Machine Learning for Engineering and Science Applications
Professor Doctor Balaji Srinivasan
Department of Mechanical Engineering
Indian Institute of Technology Madras
Expectation

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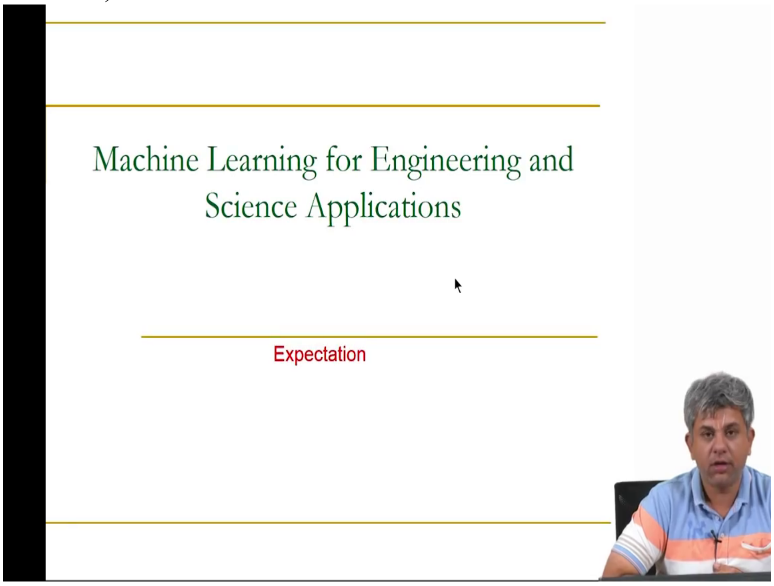

In this video we will be looking at a very simple statistical quantity called, or the statistical function called the expectation. You are familiar with expectation as

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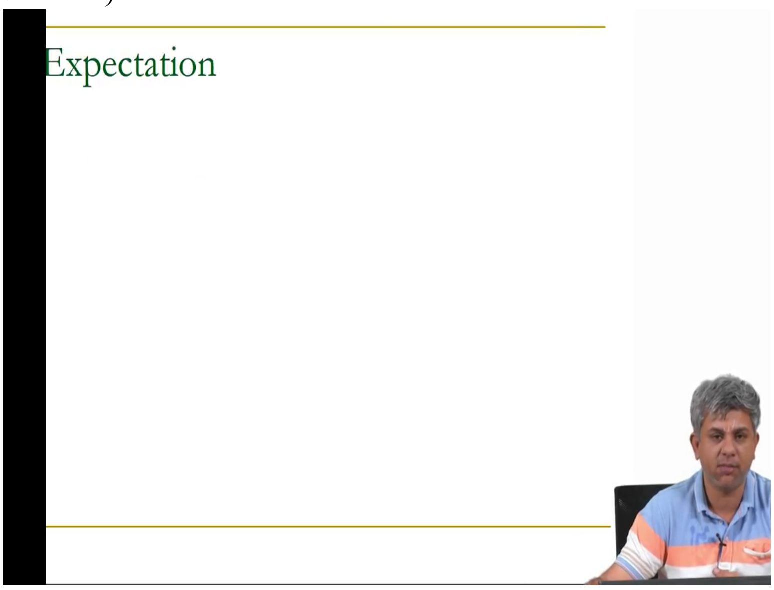

the mean or the average, Ok.

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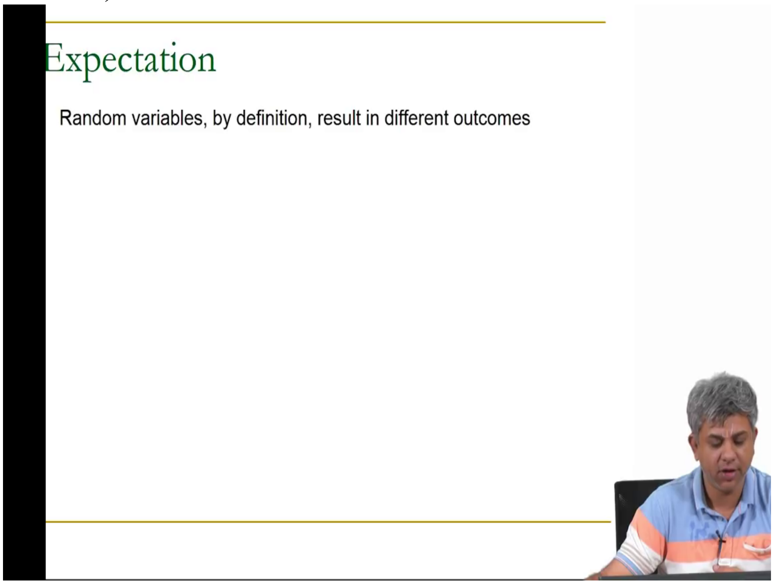

So the context is this. If you remember, we are dealing with random variables throughout. Random variables by definition will result in different outcomes.

If I throw a dice right now, sometimes it will give a 1, sometimes it will give a 2, sometimes it will give a 6. So this is obviously why it is called a random variable in the first place,

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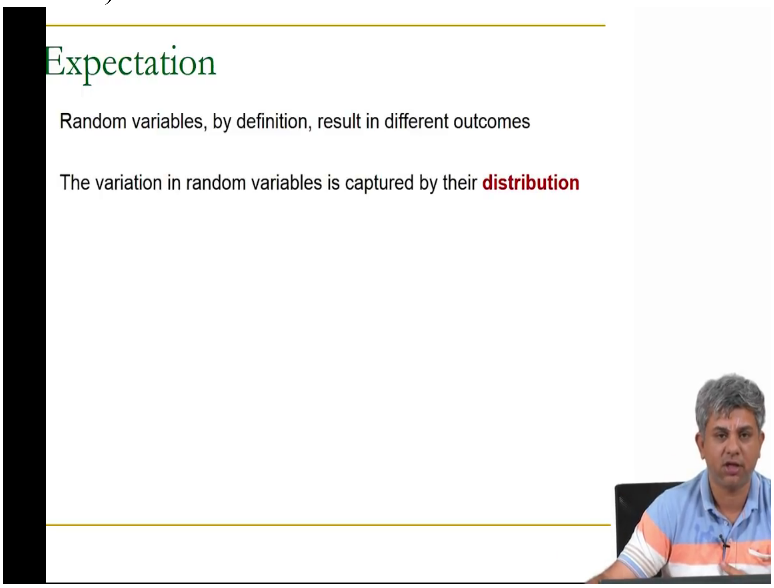

Ok.

So which way the random variable actually varies or how it gives different outcomes is captured by what is known as the distribution if you recall, if you have

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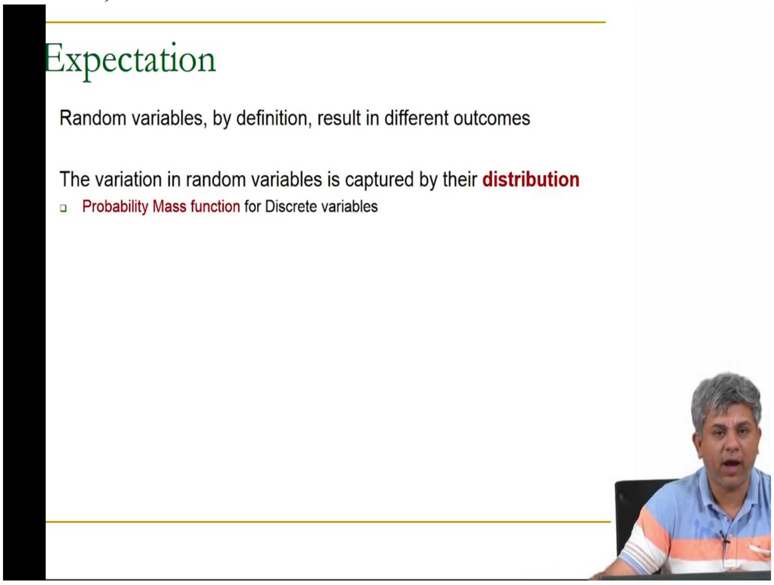

a discrete distribution such as a dice or a coin or a deck of cards, then we have something called the probability mass function that tells you how likely each one of these outcomes is.

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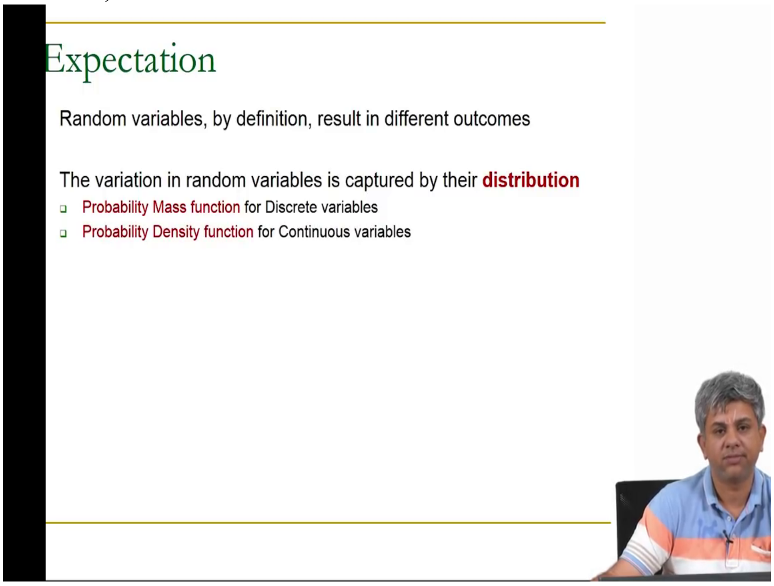

Similarly, for a continuous variable such as height, weight, temperature, pressure, stress, strain etc what you have is the probability density function that tells you how likely a range of probability is, a range of values is. So the probability that height varies between 5 point 6 and 5 point 7 is something, Ok. So that would be a probability

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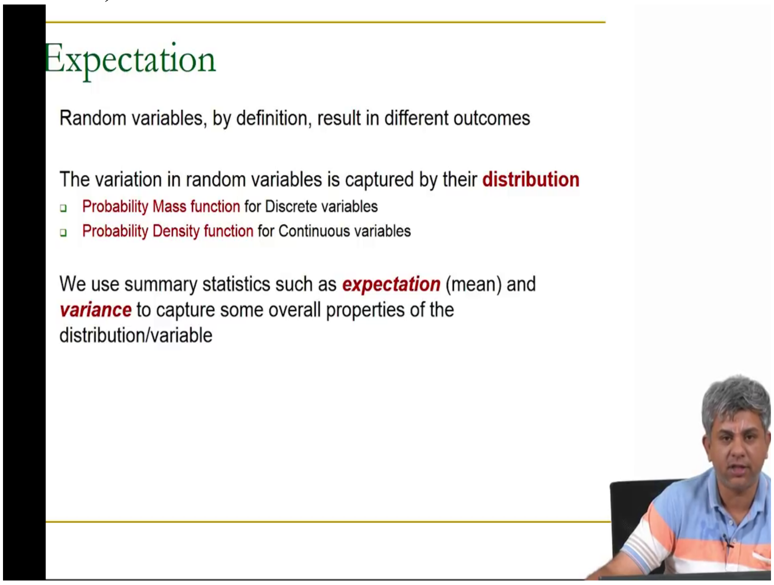

density function.

Now once you are given the distribution we start using some overall ideas, you know. It is a random variable. It has any number of, large number of values but you want to give some summary statistics, Ok. So you want to give some qualitative and quantitative picture of what the random variable is doing.

And the most common ones that we will be using at least as far as this course is concerned are two quantities called the expectation and the variance, Ok. The expectation is something that you are already familiar with. We usually call it the mean or the average. But in the context of random variables typically we tend to call it expectation, Ok.

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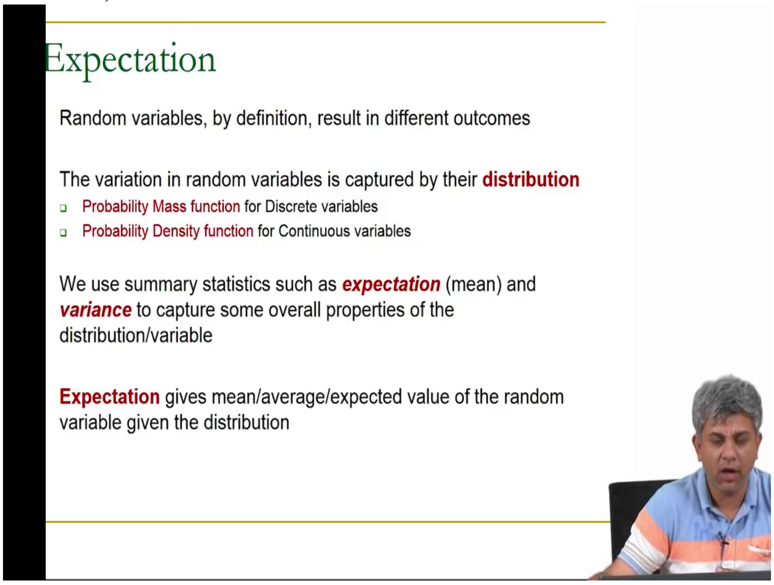

Now the expectation gives the mean average or expected value of the random variable

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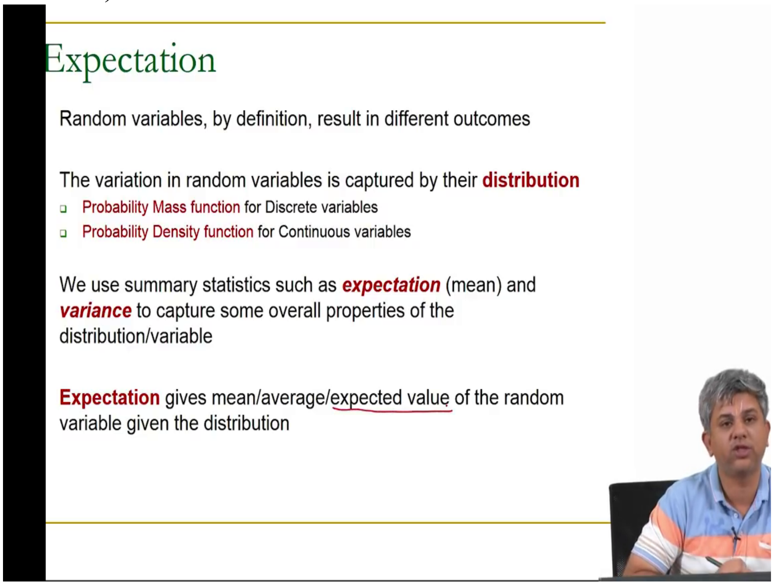

once you know the distribution, Ok, so that is important. You need to know what the distribution of the random variable is. Then you can find out the

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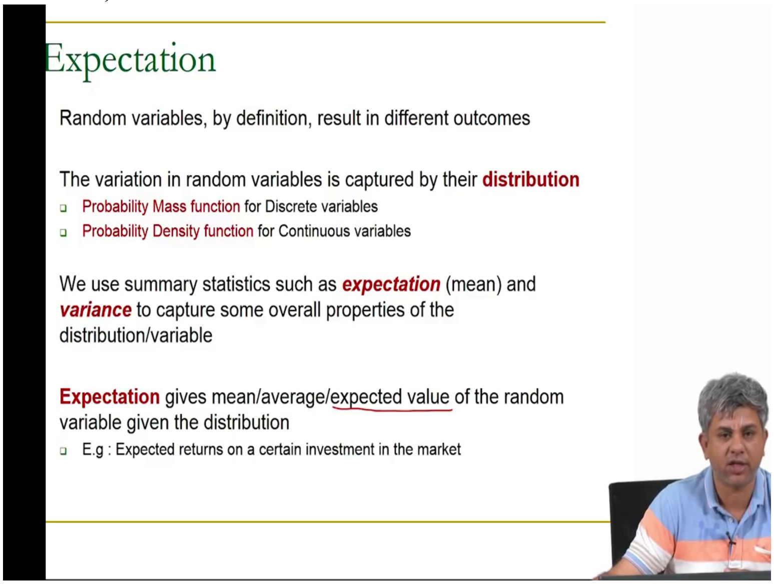

expectation.

So here are the couple of examples. You can say you know I have invested in the stock market. What are my expected returns? Obviously you know the returns are not fixed. It is the random variable. But nonetheless, given the certain investment what is my expected return from the stock market?

Another

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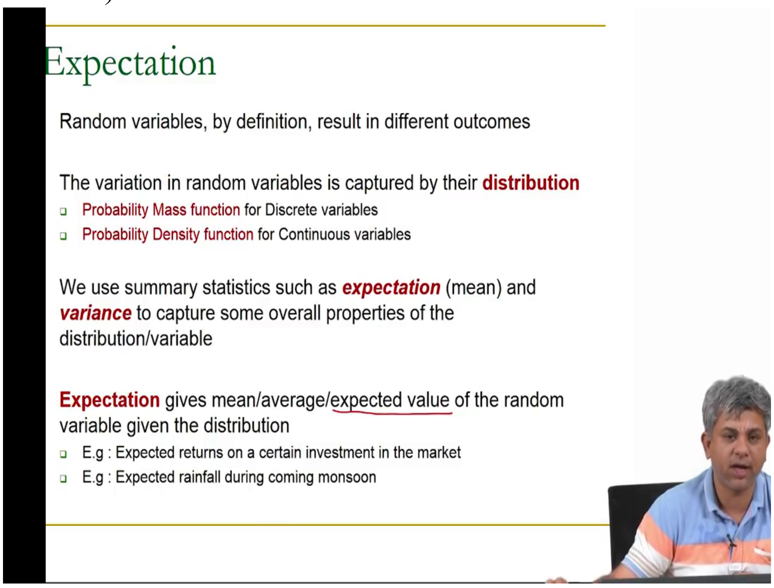

example is you know I know that monsoon is going to hit. What is going to be the expected rainfall during the coming monsoon? Ok. So you could ask questions of that sort. Again this is a random variable but overall you would still like to know what is going to be my expected crop yield, a farmer might be interested in knowing.

So here

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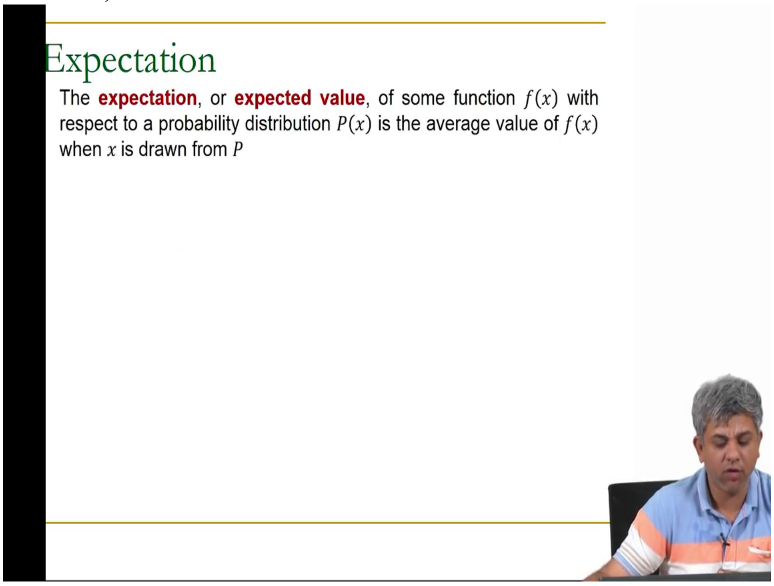

is how we define expectation, Ok. This is the average value.

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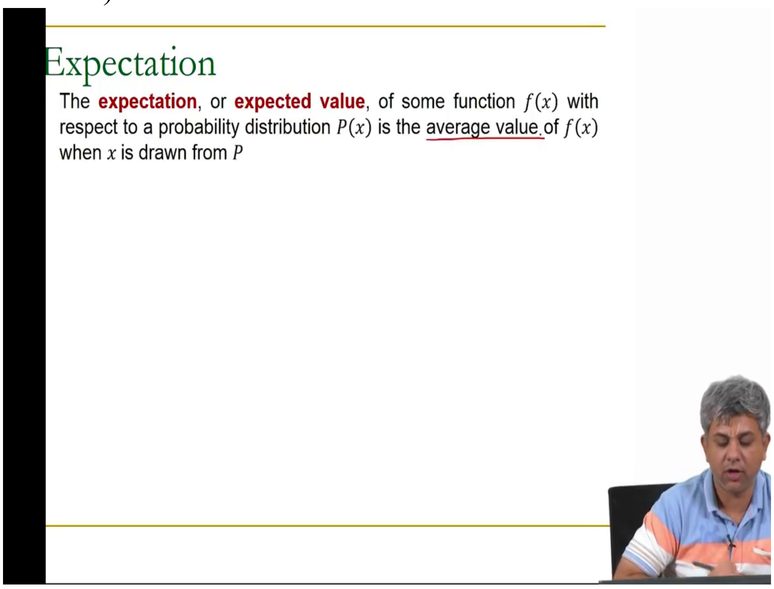

So right now I am not talking about only expectation of a random variable x, but that of a function of x. So the expectation or the expected value, both these terminologies are used for some function x of a random variable where x is a random variable,

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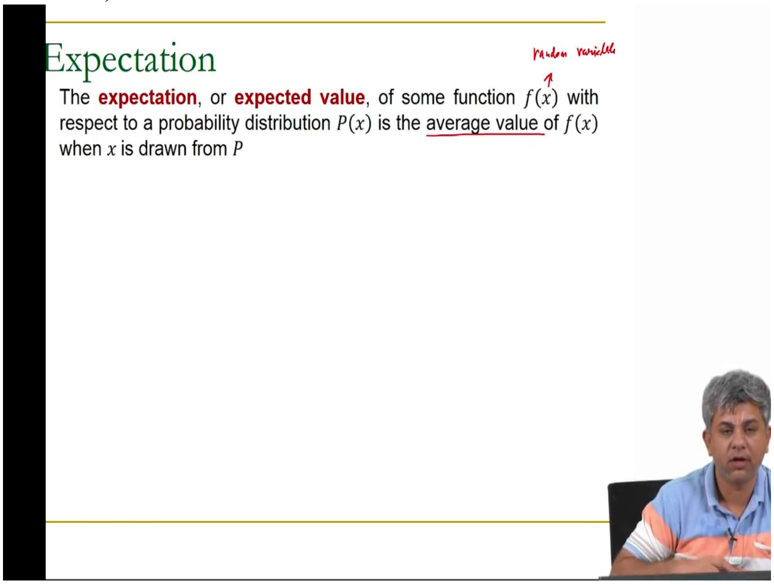

is the average value of f of x when x is drawn from a probability distribution P.

Recall the notation x is drawn from P is written as x tilde P.

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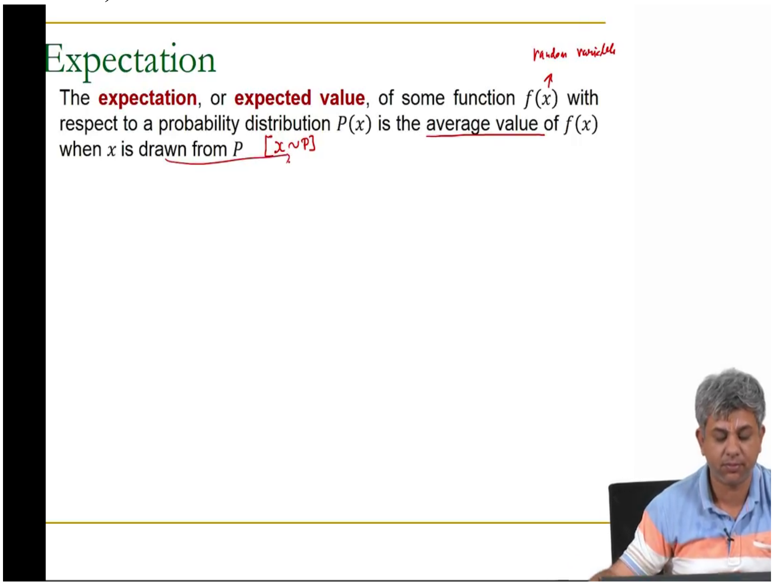

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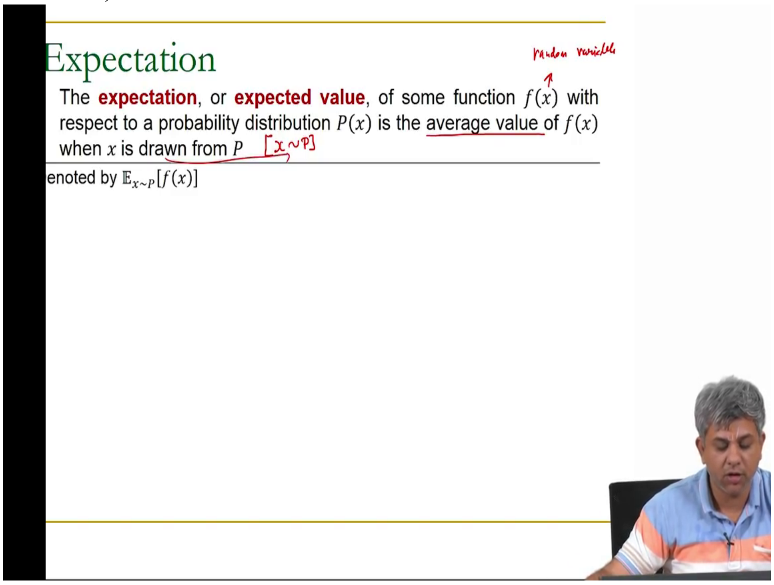

So the denotation is, how the notation that we use is x, expectation of f of x, notice the square brackets, expectation of f of x, because it is not quite a function so expectation of f of x given that x is

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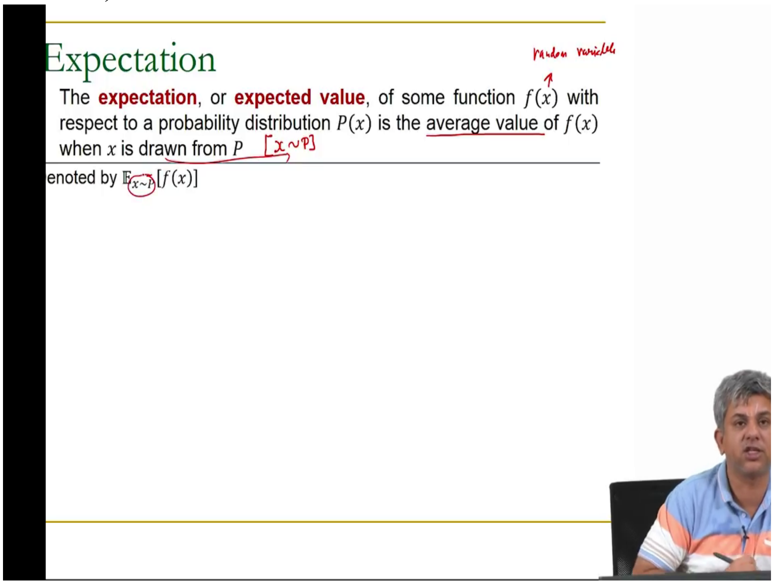

drawn from P, Ok so this is the notation that we will be using. This is the most detailed or rigorous notation,

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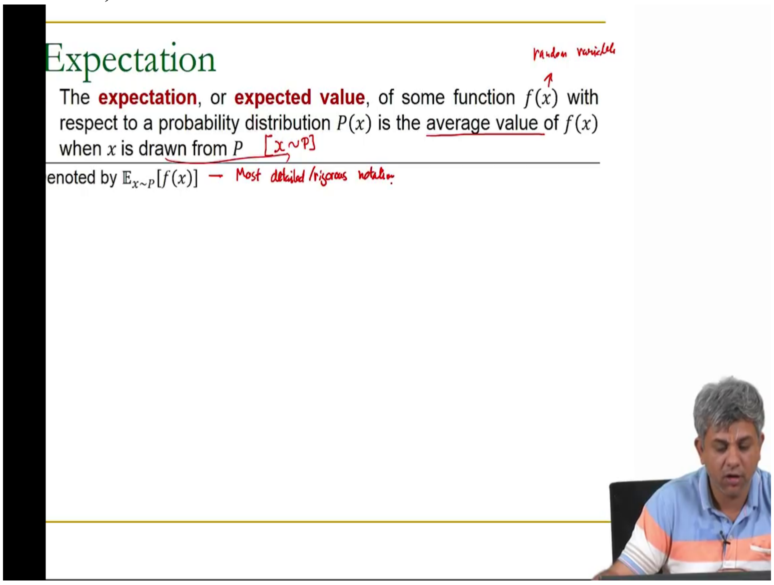

Ok.

But more often than not,

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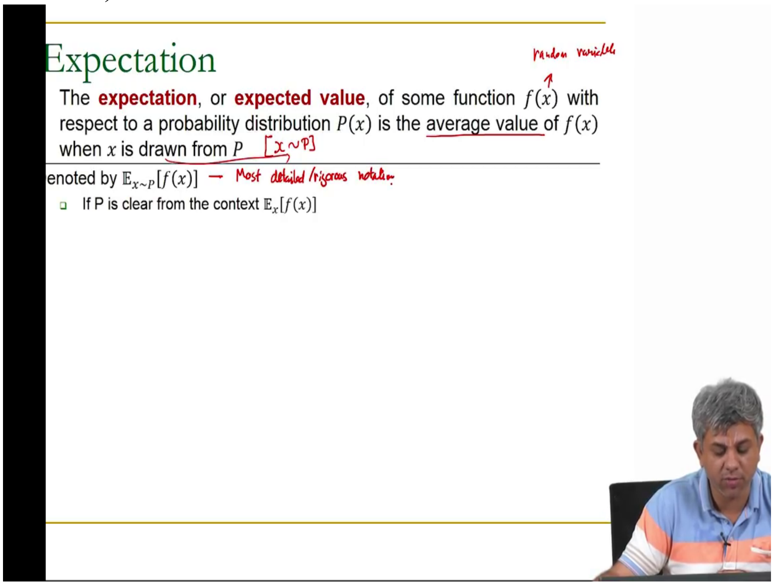

we use some shortcuts. If you know which probability distribution we are talking about, we simply say expectation of x given f of x.

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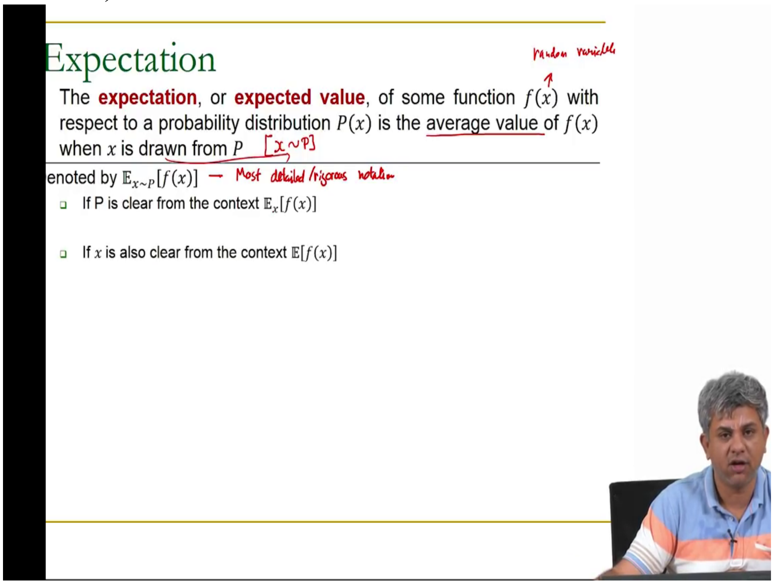

Even more simply if you also know what x is, you know which random variable x we are considering you can simply say expectation of f of x

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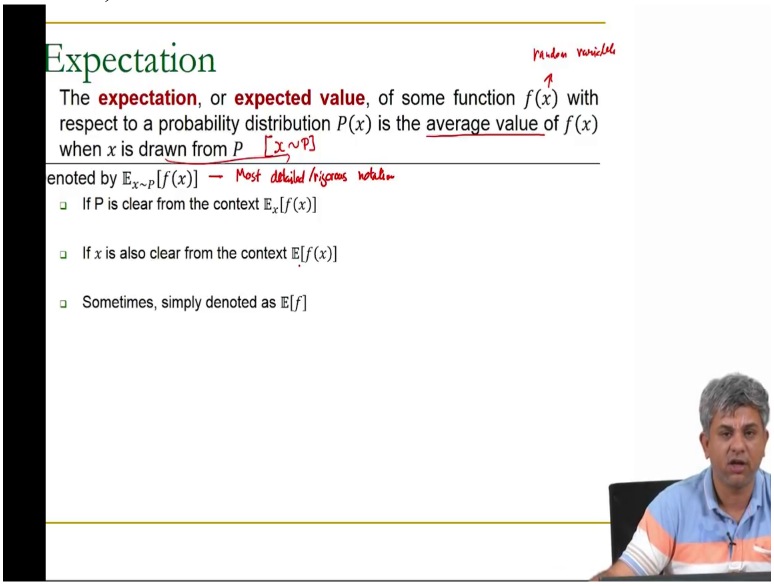

and sometimes you simply say expectation of f, Ok. So all these notations are used as far as expectation is concerned.

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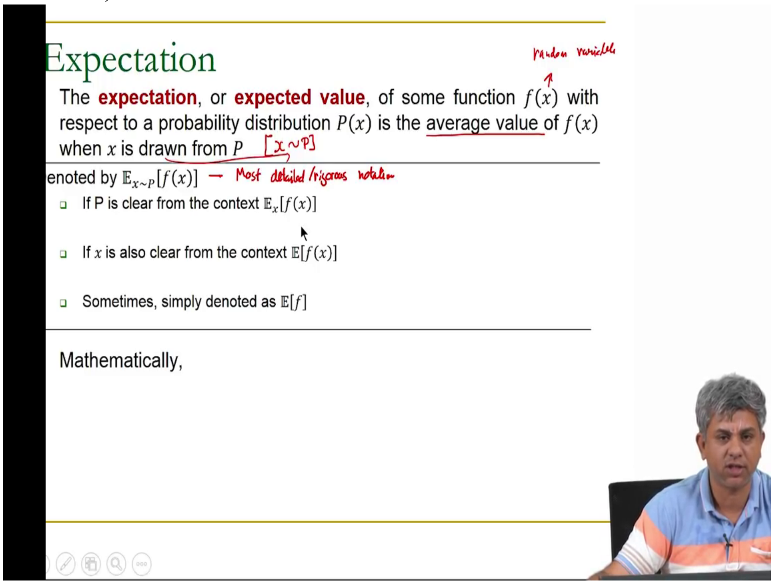

So mathematically how do we calculate expectation?

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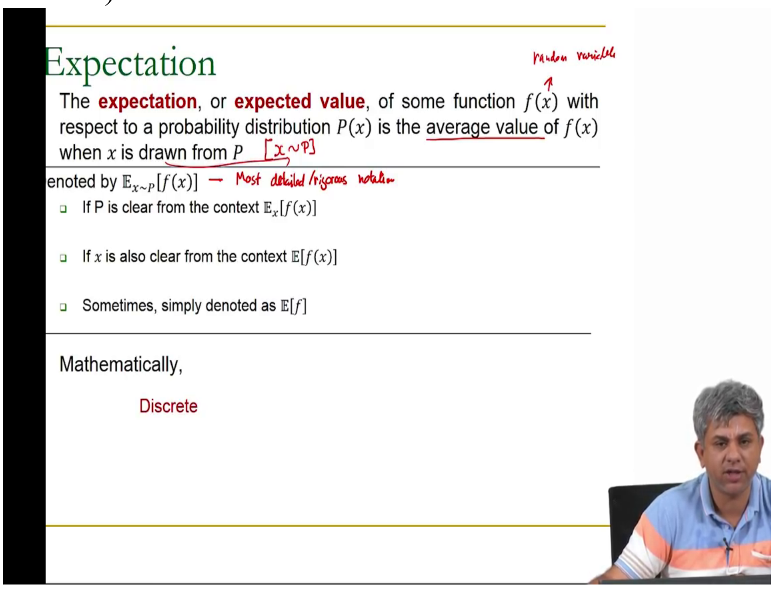

For a discrete

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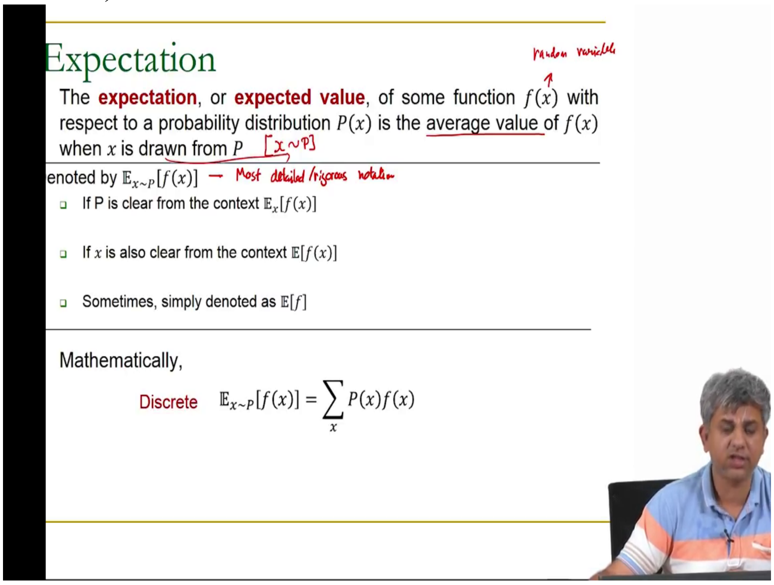

variable it is simply summation of P of f, P of x f of x, Ok and if it is a continuous

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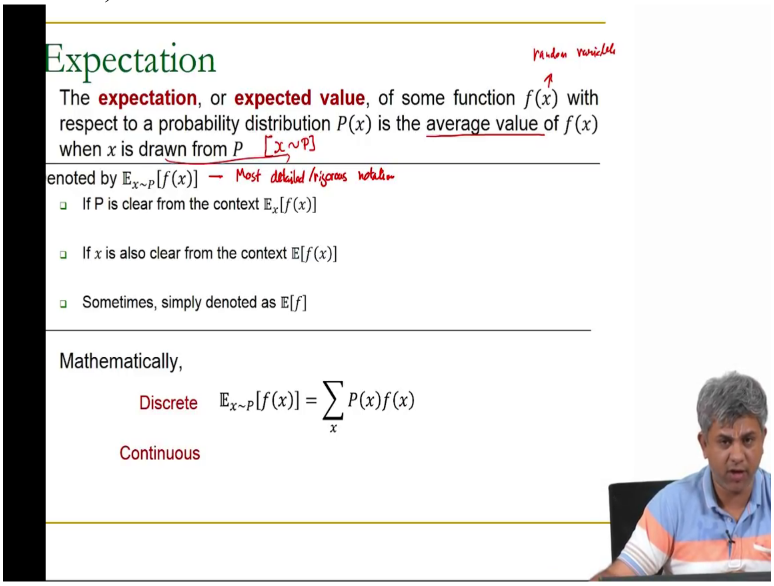

variable it

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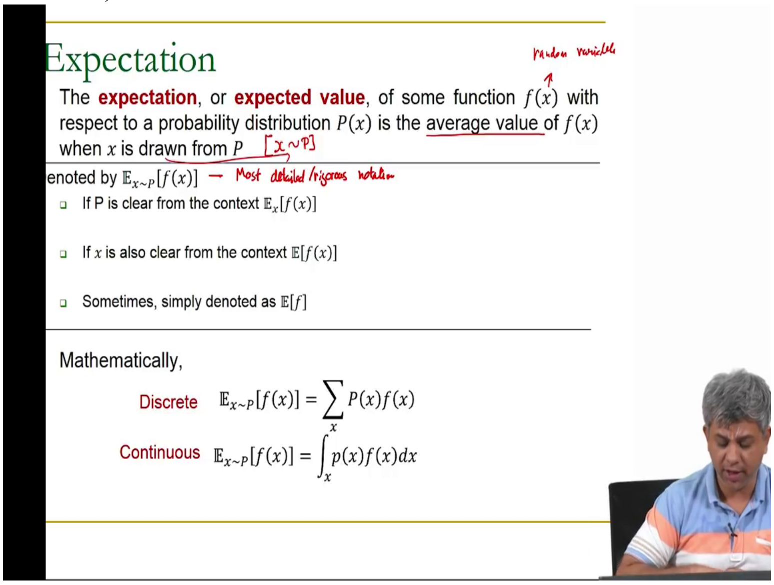

is, remember now this is a probability mass function.

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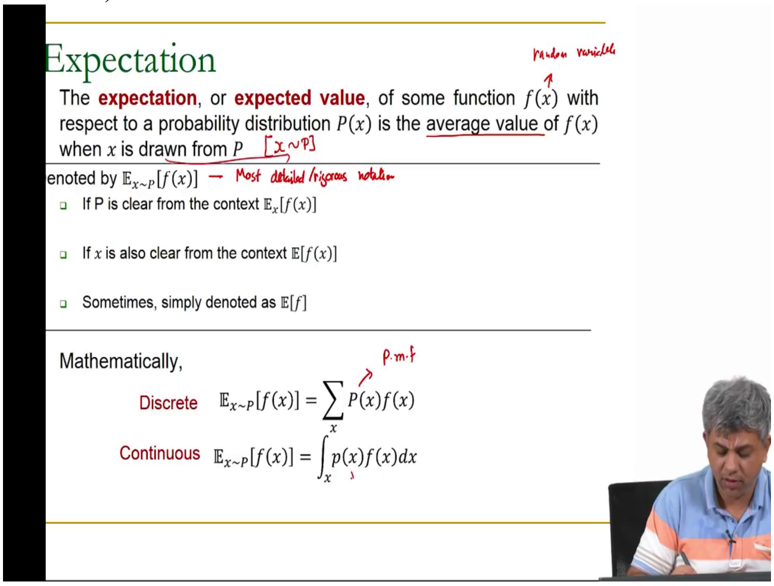

In this case it is a probability density function which is why

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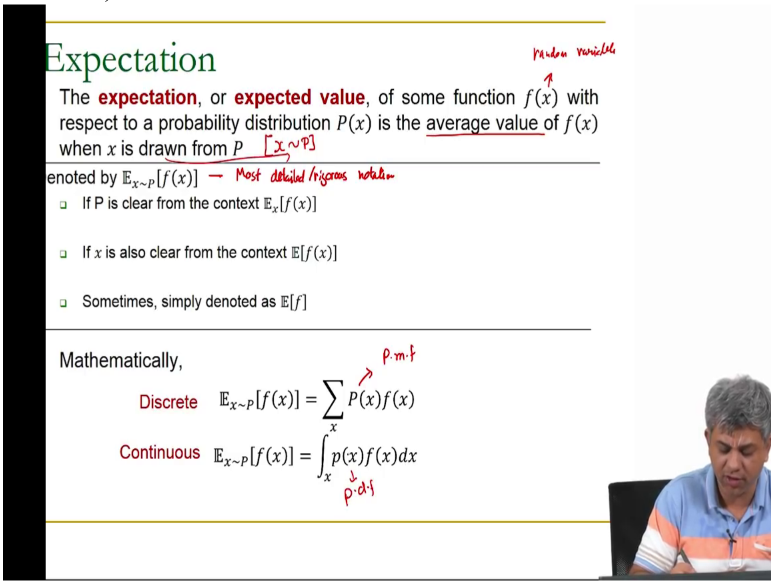

you have to multiply by d x in order to get the probability, so integral of P of x d x f of x integrated over all possible values. Now we will see a couple of

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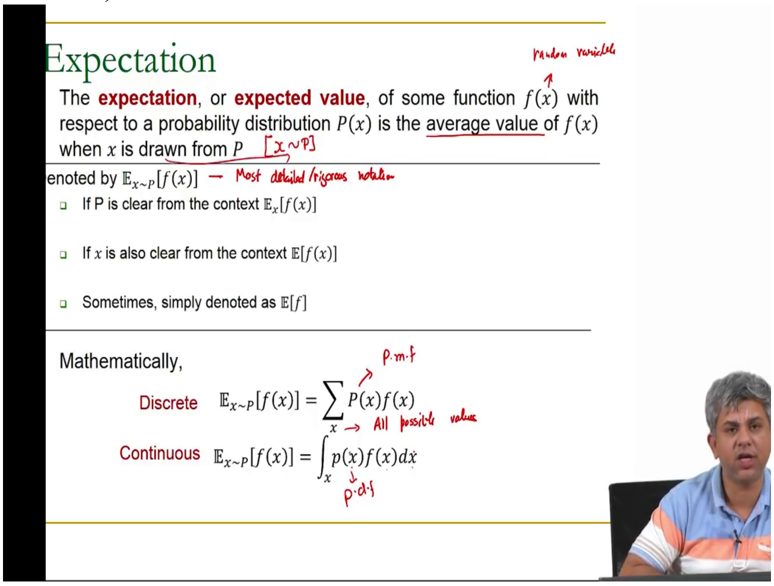

very, very trivial and simple examples on

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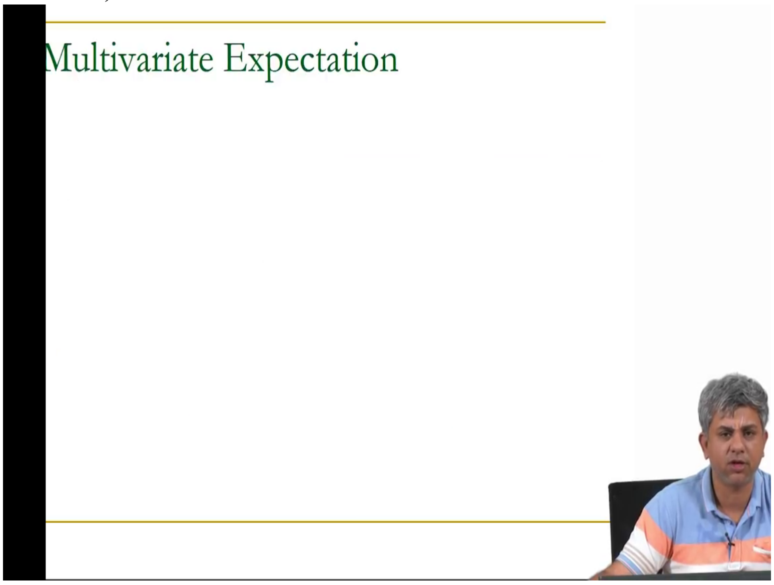

the next slide.

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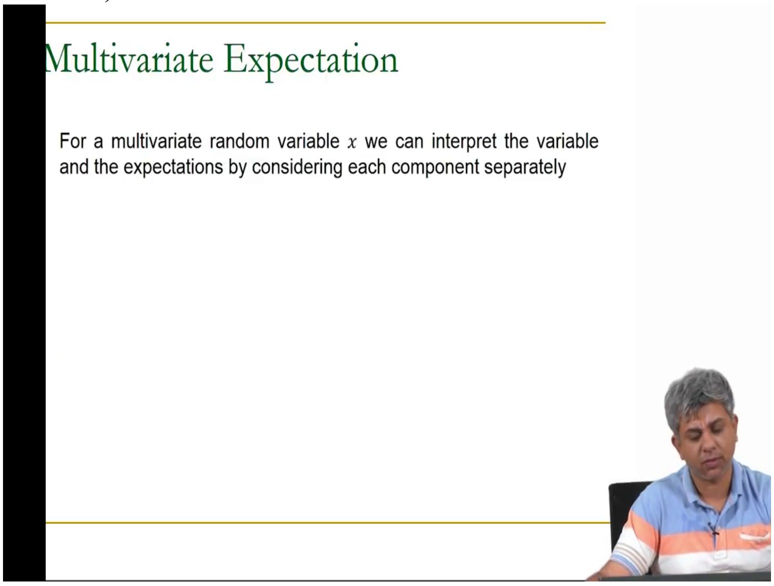

So a generalization of expectation which we just saw expectation of a single variable. But usually especially within machine learning we are dealing with vectors, Ok. So this is called multivariate expectation where basically x is now a vector consisting of x 1, x 2 so on and so forth up to x n,

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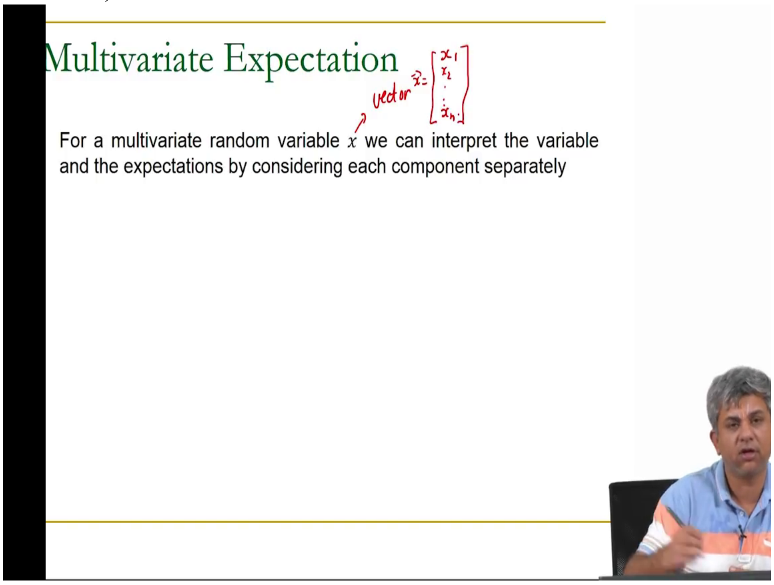

Ok.

As we saw in the previous video this could be a image, this could be temperature, pressure, velocity etc. It could be any number of variables, Ok. So if you have that then you can

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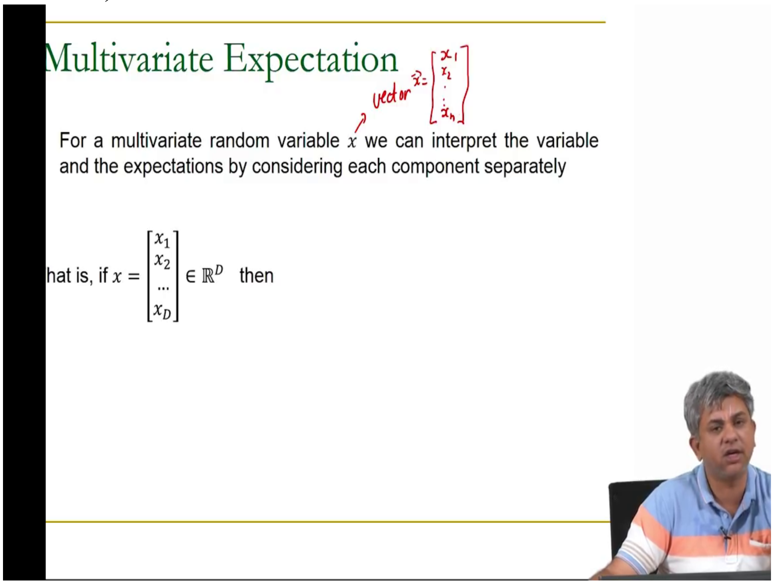

consider each component separately.

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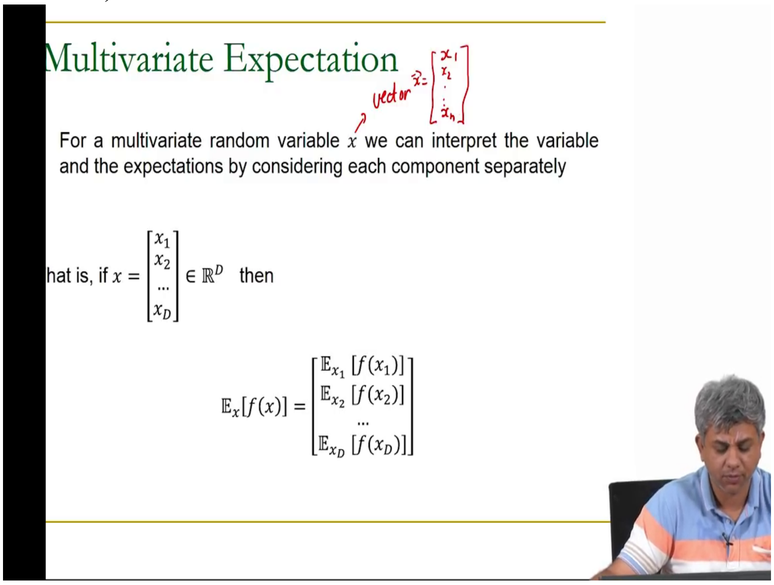

And all you will do is expectation of variable 1, expectation of variable 2, so on so forth, the reason why I wrote it here is notice

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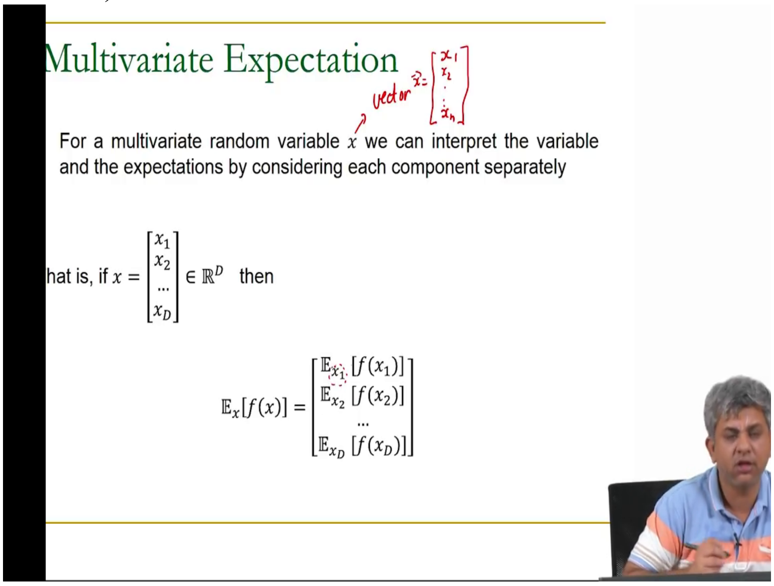

that the variable itself, the first one is expectation over variable 1, Ok.

So if I want, let us say my x vector is temperature is temperature, pressure, humidity

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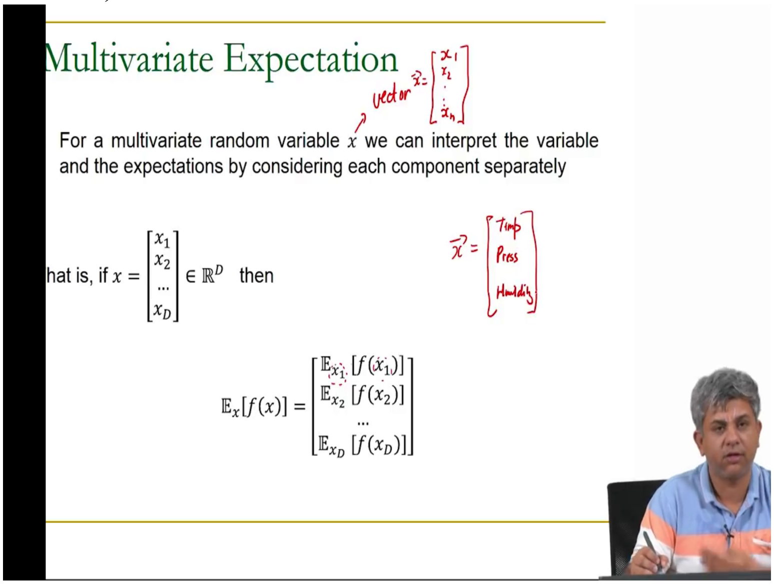

then I will take expectation over all possible values of temperature of, if you are interested in some function of temperature, so on and so forth, expectation over all possible values of pressure, function of pressure, Ok.

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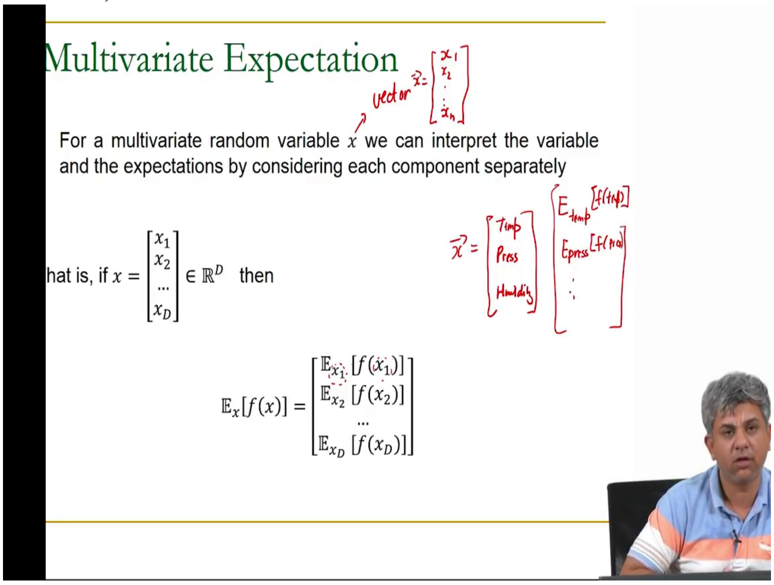

So that is multivariate expectation. Multivariate simply means multiple variables, Ok. It is not a single scalar, it is a vector.

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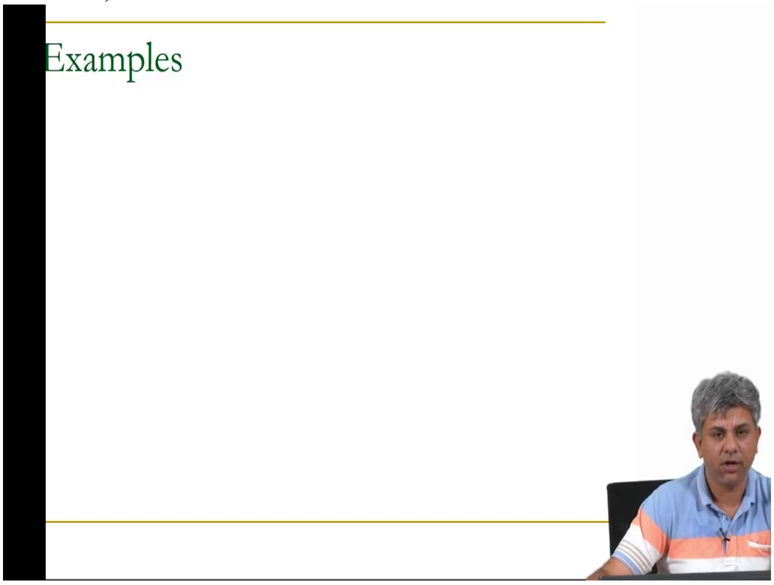

So here are some trivial examples. I am going to do univariate examples here

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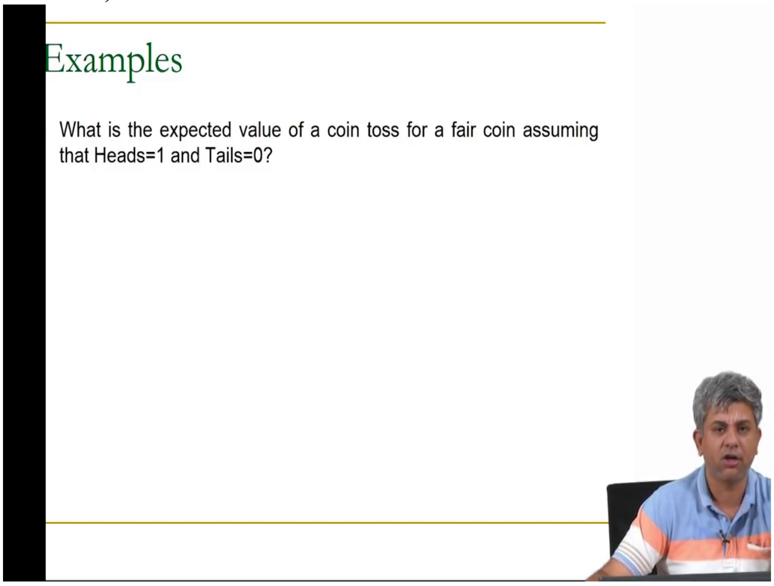

Ok.

So all of us know, let us say you want to know the expected value of a toss of a coin, for a fair coin assuming that heads has a value 1 and tails has a value 0,

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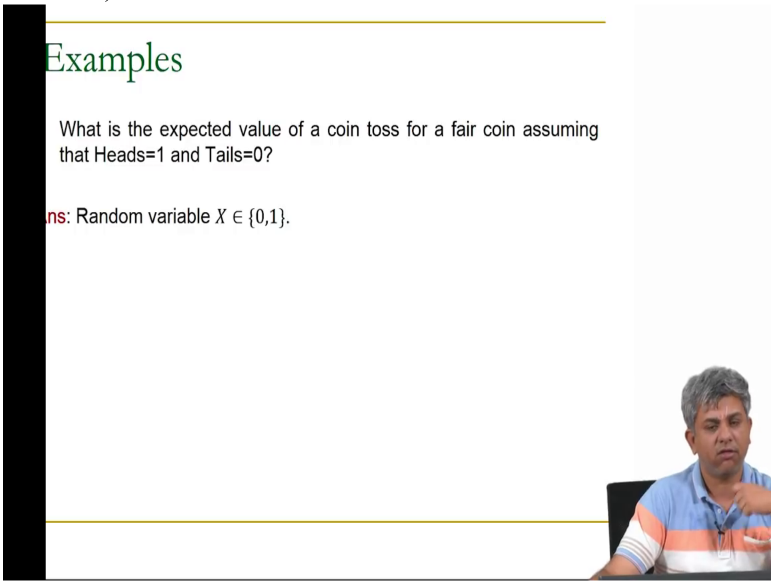

then I will just do it in detail so that you get used to this kind of calculation if you are not already used to it.

So you first identify the random variable you are considering. Here the random variable is the result of the toss of the coin and it now belongs to 0 1,

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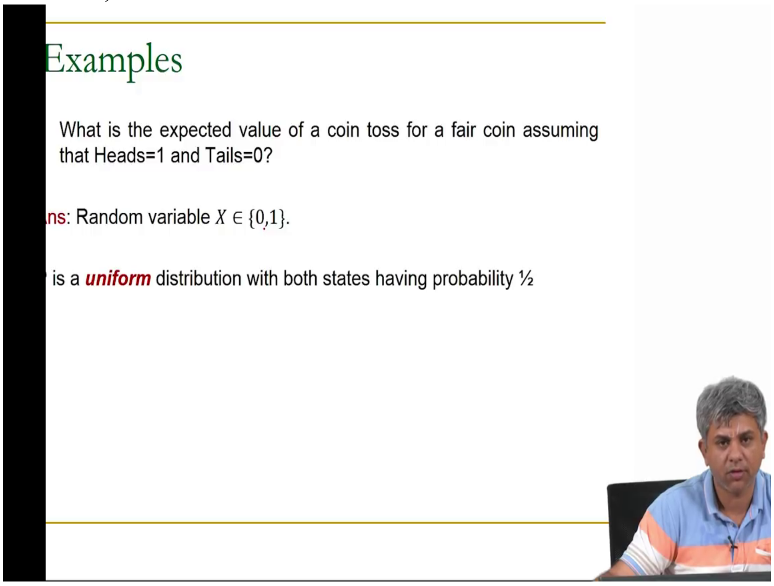

Ok. What is P, you now need to know what the distribution P is. So first identify random variable.

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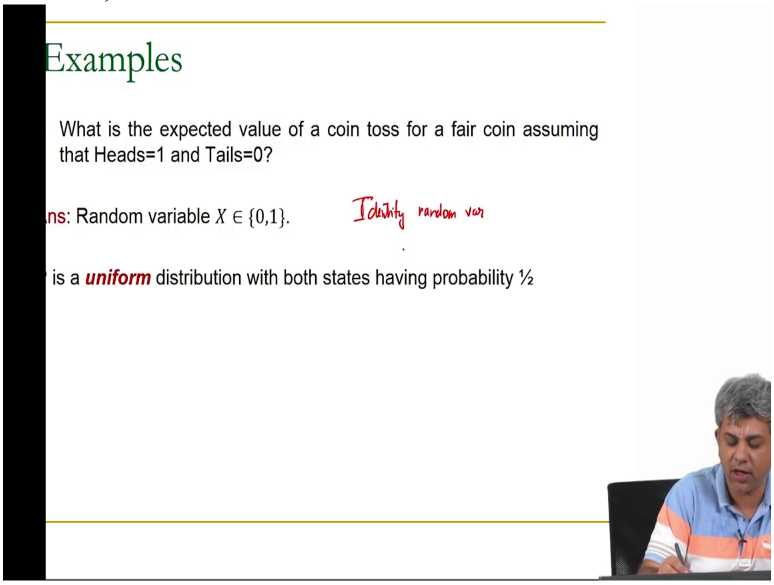

Next calculate the probability distribution. In this case it is a mass distribution

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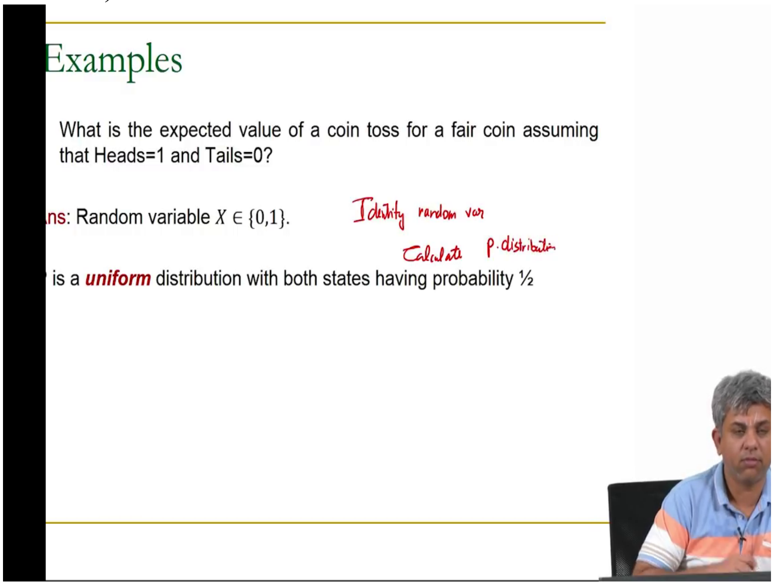

because it is a discrete random variable.

Now probability distribution is very simple. If I have x and P of x then x takes the value 0 with the probability half, x takes the value 1

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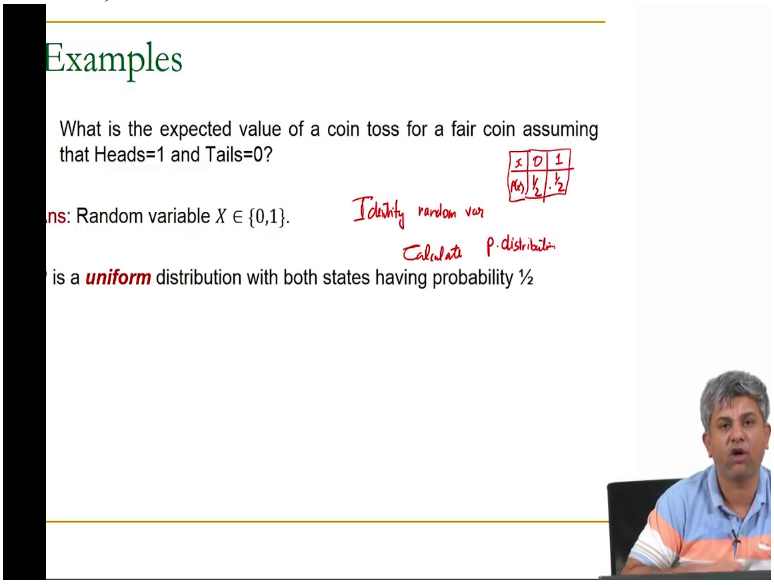

also with the probability half.

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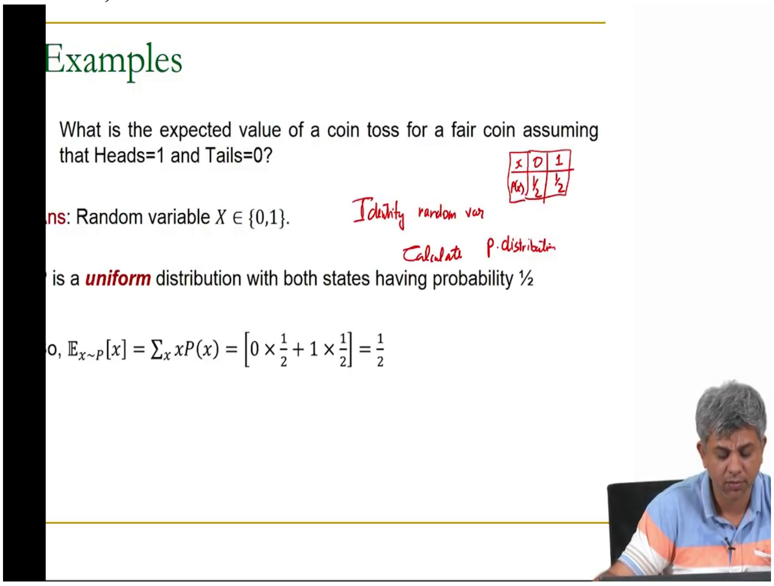

So if you find out the expectation, it is simply 0 times half plus 1 times half which is equal to half, Ok.

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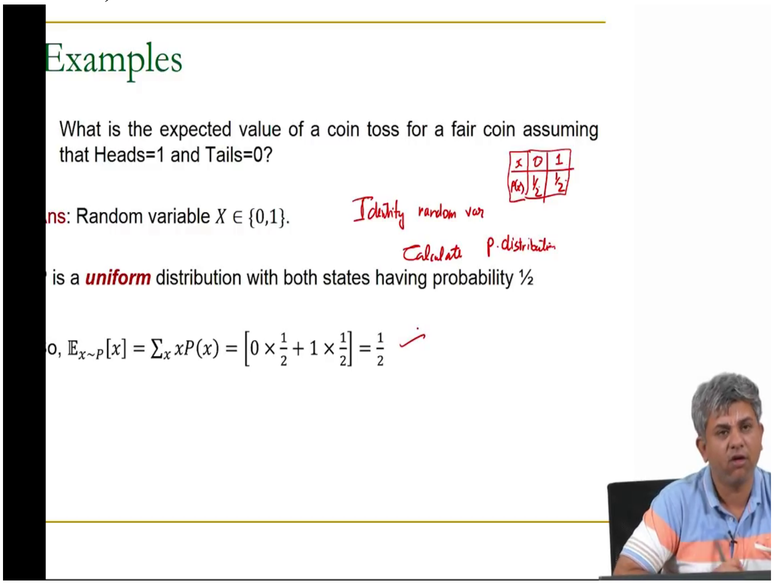

All of us know this. Another way to look at it is the average value of the toss that you will obtain is basically going to be half. So notice that the expectation, even though we have given it as the average value, or the expected value, obviously you cannot say that the expected value of a toss of a coin is half.

Because neither heads nor tails is the actual half. It just represents an average or a weighted average of values that come out, Ok.

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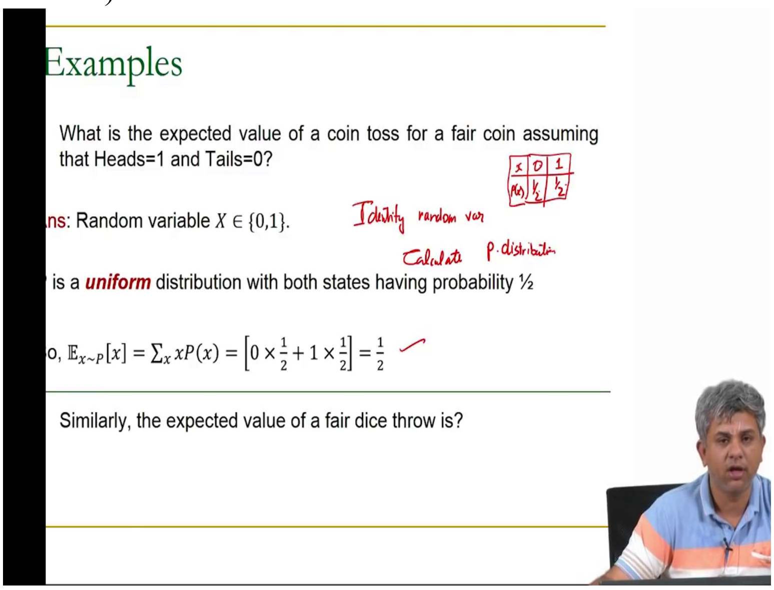

Similarly, if you want to find out the expected value of a fair dice

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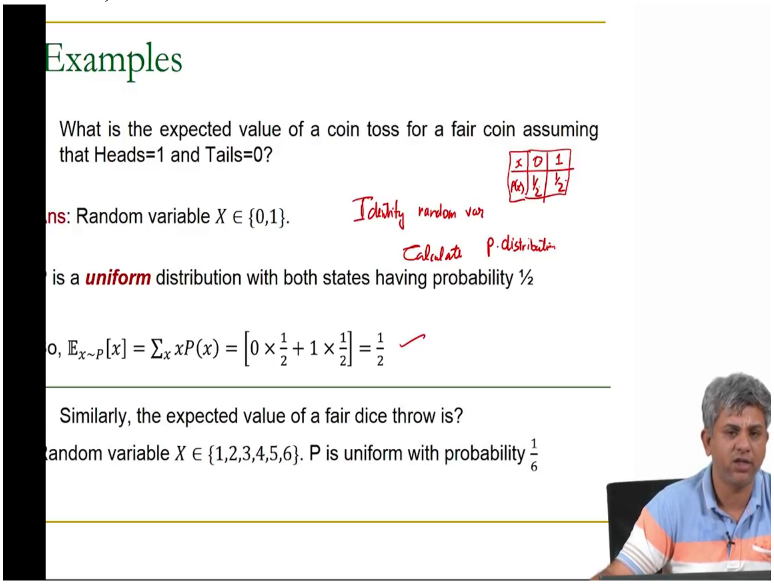

throw this is going to be the average of 1, 2, 3, 4, 5, 6.

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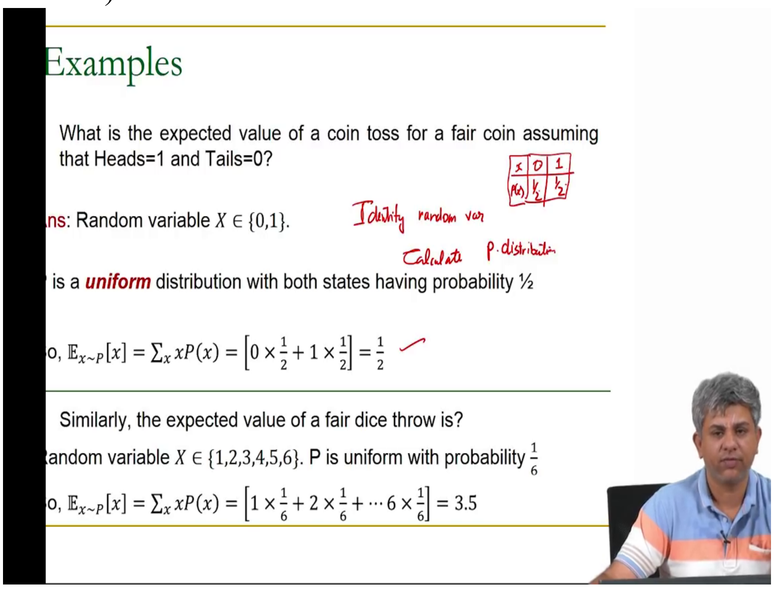

Given that all of these are equally probable, assuming this is a fair dice and it is not like sort of a loaded dice or something, again from the same idea you get 3 point 5.

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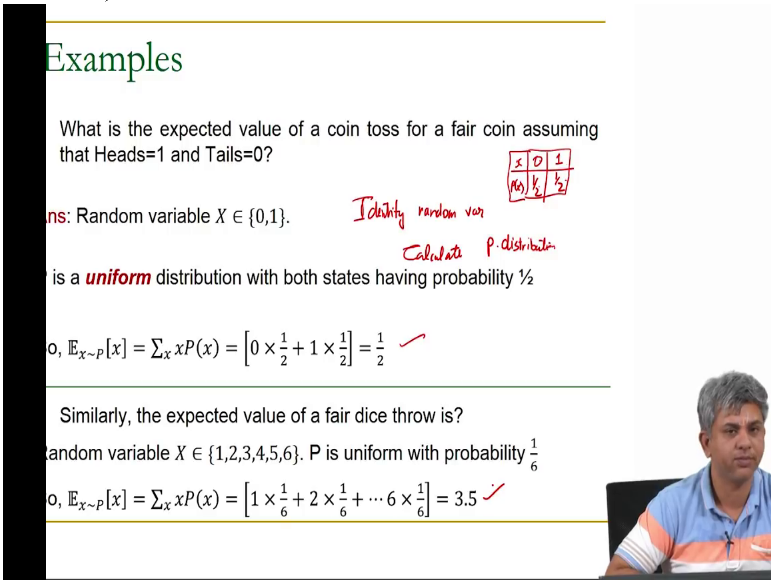

Let us look at a slightly more complex

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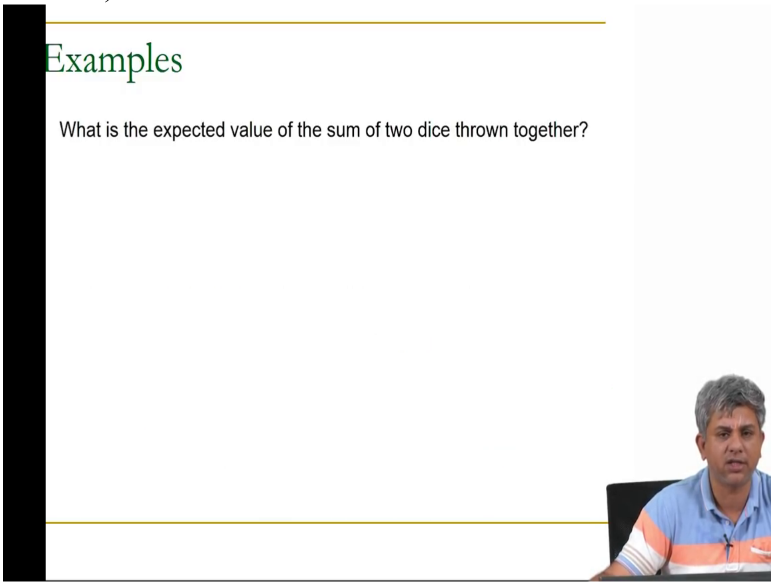

example. This is just to see, you know may be a slight increase over simple averages. Ok so what is the expected value of the sum of two dice thrown together?

Now the random variable here

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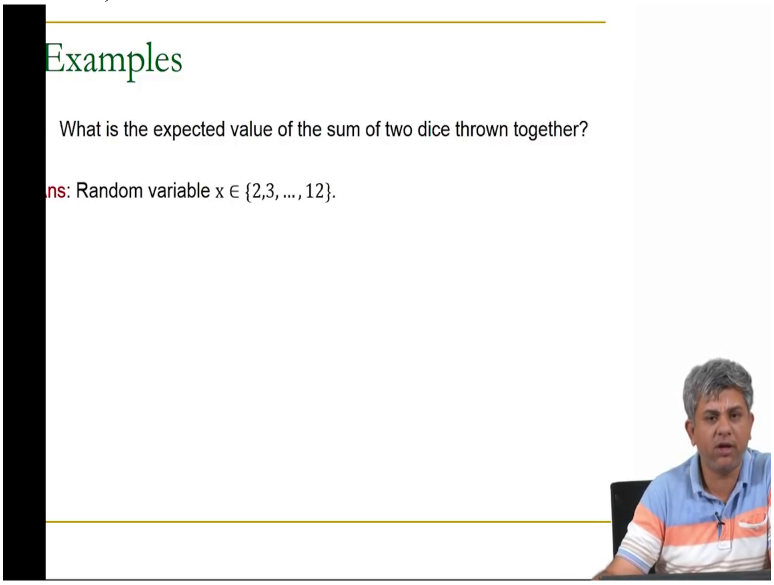

is any number between 2 and 12, obviously 1 cannot occur if you are throwing 2 dice and taking the sum.

So you have the variable that goes between

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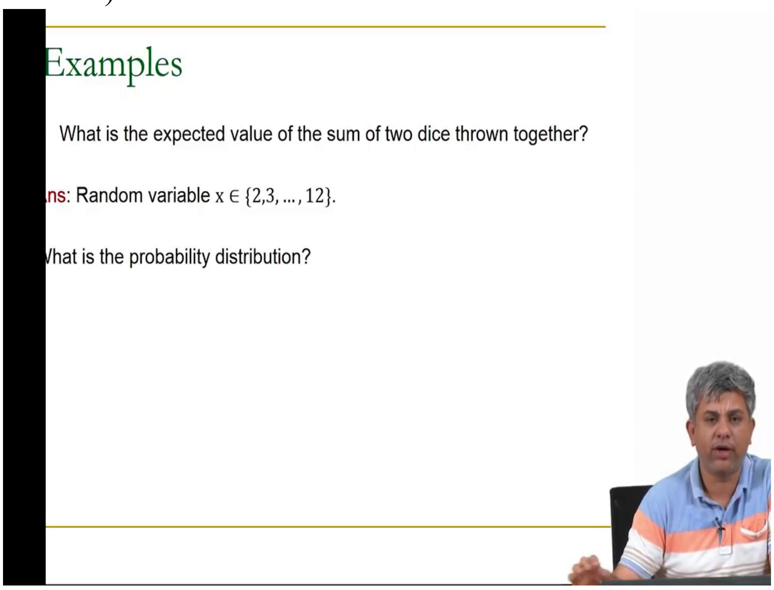

2 and 12. The probability distribution you have to be a little careful now. Ok. Now notice that the probability of 2 is not the same as the probability of 3,

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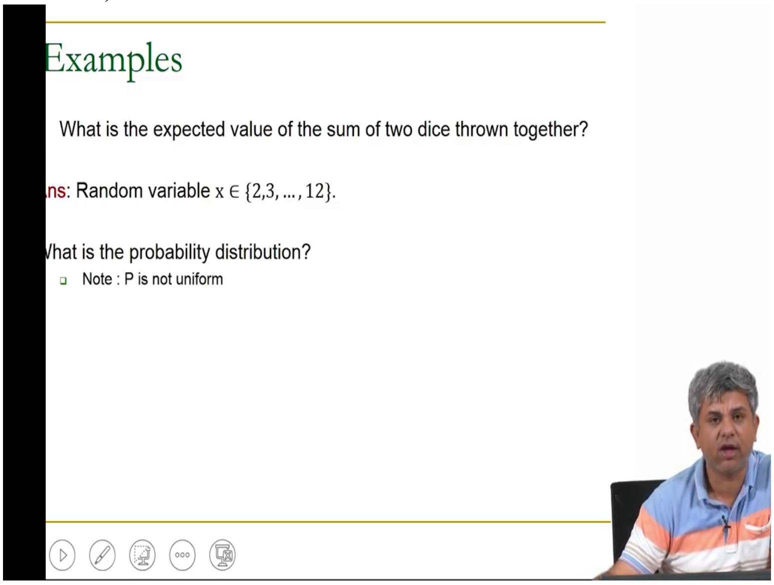

Ok, unlike the previous case where we had uniform probability distributions; in this case each of these probabilities is different, Ok.

So here is the distribution.

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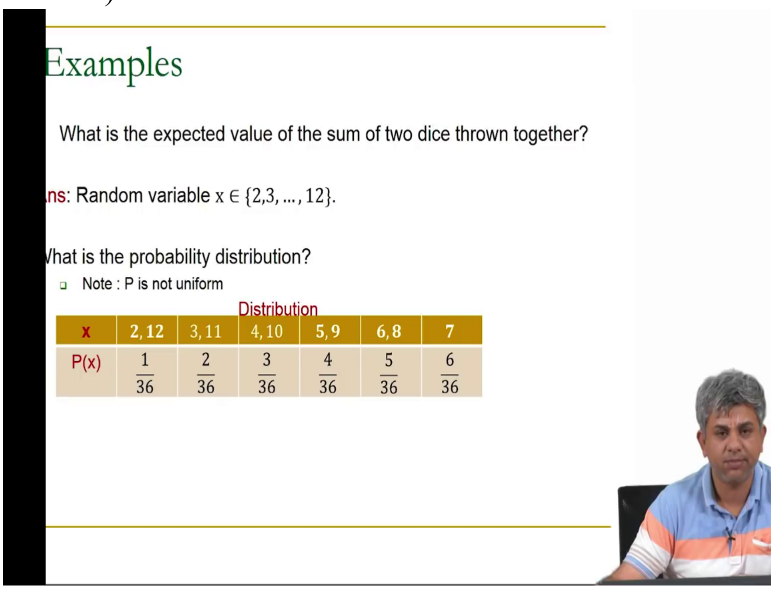

Both 2 and 12, if x is 2 as well as x is 12, both of these can occur in only one way, and so for 2, you need 1, 1 and for 12 you need 6, 6 and that can occur in 1 by 6 multiplied by 1 by 6 which is 1 by 36.

Similarly, 3 can occur

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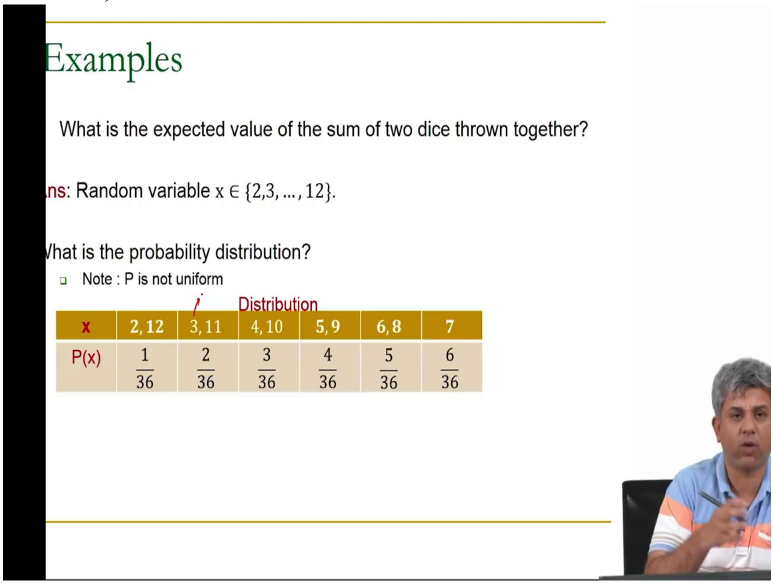

in two ways which is 1 comma 2, 2 comma 1, Ok so therefore you get 2 by 36. Similarly, for 11.

For 4 you have three ways, for 5 you have four ways, for 6 you have five ways, you know 1, 5; 2,4; 3,3 and 5,1 and 4,2, that put together. 7 can actually occur in six different ways and so these are the probabilities.

So notice that unlike the previous examples that we took, in this case, P of x is an actual non-uniform distribution.

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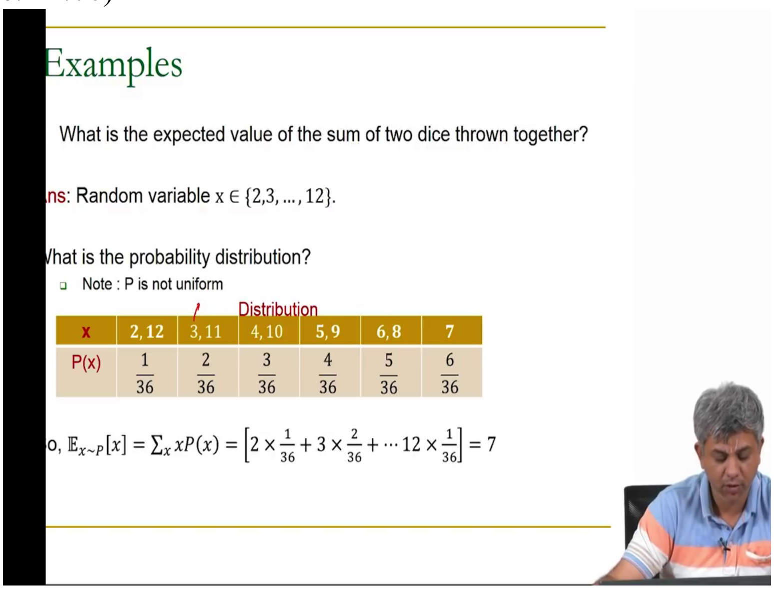

So if you find out expectation, the same thing, so 2 multiplied by 1 by 36, 3 multiplied by 2 by 36, so on and so forth up till 12 multiplied by 1 by 36, if we calculate it, it comes out to be 7.

But the calculation is a little bit lengthy, Ok. So the question is,

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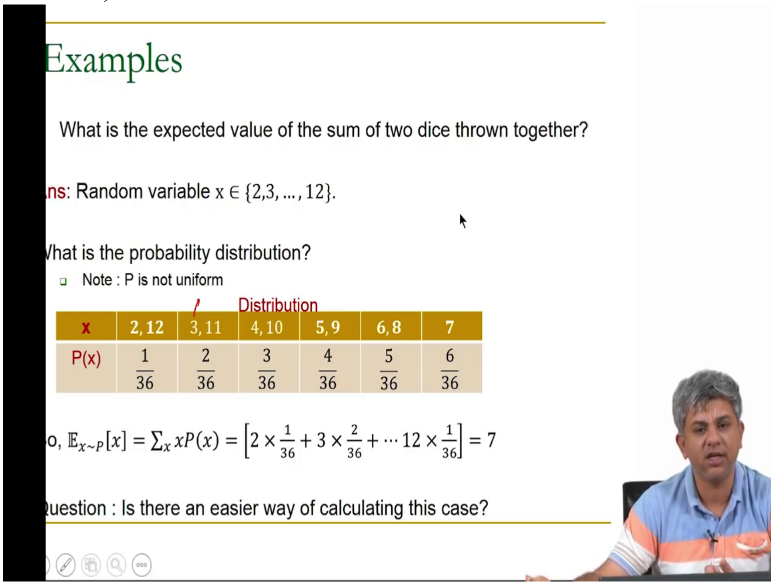

is there an easier way of calculating this case?

So for this we use a

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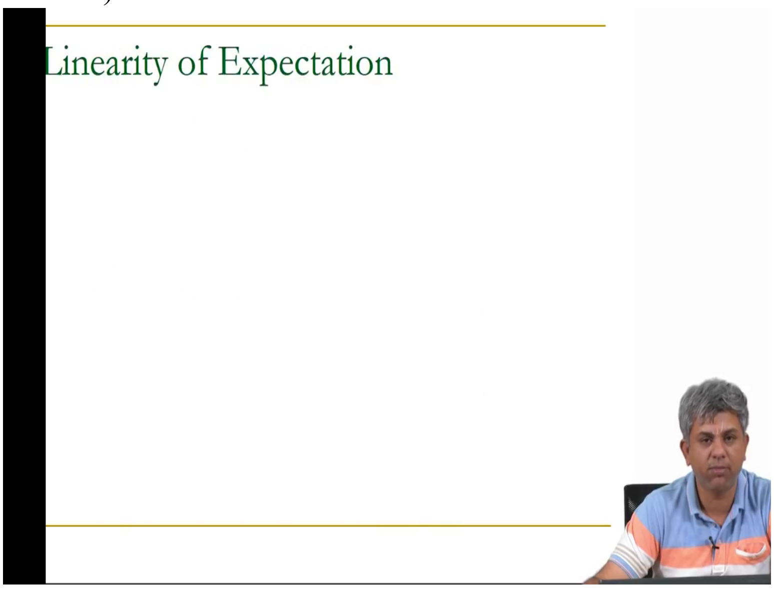

simple idea called the linearity of expectation,

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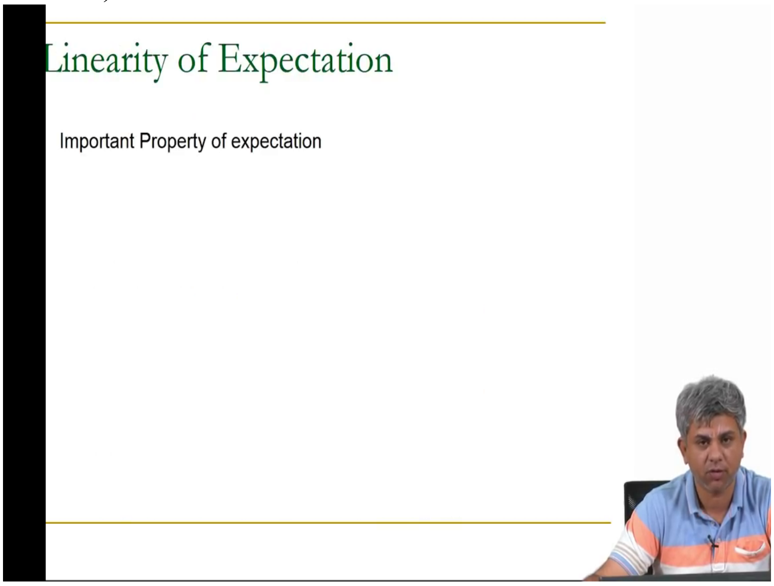

Ok. This is an extremely important property of expectations.

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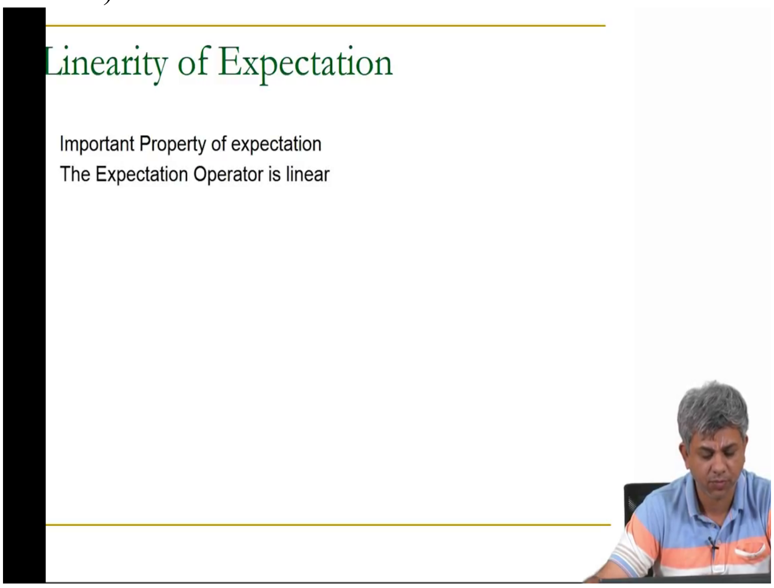

The idea is that

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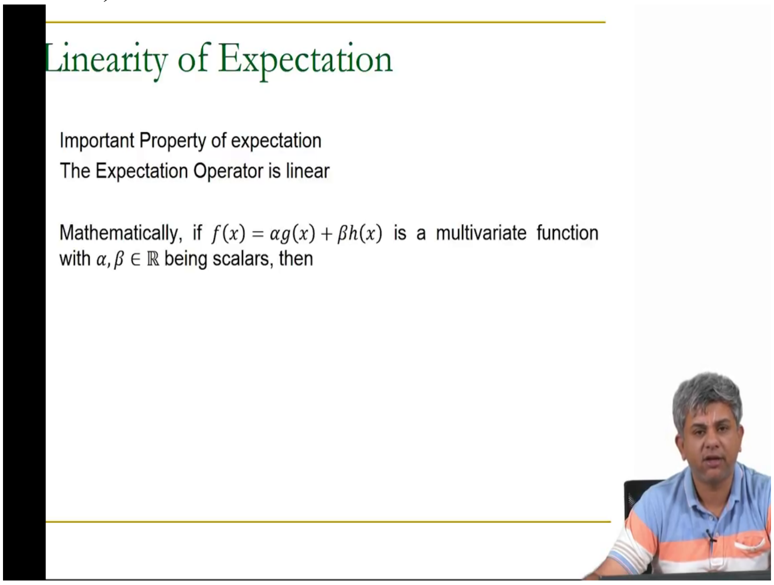

the expectation operator, so this thing, this is a linear operator.

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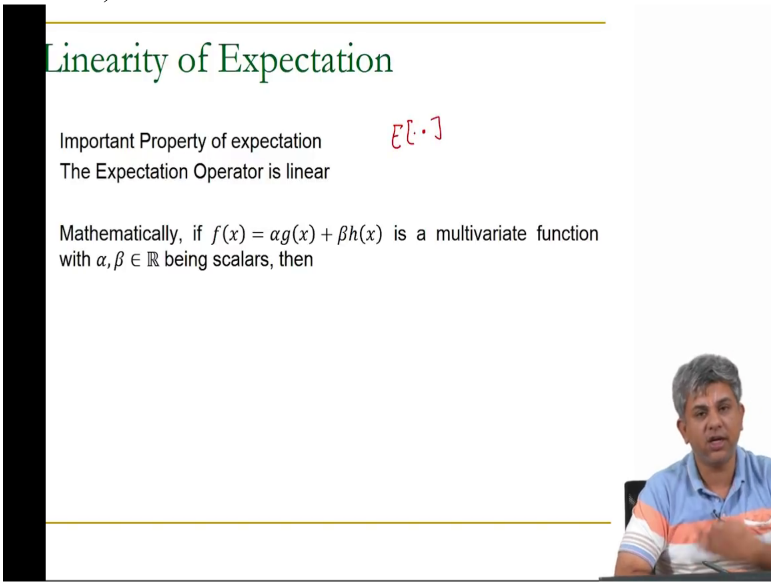

What do I mean by linear?

That is if you have f which is a linear combination, please remember linear combination from our discussion, if f is a linear combination

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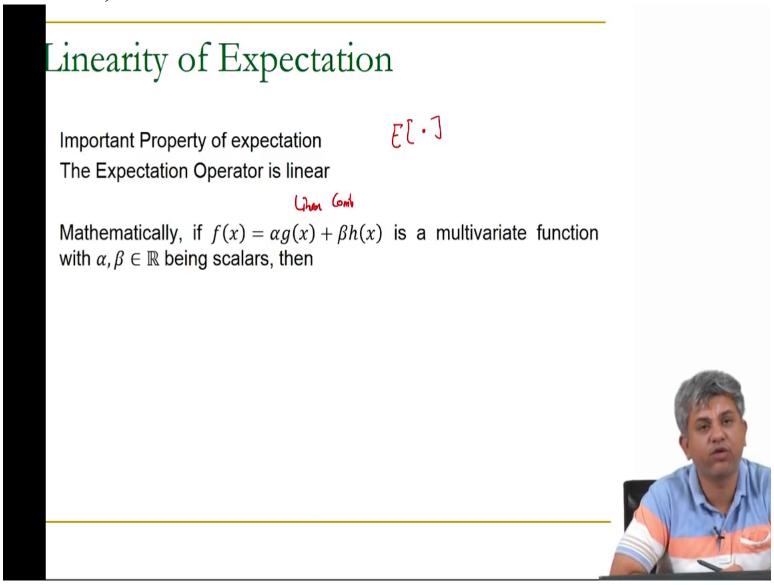

of two other functions g and h, alpha and beta let us say are scalars, so alpha times g of x plus beta times h of x is f of x

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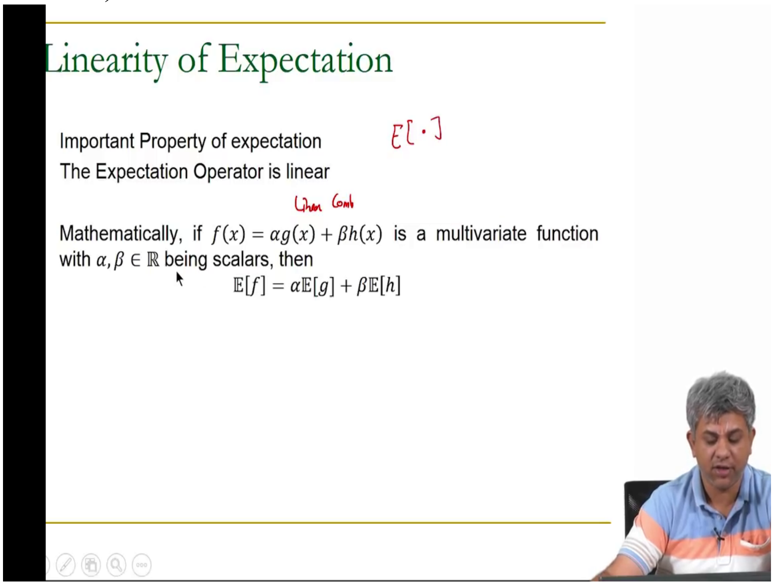

then expectation of f can be written as alpha times expectation of g plus beta times expectation of h, Ok.

So this is an important property. We will just prove it in the next slide,

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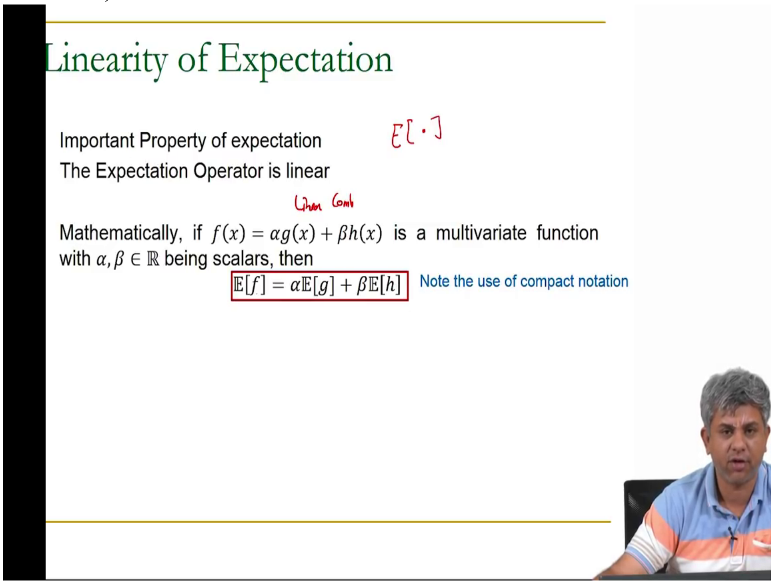

Ok. Also notice that I have used a compact notation here instead of writing expectation x tilde p f of x etc, I simply used expectation of f is alpha times E g plus beta times E h.

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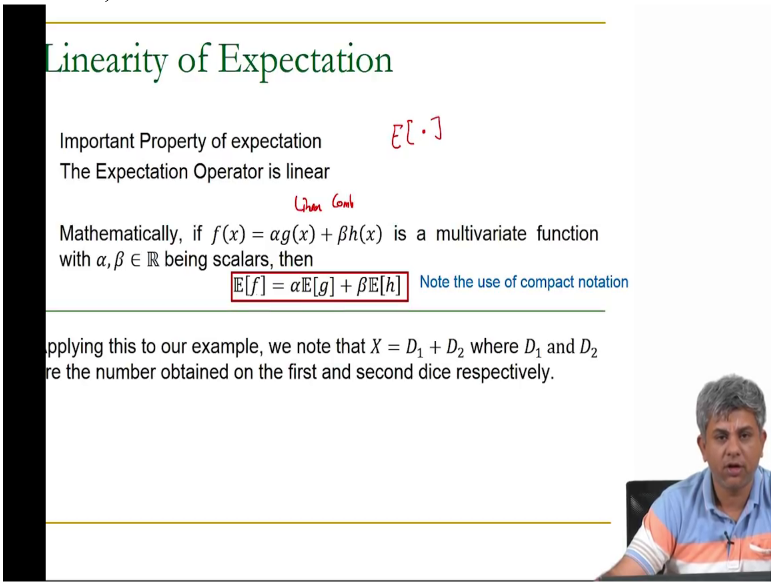

So you can apply this. I will prove this shortly but before that let us simply apply this to our two dice case. Remember that in order to find out the expectation of the two dice we actually had to find out first the probability distribution of each of those occurrences, different outcomes and then we had to do the expectation calculation, Ok.

But suppose we notice that the two dice are essentially two different random variables coming together, one is D 1 and one is D 2, where D 1 is the value that you got out of the first dice, and D 2 is the value that you got out of the second dice, Ok.

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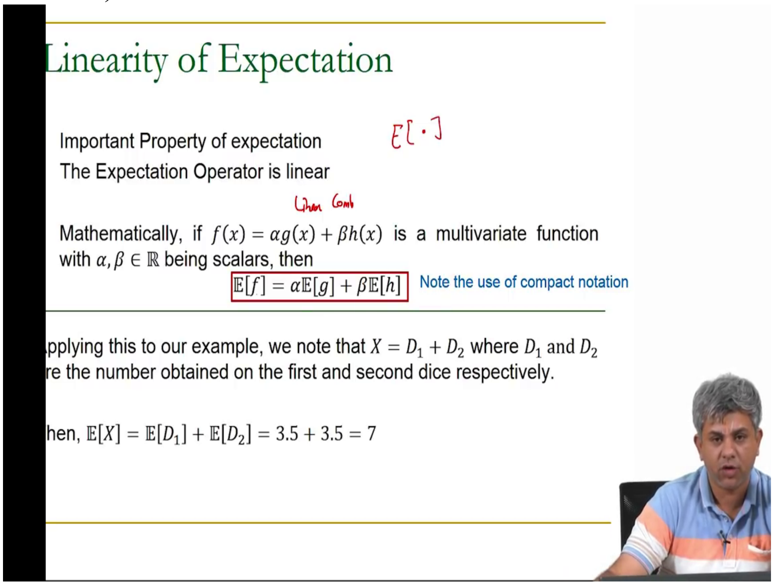

So through our linearity we can write expectation of x is expectation of D 1 plus D 2 which by linearity is equal to expectation of D 1 plus expectation of

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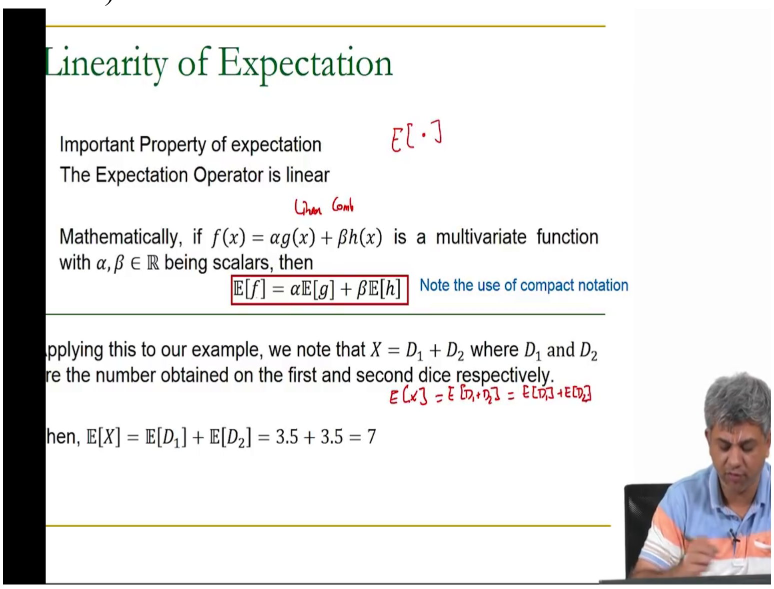

D 2 and we know this already because the expected value from one dice is 3 point 5,

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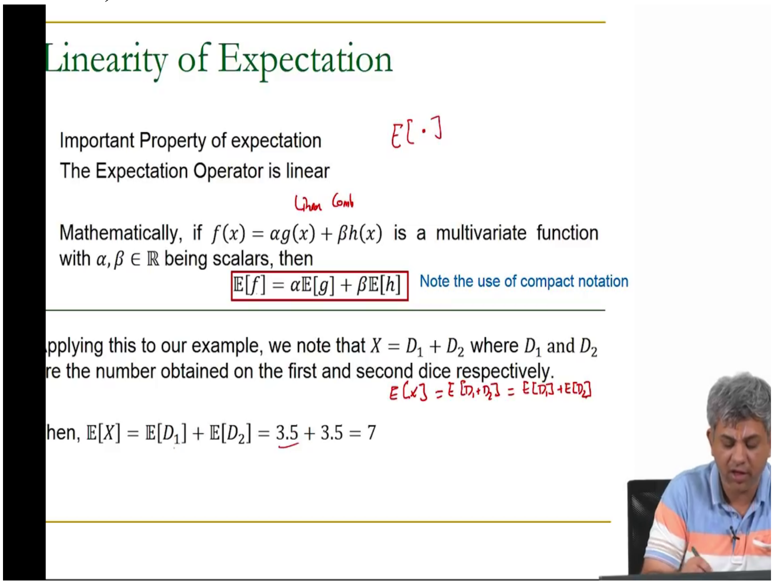

the expected value from the other dice is also 3 point 5 so the expectations add up and you can see that this is a remarkably simple calculation,

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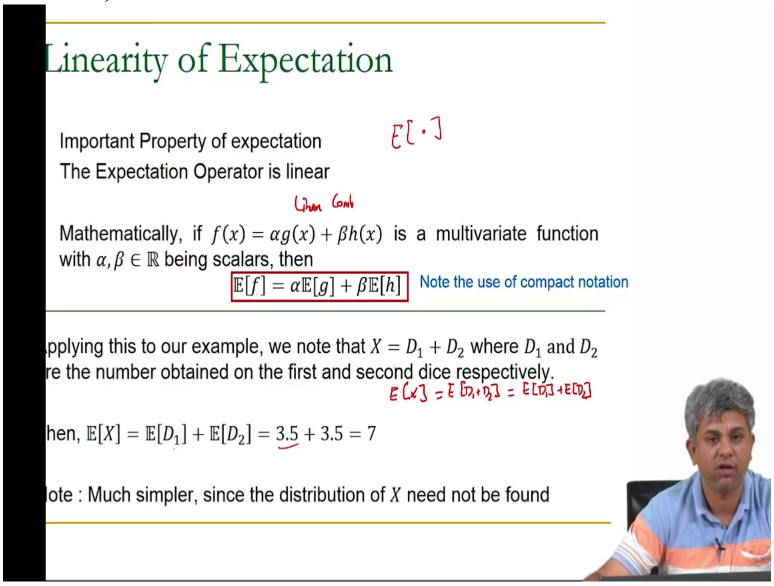

Ok.

This is a much, much simpler calculation compared to actually finding out the overall, you know probability of, probability distribution of x.

So that is the advantage of using linearity of expectations

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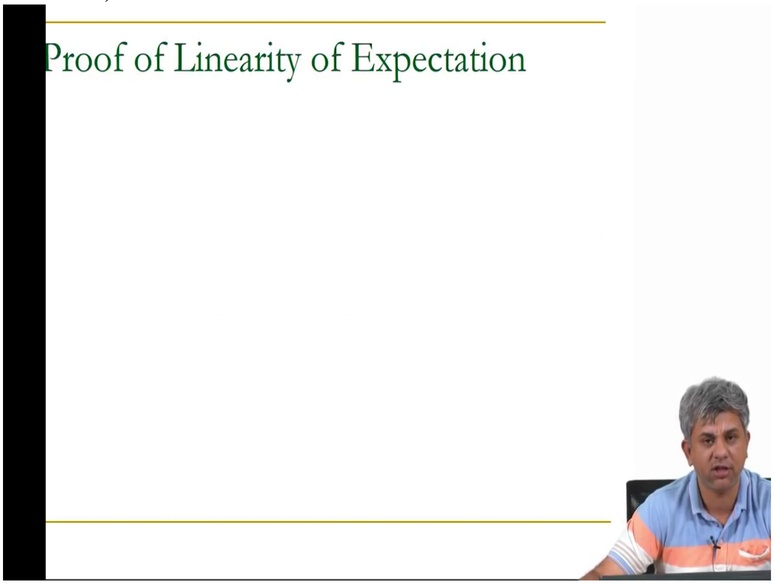

and this is a very commonly used property. Let me give a very quick

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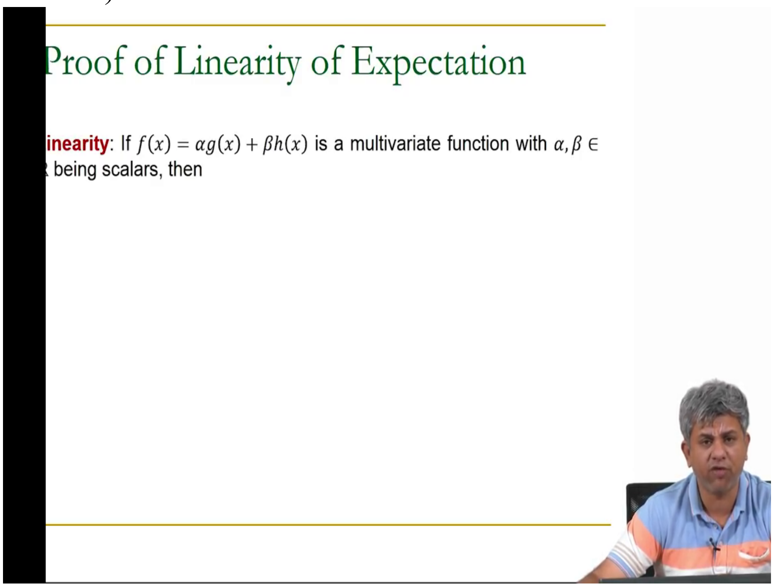

proof. I will be doing it just for continuous variables. So assume that f of x is a multivariate function. This is true not only for univariate but for multivariate functions. And

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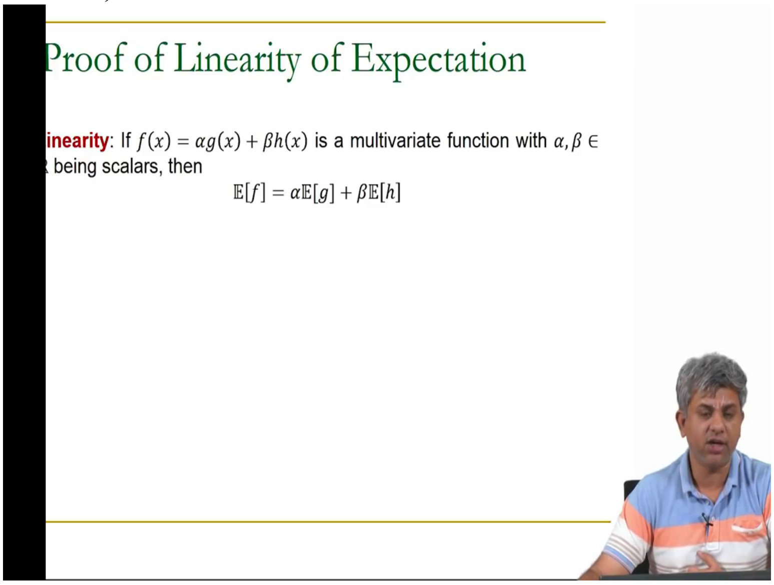

it is a linear combination of g and h.

We want to prove, this is our claim, that the expectation of

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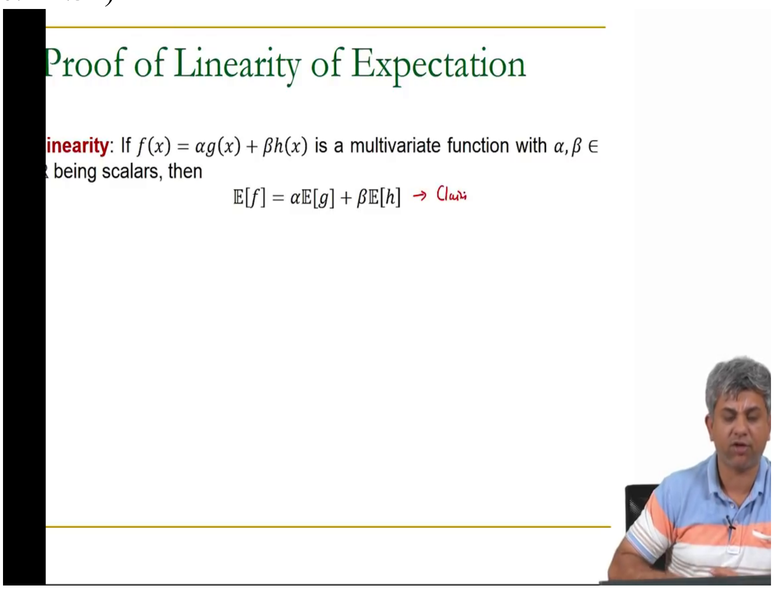

f is alpha E g plus beta E h. So here

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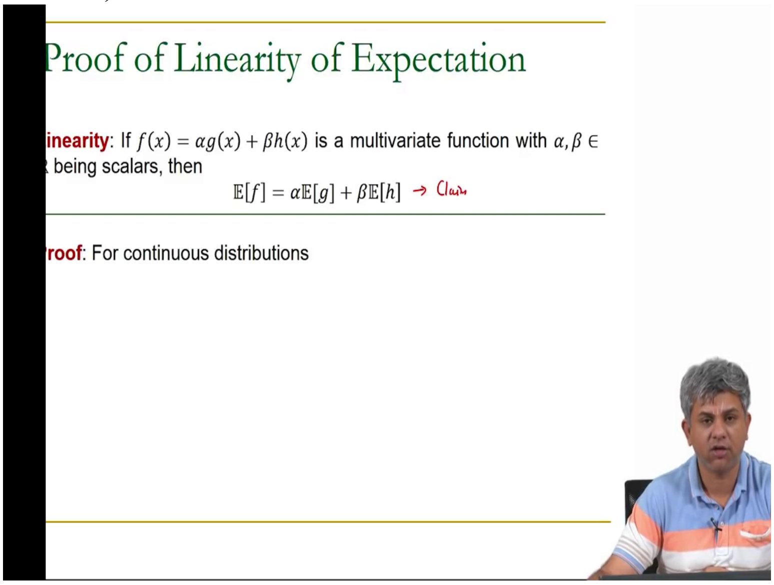

is a quick proof

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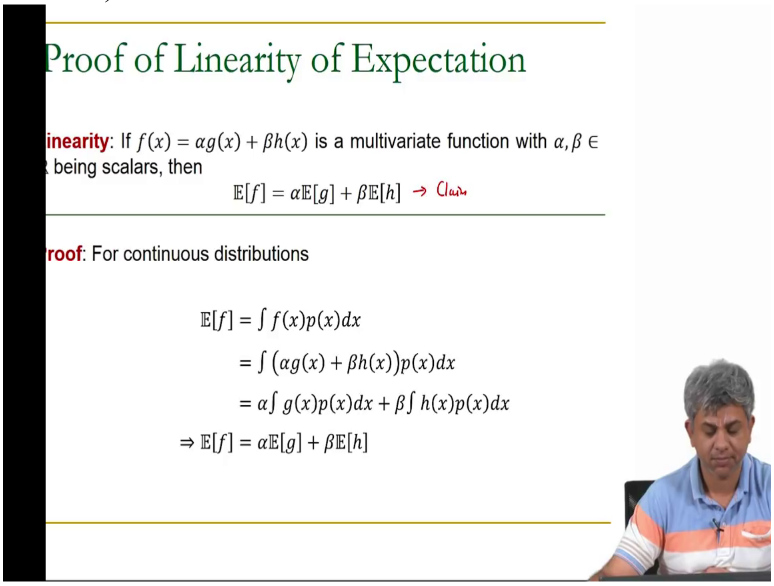

So assuming that this is a continuous distribution, you can write E of f as integral of f of x p of x d x, this is of course the definition of expectation.

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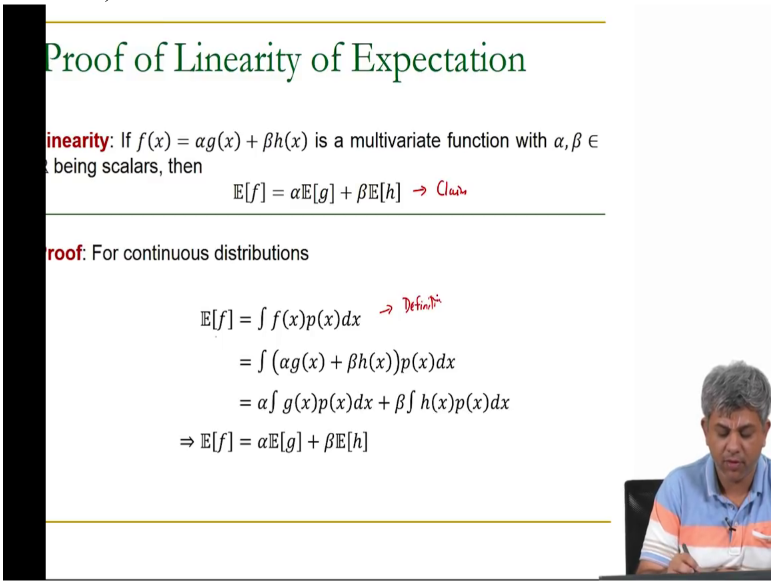

Now f of x

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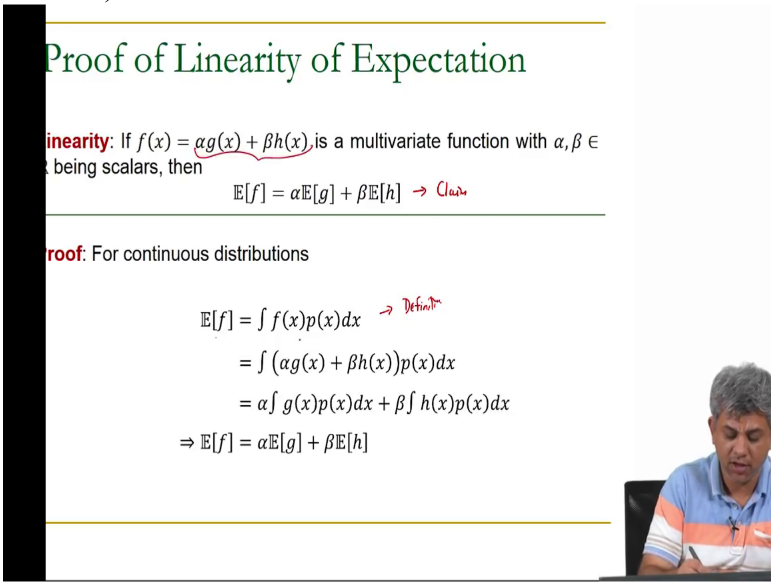

is alpha g plus beta h, Ok. Now we can split this integral into two different integrals.

Since alpha is a constant you can take it out of the integral. alpha does not depend on h, x. So alpha times integral g x p x d x plus beta times integral h x p x d x, this of course is the definition of E of g

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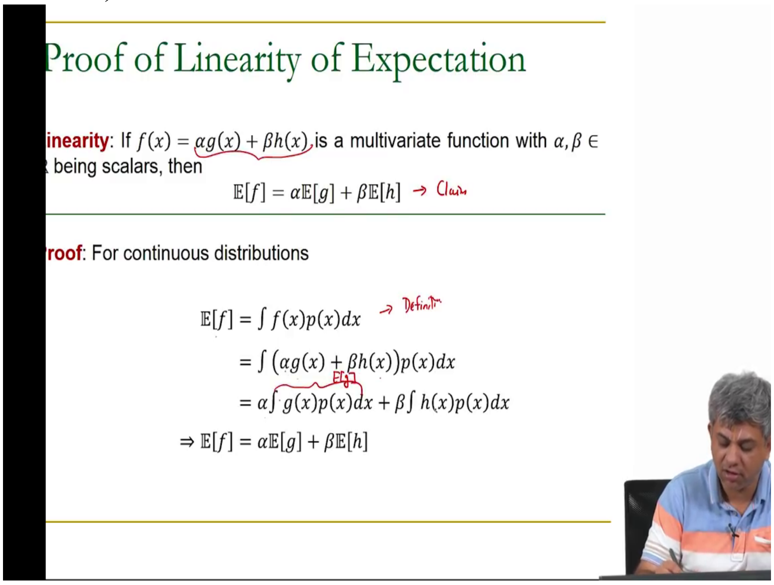

and this is the definition of E of h.

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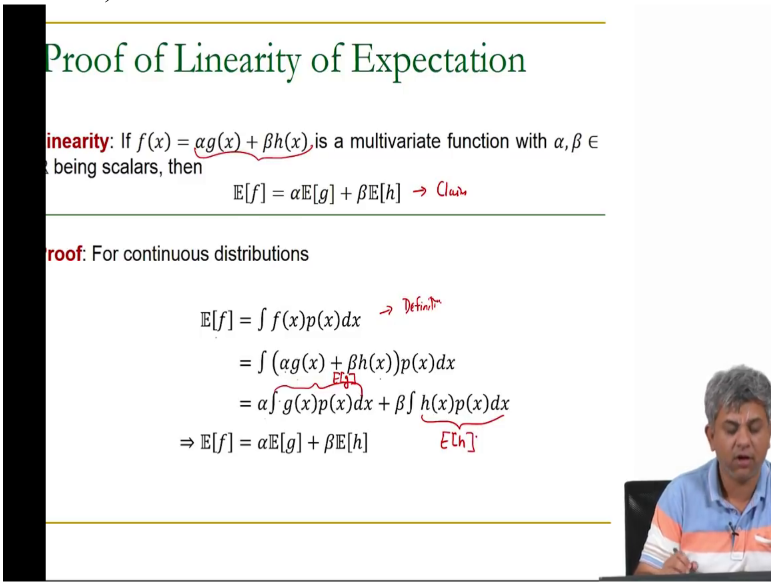

And therefore you end up at the proof that E of f

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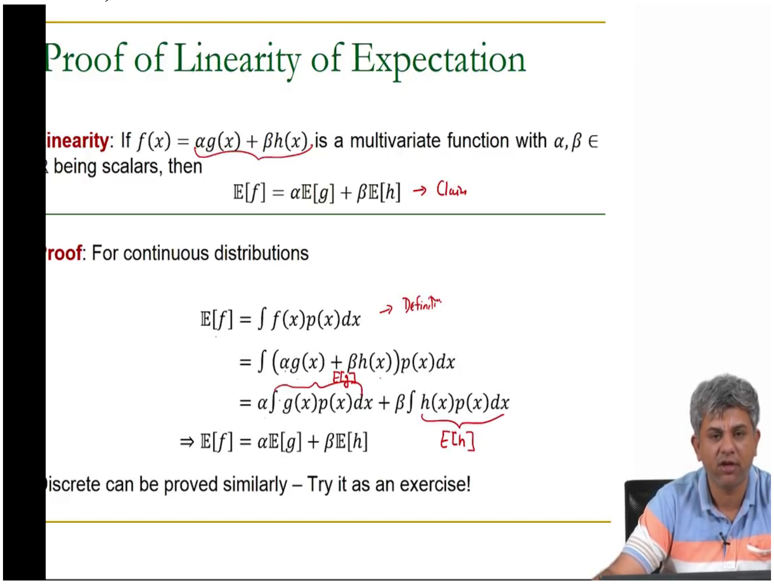

is alpha times E g plus beta times E h.

Now you can also start from the other definition. E is equal to sigma or E of f is equal to sigma f x p x d x and since this is also

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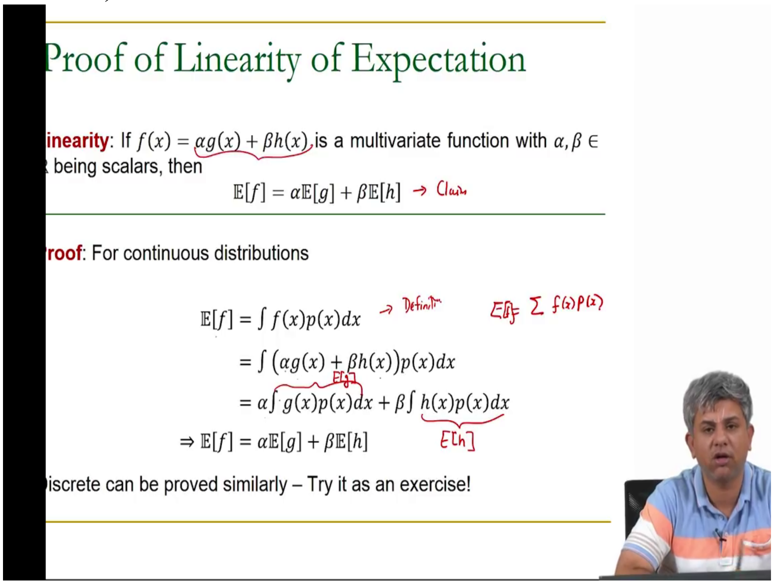

a summation you can prove a discrete also similarly. You can, I suggest that you try this as an exercise. Thank you.