called a problem x is called an NP complete problem.

If given a witness w we can verify whether indeed the witness w is a right witness for the problem instance x and polynomial amount of time specifically by performing non-deterministic computation for polynomial amount of time that is the first requirement and the reason we call this such problems as NP complete. The completeness aspect here denotes that if we have any other problem y belonging to the class NP then that problem instance y can be reduced to an instance of the problem x in polynomial amount of time.

In that sense this problem x will be called as an NP complete relation that means if you have a solution to solve problem instance x. That means if you can find out witnesses for problem instance x in polynomial amount of time then by just using the reduction of problem instances of y to the problem is tense affects you can also see you can also get solutions for your problem instances why? So that is a rough definition of NP complete relation.

So we are now interested to see whether we can come up with a zero knowledge proofs system for any relation which is NP complete. So it turns out that the graph- isomorphism is not an NP complete relation.

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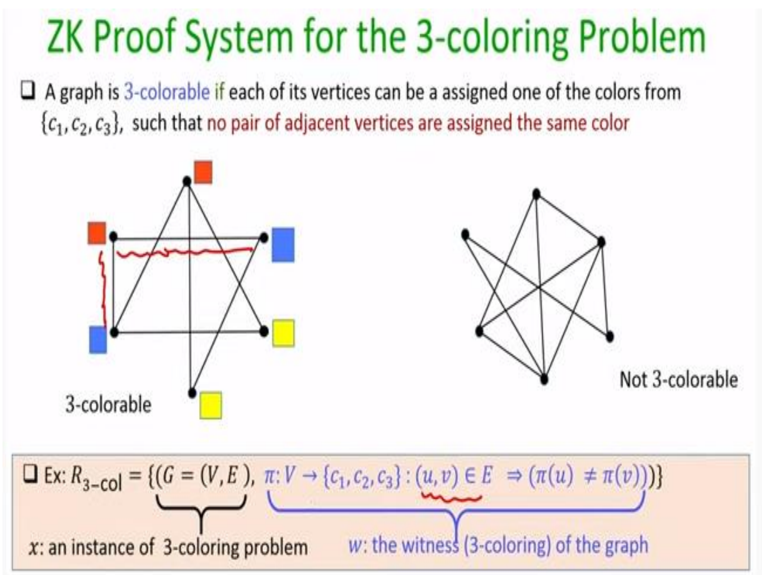

So what we are going to now do is we are going to see a zero knowledge proof system for another computational problem which we call as 3 coloring problem which is a well-known NP complete problem. So let us first see what we mean by a 3 colorable graph so we say a given graph within vertices is 3 colorable if each of its vertices can be assigned one of the colors from publicly known colors c1, c2, c3 such that no pair of adjacent vertices are assigned the same color that means we have to color the vertices in such a way that every pair every the endpoints of every edge should have different colors

So if you consider this graph for instance then this is a 3 colorable graph. So for instance if my color 1 color, 2 color 3 are red blue and yellow then we can color the vertices in this graph using these 3 colors by one of these by giving assignments by giving assigning these colors to the respective vertices and now you can see that no 2 adjacent vertices namely no 2 adjacent vertices are assigned the same color here.

So for instance if I consider these 2 vertices, they are adjacent in the sense that they are the end points of a single edge say same edge and they are having different colors. In the same way if I consider this edge the end points are getting different colors and so on. On the other hand, if I consider the graph on your right hand side then it is not 3 colorable. That means it is not at all possible to color all the vertices of this graph with just 3 colors satisfying the condition that no pair of adjacent vertices are assigned the same color.

And the 3 coloring problem or the 3 coloring in relation is a well-known NP complete relation and the x, w entry in the 3 coloring relation will look like this. So the x instance will be the public description of a graph namely the number of vertices and the vertex set and the x set of the graph will be publicly known and if indeed this x instance namely this graph is 3 colorable than the corresponding witness w will be the mapping of or the assignment of the color c1, c2, c3 to the vertex set which I call a pi.

And the coloring the witness pi should satisfy the restriction that if the edge u, v belongs to the edge set then the color assigned to the node u and the color assigned to the node v should be different. If indeed it is possible to come up with such an assignment of color pi for the given instance for the given problem instance x then x, w will be considered as a valid entry or satisfying the relationship 3 coloring. If we cannot find a witness w, then corresponding to a given x then that x, x will not be present in the 3 coloring relation.

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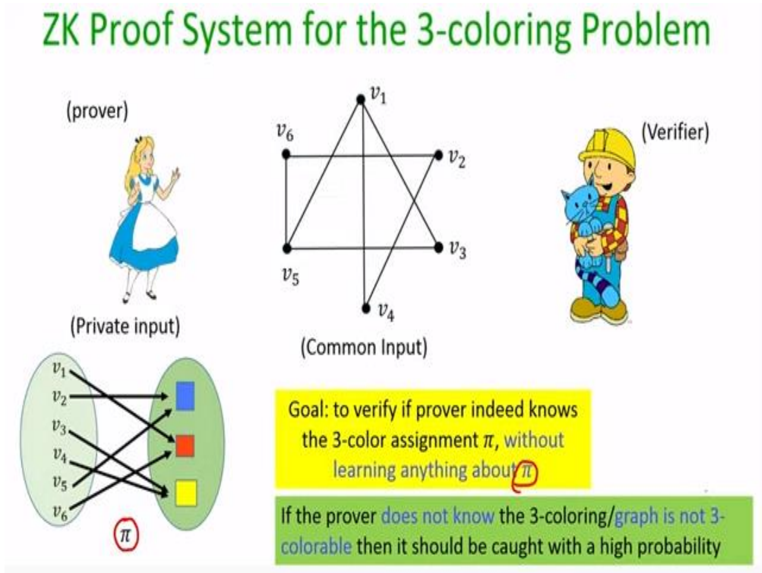

So now let us see as you know knowledge proof system for the 3 coloring problem. So imagine a Alice is the prover and Bob is the verifier and the common input for both Alice and Bob is their description of a graph and say the private input for the prover namely Alice here is a 3 coloring of the publicly known graph namely an assignment pi mapping the vertex set to the color set c1, c2, c3 and what Alice wants to prove to Bob that indeed the given graph is 3 colorable and she has an assignment pi available with her.

So the goal here is to come up with a zero knowledge proofs system which should convince Bob that indeed Alice knows the corresponding mapping pi without revealing anything about the actual mapping pi and at the same time the zero knowledge proofs system should ensure that if prover does not know the 3 coloring of the given graph or if the graph is not 3 colorable at the first place then with very high probability while giving the proof Alice should be caught by Bob.

So this is your recall that this is your soundness property the second requirement and the first property namely Bob should not learn anything about the 3 coloring is the zero knowledge property okay.

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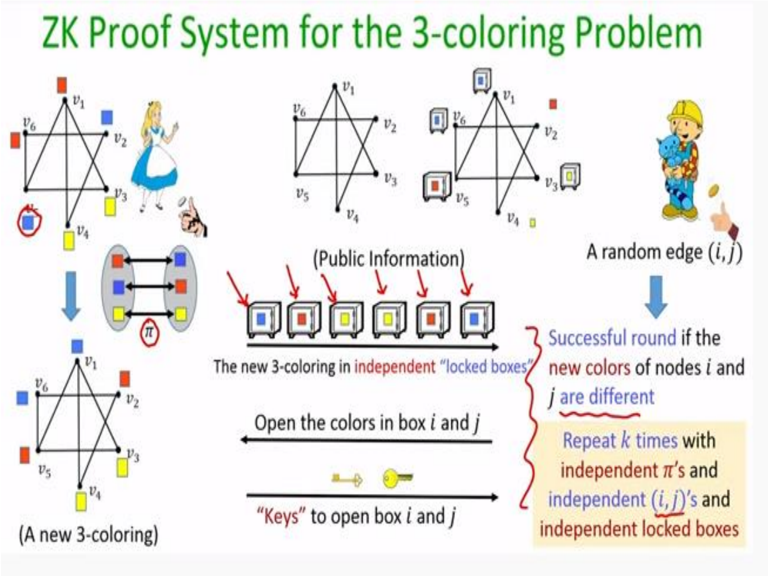

So now let us see how exactly the zero knowledge proof system for the 3 coloring problem will look like so Alice has the private 3 coloring available with her. So what she does is she randomly permits the color that she has available with her namely see she has the secret mapping pi sorry she has the original coloring of the graph. So what she does is she creates a random permutation of the color set.

So for instance she can say create a permutation we are red gets mapped to blue, blue gets mapped to red and the third color remains as it is as per the permutation and basically now what she is doing is she is creating a new 3 coloring of the graph that is available with Bob with respect to the new mapped colors okay. So wherever in the original graph whichever vertices were colored with the red color those vertices in the new coloring will be assigned.

Those vertices in the new coloring will be assigned blue color because the red color gets mapped to blue color in the same way in the old vertex in the old coloring whichever vertices were assigned the blue color those vertices in the new 3 coloring will be assigned a red color and so on so this Alice is doing at her end. Now once Alice computes the news 3 coloring of the graph what she does is she keeps the new assigned colors of the respective vertices in her locked box.

And here locked boxes and quote unquote. We will later see what exactly are the properties we required from this locked boxes and how do we instantiate it using cryptographic primitive. So on a very high level what this locked boxes means is that seems in this example we have 6 vertices.

So what Alice is doing is whatever is the new color assigned to the vertex one that is kept inside this locked box where the locked is available with Alice and its locked box in the sense that without having access to the key Bob cannot open the first box and see what is the new color of the first vertex. In the same way the new color of the second vertex is in the second lockbox the new color of the third vertex is in that third locked box and so on.

So Bob right now cannot see the colors that are available in the locked boxes. So from the viewpoint of the Bob if indeed Alice knows the original 3 coloring then from the viewpoint of the Bob, Bob will feel as if he is now seeing a new 3 coloring of the same publicly known existing graph with the exception that he actually do not know what are these exact colors now because all those new colors are in the locked box.

So this first round of message is like a commitment from Alice to Bob that may she is saying that okay I have now computed a new 3 coloring I will not show you the new 3 coloring those new 3 colorings are actually available in this locked boxes and that is the commitment from my side as per the zero knowledge proof system. Now once Bob receives the commitment of Alice what Bob does is it creates a challenge for Alice and the challenge is basically a random edge i, j from the graph.

So remember Alice will not be knowing what exactly is the random edge that Bob will challenge when Alice is committing the new colors in the locked boxes. So once Bob picks the random edge, he challenges Alice that you please open the colors new colors in the box i and the box j I want to check whether indeed they are different colors or not because if at all you know the original 3 coloring then as per your mapping the new coloring should also be a 3 coloring.

And hence since i and j are adjacent nodes the new color of the node i and a new color of the node j should ideally be different so that is what Bob is challenging Alice to show and to respond to Bob's challenge what Alice reveals is it reveals is the keys for the box i and the box j. Now I need another property from the locked box here so remember one of the properties of the locked posts is that whatever is kept inside the locked box Bob cannot see its content until and unless he gets access to the keys of the box.

The second property that I require from this locked boxes is that once Alice has kept something inside the locked box and if later, she is asked to reveal the contents of any of the boxes she later cannot change the content that she has already kept inside that particular box. So in this case Bob is challenging Alice to open the box i and box j and Alice is forced to give the keys for the box i and box j and as per the properties of the lockbox whatever she has committed inside the ith box and the jth box that she is not allowed to change.

Now Bob will verify whether indeed Bob will have take the keys for ith box and the jth box will open those boxes so they are the new colors with respect to the new 3 coloring for the graph and Bob will consider it to be a successful round if it finds that the new colors of the node i and the node j are different which should ideally be the case. If this is not the case, then the round is unsuccessful and Bob stops the protocol here itself. So this is how one round of the zero knowledge proofs system works.

Now to boost the confidence in this proof what Bob can do is Bob can repeat this process k number of times independently when in each round where in each round Alice can use a different 3 coloring with respect to the old 3 coloring techniques that means every round she will be picking an independent pi mapping the existing 3 coloring to a new 3 coloring and independently Bob will be picking fresh challenges i, j in every round.

That means it will not be the case that i, j which is picked as the challenge the edge i, j picked by picked as a challenge by the Bob will remain the same in all the rounds. They will be picked independently, and Alice will not be knowing anything about the challenges for the subsequent round in advance and if all this k rounds are successful Bob will consider that indeed Alice knows the actual or the original 3 coloring for the existing graph.

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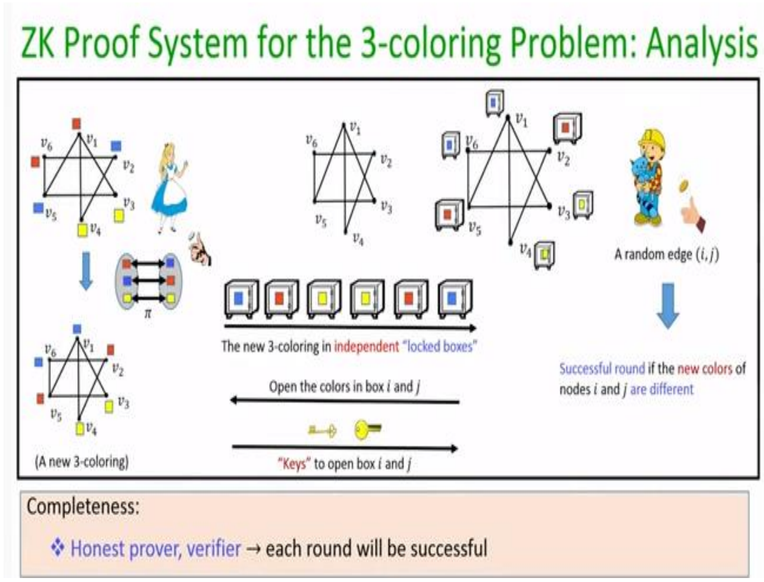

So that is a zero knowledge proof system. Now let us try to do the analysis for the zero knowledge proofs system whether it satisfies the requirement of correctness, soundness and zero knowledge. So correctness or completeness so remember completeness property means that if Alice and Bob are honest and if indeed Alice has a witness for the graph namely she has the original 3 coloring dench with high probability the proof should go through.

And Bob should be convinced and its easy to see that if indeed Alice knows the original 3 coloring then indeed in all the rounds she will be able to successfully convince Bob she will not fail and hence each round will be successful and Bob will accept the proof.

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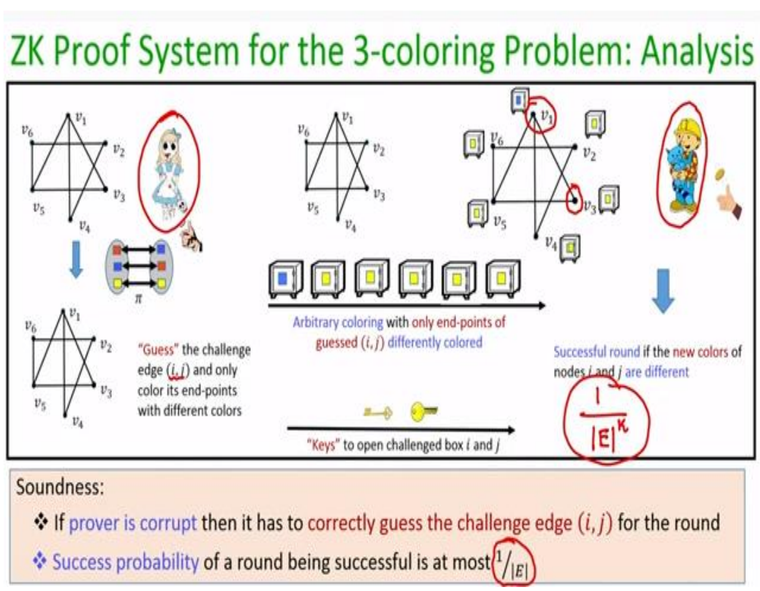

Now let us consider the soundness property. So remember for soundness property we have to consider the case when the prover is corrupt, and prover is corrupt in the sense that either the graph is not 3 colorable or it might be the case that the graph is 3 colorable but Alice does not know what exactly is the 3 coloring of the original graph. So for simplicity and without loss of generality assume that the graph is not 3 colorable.

So now let us analyze what is the probability that any round is successful with respect to a potentially corrupt Alice who does not know or who for a graph where the graph is not 3 colorable. So turns out that if Alice is corrupt and Bob is honest then the only way Alice can still successfully pass the round is when she can guess in advanced the edge i, j which is going to be picked as the challenge by Bob.

Because if the graph is not 3 colorable then Alice cannot create any new 3 coloring for the graph because the graph is not 3 colorable at the first place then the only way Alice can win is that she can guess she can pretend in her mind that this might be the edge i, j which Bob can challenge me to show. So what she can do is she can just do some arbitrary she can assign arbitrary coloring to the end points of the graph it in fact she can assign same colors to all the nodes in the graph except the end points i and j.

Because Alice can just guess that this might be the edge which Bob can ask me to open. So what Alice can do for instance she can guess that i might be a say 1 and j might be equal to anything say semantics number 3? So for instance she can imagine that she might be challenged to show the new colors for the graph for the edge v1, v3 or for the end points v1 and v3 so what she can do is for the locked box for the first locked box she can keep the color blue and the third locked box she can keep the color yellow and then in all the other boxes she can just keep the same colors right.

And if indeed she is lucky right then it might be the case that she is indeed asked to open the first locked box and the third locked box and if indeed she is asked to open the first box and the third box then she will be successfully showing Bob that hey I have assigned different colors to the node number v1 and node number v3. But what is the success probability of Alice guessing in advance that what will be the random i, j which Bob will challenge Alice to open right.

So the probability that Alice is successfully able to guess the random challenge i, j is nothing but 1 over the size of the edge set and approximately the size of edge set in the worst case can be n square that means success probability of the round being successful is bounded by 1 over the edge set namely 1 / n square and remember that there are k such rounds that means the only way Alice without even knowing the 3 coloring of the graph can successfully pass all the k rounds is when for each of the k rounds she can guess correctly in advanced that challenge i, j which Bob will be asking in each round.

And the success probability of guessing that in one round is 1/e so the probability that she can do it in all the k rounds is nothing but 1 over the size of the edge set raised to the power k and by setting k to be sufficiently large it can be ensured that this quantity 1 over the cardinality of edge set raised to power k becomes very small and hence definitively in one of the k rounds Alice will get caught and if she gets caught in any of these k rounds Bob will suspend the proof system and the claim of Alice will be rejected. So that proves the soundness property.

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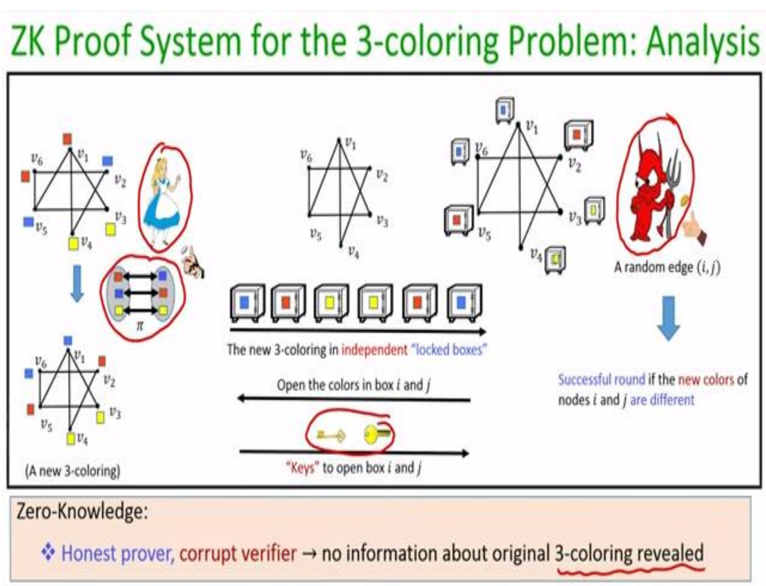

Now let us analyze a zero knowledge property and remember for the zero knowledge property we have to consider the case when our prover namely Alice is honest and indeed, she has a witness which she want to hide from a malicious verifier. So in this case the verifier is the corrupt guy and the goal of the verifier is to try to learn about the original 3 coloring. It turns out that even if the verifier is caught up it learns absolutely no information about the original 3 coloring.

This is because what he is seeing in the first round he is seeing the new 3 coloring but not the exact new 3 coloring but rather the verifier is seeing the new colors assigned to the vertices in the graph all of which are kept in the locked boxes and when the and it is only a pair of lock boxes which is opened by Alice in the third round of the in response to the challenge thrown by our verifier right.

So even if the verifier sees the color of the node i and the node j they are the new 3 coloring they correspond to the new 3 coloring and it learns only the colors for the ith node and the jth node but not the entire new 3 coloring and remember in each of the rounds Alice is picking a fresh independently chosen pi, right? That way she is randomly permuting the existing 3 coloring and creating a new 3 coloring.

So that means the new 3 coloring in the first round will be independent of the new 3 coloring in the second round and like this the new 3 coloring in the kth round will be independent of all the new 3 colorings and the previous round that means in each of the rounds verifier is just going learn that okay I will be seeing the new colors of the node i and no j which I know are going to be distinct and that is why it does not reveal anything about the original 3 coloring which was available with Alice. So that proves the zero knowledge property.

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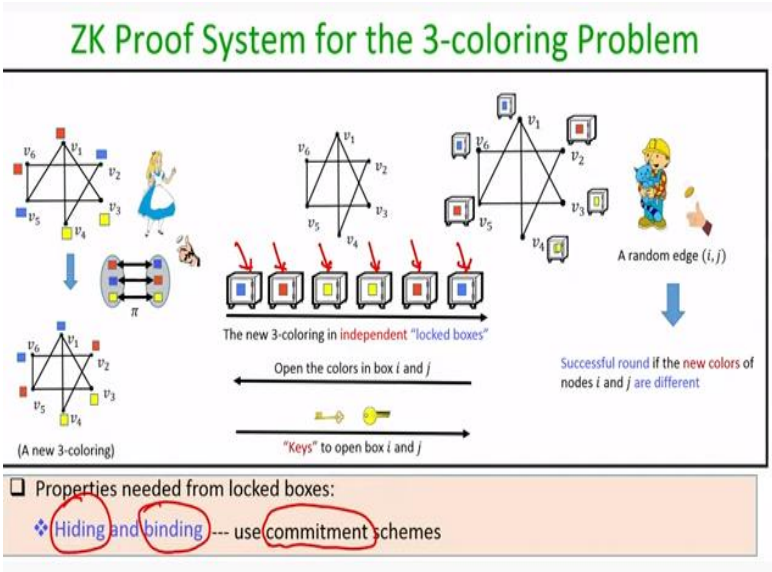

Now coming back to the question that what are the properties we need from the locked boxes? so as I said we need 2 properties we need basically the hiding property the namely if prover is honest namely if Alice is honest and if she has kept some contents inside the box then until and unless the keys for those boxes are given to Bob, Bob cannot open and see the contents that are kept inside the box.

So that is what I mean by the hiding property here and binding property means if Alice is corrupt then it should not be the case that she put something inside the box but when she is supposed to open the box, she can turn it or she can change it into any new content. So that is a binding property and now we will very well know how to instantiate these locked box both with both these 2 properties.

Basically we can use a commitment scheme so that means what Alice has to do is in each round she has to compute a new 3 coloring and a new colors for the vertices she has to commit by using any commitment scheme and when Bob challenges to open the new colors of ith node and the jth node Alice has to give the opening information corresponding to the ith commitment and the jth commitment.

So this proves that we now have a 3 coloring we now have a proof systems we now have a zero knowledge proof system for the 3 coloring problem and its well known that the 3 coloring problem is an NP complete problem that means now we have a zero knowledge proofs system for any NP relation.

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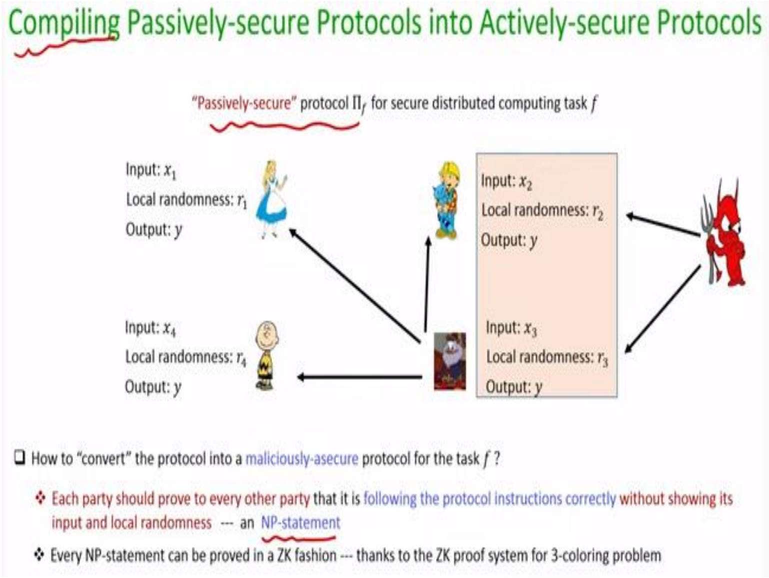

Now let us see the power of the zero knowledge proof system so the what we can now do is we can see a very nice framework namely a compiler which can compile any passively secure protocol into actively secured protocol. So imagine you have a distributed computing task say little f it can be any abstract computational task it could be say for example a task involving say multiple parties 2 parties, 3 parties or say 4 parties or any of them in number of parties where each party have some input say x1, x2, x3, x4.

I am taking the case where I have 4 parties and the goal of the parties is basically to compute f of x1, x2, x3, x4 and in such a way that even if there are some bad guys in the system they do not learn anything about the x inputs of the good guys other than what they can learn from their own input and the function output. So this is a very abstract problem this problem you also call us multiparty computation problem.

So the way any multi-party computation protocol will work as so the parties will have their own inputs and they will choose to some local randomness say r1, r2, r3, r4 respectively and then they will interact with each other as per the instructions of this protocol pi sub f and at the end they will obtain the function output y where y = f of x1, x2, x3, x4 and for the moment imagine that this protocol pi sub f is passively secure its passively secure.

In the sense that if even if there is an adversary who can control or who can see the input and the local randomness of some fraction of the same parties and whatever messages they have exchanged during the protocol still by seeing their inputs the output and the messages that they have received and they have sent during the protocol the bad guy does not learn anything additional about the inputs of the good guys.

So that is what I mean by saying that this protocol pi sub f is passively secure now imagine I want to compile this protocol compiling this protocol in the sense I want to retain the protocol pi sub f and want to ensure I want to ensure that the protocol remains secure even if there is an active adversary or a maliciously secure adversary active adversary that means I want to compile this protocol into a maliciously secure portal sorry for the typo it should be maliciously secure protocol.

So what I mean by malicious security here is that even if the bad guys who are under the control of their adversary tried to deviate from the protocol try to deviate from the instructions of the protocol they should not learn anything about the inputs of the good guy and I do not want to design a new protocol a fresh protocol I just want to retain the protocol pi sub f which is guaranteed to be secure against the passively secure passive adversary.

So how can we compile the passively secure protocol into a maliciously secure protocol? That is a question that we want to now answer and now want to be wanting to use a zero knowledge proof system here. So it turns out that a generic way to convert the passively secure protocol into a maliciously secure protocol is as follows. If each party proves to every other party that it is indeed following the protocol instructions correctly then indeed even in the presence of a maliciously secure adversary, the protocol p sub f will be achieving its task.

Now the question is what do we mean by saying that a party proving to other party that indeed it is following the protocol instruction correctly by that I mean that each party has to prove to every other party that the messages that they are sending are indeed with respect to their randomness and their input and as per the instructions given the protocol p sub f and how can the other parties verify whether indeed each party is following its protocol instruction or not? Well it can check the messages that the parties that particular party is sending and the witness whether that party is sending or performing its action properly or not will be the parties input and the local randomness.

For instance, in this case if Alice wants to verify whether indeed this third party is following his protocol instruction correctly or not then one way of verifying that is Alice checks the messages which this third party is sending and along with that if this third party shows his input x3 and his randomness r3 which he has used as part of this protocol pi sub f to Alice then indeed Alice can performs the action of this third party.

Because the description of the protocol is known what was not known was x3 and r3. But now x3 and r3 is also given to Alice so she can her self-compute the messages which this third party is supposed to send as per the protocol pi sub f and if those messages matches the messages that indeed this third party has sent or communicated during the real execution of the protocol that proves that indeed this third party is following its step as per the protocol pi sub f and like this every party can verify every other party's action whether they are performing their actions as per the protocol pi sub f if that sending party reveals its input and local randomness.

But it turns out that the input and the local randomness of every party cannot be given to other parties because that is what ensures the security of the protocol pi sub f if I learn the input and the randomness of every other party then there is no way I can guarantee the security of the protocol pi sub f. So it turns out that the statement which every party wants to prove to every other party namely I am following the protocol instruction is nothing but an instance of an NP statement.

The problem instances the set of messages that I have sent, and I want to prove to you that corresponding to these messages I have some randomness and some input such that these messages are indeed computed consistently as per those input and the randomness as per the protocol instruction pi sub f. So what each party now has to basically do to convince to other party that it is indeed following the protocol instruction it has to basically prove an NP statement.

And interestingly we now have a zero knowledge proof system for the 3 coloring problem which is an NP complete statement and since it is NP complete statement acmes any instance of NP problem or any NP statement can be reduced to an instance of this 3 coloring problem. That means we can now use the zero knowledge proofs system for the 3 problem, right? And each party can transform an instance of the NP statement namely that it is following the protocol instruction correctly into an instance of the 3 coloring problem and give us zero knowledge proof for the existence of 3 coloring and convince to the other parties that indeed it is following the protocol instructions.

If the proof gets satisfied that means that gives the guarantee that every party is following the protocol instructions correctly if the proof does not go through, we can simply stop the protocol there itself. So in that sense the zero knowledge proof system it gives you a very powerful paradigm of compiling a passively secure protocol into a maliciously secure protocol. So that brings me to the end of this lecture.

Just to summarize in this lecture we have seen zero knowledge proof system for the 3 coloring problem and 3 coloring problem is a well-known NP complete problem and we have seen that how using zero knowledge proof system we can compile any passively secure protocol for this for any distributed computing task into a protocol which will be secure even against a malicious adversary. Thank you.