Second Level Algorithms

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Lecture 37

Welcome to the 37th lecture of the second-level algorithm course. In the last lecture, we started seeing the main proposing deferred acceptance algorithm, thanks to Gale and Shapley. We have seen the pseudocode and working of the algorithm, which we dry-ran, and we started the proof of correctness of the algorithm. So, we will complete that in this lecture, okay? So, let us begin.

Proof of correctness. We have already seen that the main proposing deferred acceptance algorithm makes at most $n \times (n-1)+1$ iterations and outputs a perfect matching. This is, of course, under the assumption—which we made without loss of much generality—that the number of men is the same as the number of women in all our instances.

In this lecture, we will show that the perfect matching output by the main proposing deferred acceptance algorithm is a stable matching. In particular, we will prove this theorem. The perfect matching output by the main proposing deferred acceptance algorithm is a stable matching. Proof.

So, it is a proof by contradiction. So, suppose the perfect matching output by the main proposing deferred acceptance algorithm is not a stable matching. for some instance, ok.

So, there must exist a blocking pair, say (m, w), for this matching. There can be four ways in which a pair of men and women can block a matching. One is both of them are unmatched. The other is one of them is unmatched and the other prefers the other person compared to his or her current partner.

Or the third way is both of them are currently matched, but both of them prefer each other compared to their current partner. So, let us give a name to the perfect matching which is output by the algorithm, which we are assuming is not a stable matching, to be M capital M. Now, because M is a perfect matching both men M and the women W are matched with some women and men, respectively. in M, but for M and W to form a blocking pair, we must have that both the man M prefers W over his partner under the matching capital M and the woman W preferred the man M over her partner under the matching capital M. Now, there can be two cases either man either the man M has proposed to the woman W or the man M has not proposed to the woman W in then execution of the main proposing deferred acceptance algorithm. So, consider the following two exhaustive cases. Exhaustive cases means that at least one of them should happen.

The man M has proposed to the woman W in an execution of the main proposing deferred acceptance algorithm. Notice that the main proposing deferred acceptance algorithm is not fully specified. For example, if there are multiple unmatched man, then the algorithm does not specify which unmatched man to pick. So, it can happen in principle that there are two different executions of the same instance of the problem for the algorithm the main proposing deferred acceptance algorithm and so to argue we need to fix an execution of the algorithm. So, let us fix an or any execution of the main proposing deferred acceptance algorithm. Although let it is a fact that the number of iterations and even more the number of proposals each man makes is exactly the same. irrespective of how we pick an unmatched man in any iteration.

But because we have not proved it, so let us fix an execution of the men-proposing deferred acceptance algorithm and with respect to this execution, let us argue and find a contradiction. So, the first case is the man M has proposed to the woman W in that particular execution of the main proposing deferred acceptance algorithm. Notice that if the man M has proposed the woman W at that point the woman W is matched either with man M or any other men whom women w prefer more.

So, observe, this is a crucial observation that after the where the main aim has proposed the woman W, the woman W is matched with him or any other man whom the woman W prefer more. Every woman can change her partner, but whenever she changes her partner, she is matched with some men whom she prefer more.

But notice that this implies that the woman W preferred her partner under the matching M because matching M is the output of the execution of the main proposing deferred acceptance algorithm. So, this implies that the woman W prefers her partner over the man M. However, this contradicts our assumption that the woman W prefer M more than her partner under the matching capital M. So, we get a contradiction in this case.

In case 2, the man M has not proposed to the woman W in the execution under consideration. In this case, we observe that the man M prefers his partner under the matching M over the woman W. This is so because every man proposes to his most preferred woman, followed by the second most preferred woman, and so on. So, in this case, the man M prefers his partner under the matching capital M over the woman W. However, this contradicts our assumption that the man M prefers the woman W over his partner under the matching capital M. So, we get a contradiction, which concludes the proof. So, in both cases, we get a contradiction; hence, the perfect matching output by the man-proposing deferred acceptance algorithm is indeed a stable matching. Now, we will explain a few more facts without proof about stable matching.

It has a lot of structure. The first fact is the number of proposals any man makes in any instance of the man-proposing deferred acceptance algorithm is the same, irrespective of the choice of the unmatched men picked to propose a woman in any iteration. the amount of non determinism present in the algorithm namely which unmatched man to pick in every iteration does not affect the either the output or the number of iterations of the algorithm . The second important fact the number of stable matchings in an instance can be more than one. So, let us see an example of an instance of stable matching with more than one stable.

So, consider 2 men and 2 women, man 1 prefers W_1 over W_2, man 2 prefers W_2 over W_1, woman 1 prefers M_2 over M_1. and woman 2 prefers M_1 over M_2. So, consider a matching where M_1 is matched with W_1 and M_2 is matched with W_2, this is a stable matching. Why? Because both men are matched with their most preferred women.

hence they cannot be part of any blocking pair . So, no man can participate in a blocking pair, hence no woman also cannot participate in a blocking pair and thus there is no blocking pair for that matching. But you see this is not the unique stable matching. Consider this matching where M_1 is matched with W_2 and M_2 is matched with W_1. These also are stable matching because both the women are matched with their most preferred men.

So, hence we see that in this instance there are two stable matchings. Okay, so let us stop here. Thank you.