

## Second Level Algorithms

Prof. Palash Dey

Department of Computer Science and Engineering

Indian Institute of Technology, Kharagpur

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Lecture 45

Welcome to the 45th lecture of the second-level algorithm course. In the last class, we started seeing the Edmond's blossom algorithm. We have formally defined what a flower is—it consists of a blossom and an optional stem—and how to modify the given matching with another matching of the same cardinality to get rid of the optional stem. We were discussing how we can compute an M-augmenting path or conclude that M is a maximum cardinality matching of G by reading the output of the recursive call on  $G/B$  and  $M/B$ . In particular, we stated the following lemma, which we will prove today. So, let us begin. So, this is the lemma that we stated in the last class.

M is a maximum cardinality matching in G if and only if  $M/_{GB}$  is a maximum cardinality matching in G. Moreover, given an M-augmenting path in  $M/_{GB}$ , we can compute an M-augmenting path in G in polynomial time.

So, this justifies the use of the recursive call. After that, we will also need to show that if the queue becomes empty, then the current matching M is a maximum matching, which will conclude the proof of the algorithm's correctness. So, let us begin with the proof of this lemma. So, let us start with an M-augmenting path of G and with that, let us compute an  $M/B$  augmenting path. So, it is an if-and-only-if statement. So, we have to prove both directions.

So, let  $M/B$  is a maximum cardinality matching in  $G/B$ , but if possible M is not a maximum cardinality matching in G. So, it is a proof by contradiction. So, let us assume that  $M/B$  is a maximum cardinality matching in  $G/B$ , but M is not a maximum cardinality matching in G. Then we have proved that M is not a maximum cardinality matching if and only if it has an augmenting path, then there exists an M augmenting path let us call it P in G ok. observe that P has 2 unmatched vertices both the end points both the end points of P are unmatched vertices and B is a flower it has exactly one unmatched vertex

let us call it vertex of the blossom B be V, then at least one of the unmet vertices of P cannot be V. let us call it U. Now, two cases, if P does not contain any vertex of B of the blossom B then P is also an  $M/B$  augmenting path in  $G/B$  contradicting our assumption that  $M/B$  is a maximum cardinality matching in  $G/B$  because using an  $M/B$  augmenting path in  $G/B$ , we can obtain another matching in  $G/B$  whose cardinality is 1 more than the cardinality of  $M/B$ . So, now, assume that the path p intersects with b. So, how does the path look like?

It starts with an unmatched vertex u which is not part of the part of the blossom and we keep traversing the path it is an alternating path and we keep traversing the path and till we encounter the first vertex w which belongs to b. So we traverse the path P from U until we encounter a vertex in B any vertex in B let us call it W. Now, we look at the prefix of the path P from u to w that is the part of the path P from u to w. We consider the prefix let us give it a name say u which is say equal to  $x_0, x_1, \dots, x_k$  which is equal to w. Then we consider the prefix of the path and observe the path. Now, look at the prefix of the path in the contracted graph  $G/B$  all the vertices  $x_0, x_1, \dots, x_{k-1}$  because they are not part of the blossom B those vertices are present in the contracted graph  $G/B$ , but  $x_k$  is a part of the blossom B and that blossom B. So,  $x_k$  this vertex is not present in the contracted draft  $G/B$ . So, let us replace this vertex  $x_k$  with the contracted vertex corresponding to the blossom. So, now, that is the path let us look at Q equal to  $x_0, x_1, \dots, x_{k-1}$  and then the vertex corresponding to the blossom B. Now, we claim that this is an  $M/B$  augmenting path why the edges alternate it starts with an unmatched vertex  $x_0$  and observe that  $v_B$  is an unmatched vertex because if you look at the graph G and this blossom B because this blossom has one unmatched vertex and every other vertex is matched so all the edges incident on B from any vertex outside B all the boundary edges of B must be unmatched edges, they are not part of matching. Hence, in the contracted graph when B becomes one vertex it is not incident on any of the matching edges in  $M/B$ . So, the path  $x_0, x_1, \dots, x_{k-1}$  to  $v_B$  is an  $M/B$  augmenting path in  $G/B$ . However, this contradicts our assumption that  $M/B$  is a maximum cardinality matching in  $G/B$ . So, in both cases, we get a contradiction. So, this proves that one direction of this lemma: if  $M/B$  is a maximum cardinality matching in  $G/B$ , then M must be a maximum cardinality matching in G. So, for the other side, let us again assume. Let us assume that M is a maximum cardinality matching in G, but there is an  $M/D$  augmenting path

in  $G/B$ . Let us call that path P. Again, two cases: if P does not contain the super vertex  $v_B$ , then P is also an M-augmenting path in G, contradicting our assumption that M is a maximum cardinality matching in G. So, let us assume  $v_B$  is part of the path P.

$v_B$  is an unmatched vertex in  $G/B$ . The path P must look like: It starts with an unmatched vertex  $x_0, x_1, x_2$ . Edges alternate from within matching and outside matching. and it ends with another unmatched vertex, and this  $v_B$  must be one of the endpoints. So, without loss of generality, let us assume  $x_k$  is equal to  $v_B$ . Then, since this edge  $x_{k-1}x_k$  belongs to the edge set of  $G/B$ . You see here is the super vertex  $v_B$ , and this is  $x_{k-1}$ . Why is this edge present in  $G/B$ ? The only reason it is present in  $G/B$  is because in the blossom B, there exists a vertex—let us call it y—such that there is an edge between  $x_{k-1}$  and y in G. So, there exists say y in the blossom B such that the edge  $x_{k-1}y$  is present in the graph G. Now, let us recall: how does the blossom look like?

It is an odd cycle with edges alternating between matching and out of matching, except for one vertex, which is an unmatched vertex. Now, whatever the vertex y be in the odd cycle, because it is an odd cycle, the edge between y and  $x_{k-1}$  does not belong to  $M/B$ . Because  $v_B$ , the super vertex, was an unmatched vertex. So, we take the path from y to the unmatched vertex—let us call it v—which starts from starts using an unmatched edge and ends at B. So, let us call this path Q. Then the path  $x_0, x_1, \dots, x_{k-1}$  and then concatenated with Q is an M-augmenting path in G, contradicting our assumption that M is a maximum cardinality matching in G. So, this proves the first part of the algorithm or the lemma: M is a maximum cardinality matching in G if and only if  $M/B$  is a maximum cardinality matching in  $G/B$ . So, if somehow the recursive call concludes that  $M/B$  is a maximum cardinality matching in  $G/B$ , then the algorithm rightly concludes that M is also a maximum cardinality matching in G. And the proof gives a polynomial-time algorithm to compute an M-augmenting path in G from an  $M/B$  augmenting path in  $G/B$ . So, this concludes the proof of the lemma. In the next class, we will see how the algorithm is right when it outputs that the current matching is a maximum matching if the queue is empty, OK. So, let us stop here. Thank you.