Second Level Algorithms

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Lecture 34

Welcome to the 34-th lecture of the second-level algorithms course. In this lecture, we will see an important decomposition result regarding flows, okay? So, let us begin. So, we define two kinds of elementary flows. The first one is path flow.

A path flow is a flow f such that there exists an s-to-t path. P with the following conditions. There exists an s-to-t path P and a constant C greater than 0 such that f of e is C if the edge e is part of the path P and 0 otherwise.

Pictorially, if this is the graph G, this is the source S, the sink T, and here is a path P. We are sending some amount of flow C along this path; every other edge does not carry any flow. The second one is A flow is called a cycle flow. A flow F is called a cycle flow if there exists a cycle C in the input graph G and a constant d greater than 0 such that f of e is d.

If the edge E is part of the cycle C and 0 otherwise. The flow path decomposition theorem states that every flow can be written as a sum of some flow paths or path flows and cycles. Every s to t flow f can be written as a sum of path flows and cycle flows. Moreover, the total number of path flows and cycle flows is at most m, the number of edges. Let us prove it. Let f be an arbitrary s to t flow. We will prove this result by induction on the number of edges that carry positive that carry a positive amount of flow. Base case. There is only one edge that carries a positive amount of flow. That is, by the conservation of flow property. It must be from s to t. Hence, the claim holds true.

F itself must be a path flow for the s-t path. Consisting only of the edge s-t. Inductive hypothesis, the result for every s-t flow with at most m minus 1 edges carrying some positive amount of flow. Now, inductive step: let f be any s-t flow with m edges carrying some positive amount of flow. The idea is we will start a walk from s and either reach t or

we end up finding a cycle. We start a walk which starts with s and continues until we reach t or we visit any vertex twice.

That is, u_i is the same as u_j for some j greater than i. And u_i, u_{i+1} till u_{j-1} are all different. Reach t, then the walk is actually an s-t path P equal to $u_0, u_1, \ldots, u_k$, which is equal to t. Let c be the minimum flow of any edge along this path. P and f_p be the path flow defined as follows.

f_P of e is C if the edge e is part of the path P and 0 otherwise. We observe that $f - f_P$. is a valid s to t flow. Moreover, the number of edges that carry some positive flow in $f - f_P$ is at most $m-1$ because there exists at least one edge in the path P where the flow value is C. So, for at least one edge along the path P, the flow value will be 0 in the flow $f - f_P$. But now, $f - f_P$ is an s-t flow with at most $m-1$ edges carrying some positive amount of flow. By the induction hypothesis, $f - f_P$ can be written as a sum of at most $m-1$ path flows and cycle flows. Hence, f can be written as a sum of at most m path flows and cycle flows. So, while performing a walk from S, if we discover a path from S to T, then we have the result. Now, let us see what if we discover a cycle? If the walk from S discovers a cycle C equal to $u_i, u_{i+1}, \ldots, u_{j-1}, u_j$, which is the same as u_i. So, in the work, we use only those edges which carry some positive amount of flow. That is very important. So, all the edges in cycle C carry a positive amount of flow. The same holds for the path also, which is why this C is strictly greater than 0. So, if the work from is discovered, the cycle C all of its edges carry some positive amount of flow, then we define a constant d greater than 0 and a cycle flow f_C as follows. D is the minimum of the flow value of all the edges of the cycle, which is strictly greater than 0, and f_C of e is D if the edge E is part of the cycle and 0 otherwise. We consider the s-t flow $f - f_C$ and observe that the number of edges carrying a positive amount of flow in $f - f_C$ is at most $m-1$. Using the same argument as the path, it follows from the induction hypothesis that f can be written as a sum of at most m path flows and cycle flows. So, this proves the result.

Moreover, from the proof of this result, we can derive this corollary. For every s-t flow f, there exists an s-t flow f_1 such that the value of f is the same as the value of f_1, and f_1 can be written as a sum of at most m path flows. Okay, so now let us see a concrete example. Consider this flow, this is an s-t flow. So, this flow can be decomposed into two flows. First, let us perform a walk from s. Suppose we get the s-to-t path. So, one flow could be sending the minimum flow value along this path—the minimum flow that any edge is carrying—which is 1, plus the remaining 1.

So, here you see, if we start a walk from s, we cannot find a cycle. If the total amount of flow leaving s is 0, then to find the cycle, we use any cycle-finding algorithm in the graph, considering only those edges with a positive amount of flow. So, using that, we find this cycle, and send one unit of flow along this cycle. So, that is how a flow can be written as a sum of possibly more than one path flow and more than one cycle flow, but the total number of path flows and cycle flows used in the sum is at most the number of edges.

So, let us stop here. Thank you.