Second Level Algorithms

Prof. Palash Dey

Department of Computer Science and Engineering

Indian Institute of Technology, Kharagpur

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Lecture 42

Welcome to the 42nd lecture of the second-level algorithms course. In the last class, we started seeing the maximum matching problem for general graphs, and we will continue the study of the maximum matching problem in this lecture, ok? So, let us begin. In the last class, we made the following claim, which lets us formally write as a lemma. A matching M is a maximum cardinality matching in G if and only if there is no M-augmenting path. in G. Proof: For the 'if' part, we have to show that if there is no M-augmenting path in G, then M must be a maximum matching. So, we will prove this using contradiction. Suppose there is no M-augmenting path in G and M is not a maximum cardinality matching. Ok, then there exists another matching M' of G such that the cardinality of M' is 1 more than the cardinality of M. We consider the set edges where H is M symmetric difference M'. Both are sets. Let us recall the symmetric difference of two sets A and B is defined as $(A \setminus B) \cup (B \setminus A)$, ok. So, which can also be written as $(A \cup B) \setminus (A \cap B)$. So, from $A \cup B$, we remove the common elements.

So, the The symmetric difference of two matchings is the union of the two matchings minus the common edges. So, how does the graph G with the set H of edges look? We observe that The induced subgraph of G, induced by the set H of edges, is a disjoint collection of paths and cycles where paths and even cycles, where the edges of paths and cycles alternate between M and M'. Every cycle has the same number of edges. from M and M', ok. However, the cardinality of M' is 1 plus the cardinality of M, ok.

So, we there exists a path P in $G[H]$ whose both end edges belong to M', ok. However, such a path is an M-augmenting path.

Contradicting our assumption, that there is no M-augmenting path in G. So, let us see pictorially how it looks like. We must have a path which starts with an edge in M', then we have an edge in M, which I am denoting with black color, and this is how it alternates.

And it ends with an edge in M'. Okay, so such a path must exist because the number of edges in M' is one more than the number in M. But such a path is an M-augmenting path because the end vertices must be free vertices in M. If it is not, then this path could have been extended. Okay, so this shows the 'if' direction.

The 'only if' direction we have already shown. The 'only if' direction says that a matching M is a maximum cardinality matching, then there must not be any M-augmenting path. Indeed, if there exists an M-augmenting path P, then we can use P to construct another matching M', which is defined as M symmetric difference P, with cardinality of M' being 1 more than the cardinality of M. Hence, M cannot be a maximum cardinality matching. Okay.

So, this concludes the proof of our lemma. Look for another characterization of maximality. Max of maximum matching, maximum cardinality matching. For that, let us consider any subset say U, where U is a subset of vertices, and let us delete U from the graph. In particular, let us look at the induced subgraph, the subgraph induced by the vertices in $V \setminus U$. It breaks into various components.

many of them are of even cardinality, many of them are of odd cardinality. Now, we only focus on those components having an odd cardinality. Then you see that the component whose number of vertices is an odd number, all the vertices cannot be matched together. to match all the vertices in any odd component we need at least one vertex from u. So, let us write we consider the odd components of the graph induced by $V \setminus U$ any odd component or for any odd component not all its vertices can be matched themselves to match all the vertices of any odd component at least one vertex from U is required ok. So, let this gives a natural bound natural upper bound on the cardinality of M. For that let $o(V \setminus U)$ be the number of odd in $G[V \setminus U]$. So, you see if I look at the matching M for every vertex at least one edge one vertex from U is required to match all the vertices in an odd component. So, if the number of odd component is more than the cardinality of U then the all the vertices in every odd component cannot be matched and hence the extra number of odd components those many vertices has to remain unmatched for any matching of this graph. So, the number of vertices that can be matched is at most then from Set of all vertices minus the number of vertices that must remain unmatched, which is the number of odd components of $V \setminus U$ minus the cardinality of U. So, at most these many vertices can be matched, and hence the size of the maximum matching can be at most half of the maximum number of vertices that can be matched. Because this inequality—ah, this is the cardinality of M. Because this inequality holds for every subset

U of V, we have cardinality of M is less than or equal to the minimum over U subset of V of this expression.

The celebrated Tutte-Berge theorem says that this inequality is actually tight. For any maximum cardinality matching $M[G]$, we have the cardinality of M is equal to the minimum over subsets of vertices of the upper bound. Okay, so one direction is obvious, as we have argued—this is fine. We will show the other direction, thereby proving the Tutte-Berge theorem. The proof of the other direction is algorithmic: we will prove M we will exhibit a matching M of G such that the cardinality of M is greater than or equal to half the cardinality minus $o(V \setminus U)$ minus the cardinality of U, okay, thereby proving the Tutte-Berge theorem. Actually, we will design a polynomial-time algorithm to compute the above matching M, okay, and the algorithm is the celebrated Edmond's blossom algorithm. This is an iterative algorithm; in every iteration.

It calls a recursive process or recursive algorithm. Matching M, the recursive algorithm returns matching $M^{'}$. With cardinality $M^{'}$ being the same as the cardinality M and an $M^{'}$ augmenting path. So, the recursive algorithm either returns either returns another matching $M^{'}$whose size is the same as M and an $M^{'}$ augmenting path, or it returns a subset U of vertices such that the cardinality of $M^{'}$ is at least the Tutte-Berge bound, ok. So, we will see this Edmonds' blossom algorithm in the next class, ok. Thank you.